Quantum Meanfield Theory with Feedback

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Introduction

Our goal is to investigate the dynamical spinor electrodynamics with feedback. In the community the following approach is discussed:

- The first approach is based on spinor electrodynamics for purely time-dependent electric fields. An equation of motion for the vacuum expectation value of the one-particle Wigner function is derived.
- Taking feedback into account, which implies solving Maxwell's equations inline with the transport equation leads to divergencies, which can be removed with the help of a renormalisation procedure.

Due to the fact that only the vacuum expectation value of the one-particle Wigner correlator is considered the approach lacks generality. A more advanced theory for arbitrary one-particle expectation values in purely time-dependent electric fields can be derived.

Quantum Transport Theory in Meanfield Approximation

In the paper by Smolyansky et al.[1] it is assumed that there is only a time-dependent electric field given by

$$E(t) \; = \; -\frac{d}{dt}A^3(t) \; , \tag{1}$$

while spatial variations and magnetic fields are neglected. The fermions can be expressed using the spinor

$$\Psi^{(\pm)}_{\boldsymbol{p}r}(x) \; = \; L^{-3/2}\left(i\gamma^0\partial_0 + \gamma^k p_k - e\gamma^3 A_3(t) + m\right)\chi^{(\pm)} R_r e^{i\boldsymbol{px}} \; , \tag{2}$$

where R_r, $r = 1, 2$, are the eigenvectors of the matrix product $\gamma^0\gamma^3$ and χ is the solution of the differential equation

$$\ddot{\chi}^{(\pm)}(\boldsymbol{p}, t) \; = \; -\left(\omega^2(\boldsymbol{p}, t) + ie\dot{A}_3(t)\right)\chi^{(\pm)}(\boldsymbol{p}, t) \; , \tag{3}$$

where $\pm$ denote the positive and negative frequencies of the eigenstates of (3). With the help of the spinor it is possible to define the ladder operators $c^{(\pm)}_{\boldsymbol{p}r}$, $\overset{*}{c}{}^{(\pm)}_{\boldsymbol{p}r}$, which are subject to the following equation of motion

$$\frac{d}{dt}c^{(\pm)}_{\boldsymbol{p},r}(t) \; = \; \pm\frac{eE(t)\epsilon_\perp}{2\omega^2(\boldsymbol{p}, t)}c^{(\mp)}_{-\boldsymbol{p},r}(t) + i\left[H(t), c^{(\pm)}_{\boldsymbol{p},r}(t)\right] \; , \tag{4}$$

[1] Smolyansky et al., **Dynamical derivation of a quantum kinetic equation for particle production in the Schwinger mechanism**, arXiv:hep-ph/9712377

Quantum Transport Theory in Meanfield Approximation

and the vacuum expectation value

$$f_r(\boldsymbol{p},t) \; = \; \langle \, 0_{\rm in} \mid \overset{*}{c}{}^{(+)}_{\boldsymbol{p}r}(t) \; c^{(-)}_{\boldsymbol{p}r}(t) \mid 0_{\rm in} \, \rangle \; . \tag{5}$$

The structures of the fermion and boson distribution functions are similar and can be combined into one expression, where $\pm$ now denote the bosonic $(+)$ and the fermionic $(-)$ cases (see Bloch et al.[2]). The equation of motion of the vacuum expectation value of the one-particle correlation function is given by

$$\frac{d}{dt} f_{(\pm)}(\boldsymbol{p},t) \; = \; \frac{1}{2} \mathcal{W}_{(\pm)}(t) \int_{-\infty}^{t} \mathcal{W}_{(\pm)}(t) F_{(\pm)}(\boldsymbol{p},t) \cos\big(x(t',t)\big) \; = \; S_{(\pm)}(\boldsymbol{p},t) \; , \tag{6}$$

where

$$\mathcal{W}_{(\pm)}(t) \; = \; \frac{eE(t)p_3(t)}{\omega^2(\boldsymbol{p},t)} \Big(\frac{\epsilon_\perp}{p_3(t)} \Big)^{g_{(\pm)}-1} \; , \tag{7}$$

$$F_{(\pm)}(\boldsymbol{p},t) \; = \; \big[1 \pm 2 f_{(\pm)}(\boldsymbol{p},t) \big] \; , \tag{8}$$

$$x(t',t) \; = \; 2 \big[\theta(t) - \theta(t') \big] \tag{9}$$

with $\theta(t)$ being the dynamical phase.

[2] Bloch et al., **Pair creation: Back reactions and damping**, Phys. Rev. D 60, 116011

Renormalisation Procedure

Simpler equations of motion are obtained by introducing

$$v_{(\pm)}(\boldsymbol{p},t) = \int_{t_0}^{t} dt' \mathcal{W}_{(\pm)}(\boldsymbol{p},t')\, F_{(\pm)}(\boldsymbol{p},t') \cos\big(x(t,t')\big)\,, \tag{10}$$

$$z_{(\pm)}(\boldsymbol{p},t) = \int_{t_0}^{t} dt' \mathcal{W}_{(\pm)}(\boldsymbol{p},t')\, F_{(\pm)}(\boldsymbol{p},t') \sin\big(x(t,t')\big)\,. \tag{11}$$

Inserting (10), (11) into the transport equation (6) a set of coupled linear differential equations in $f_{(\pm)}(\boldsymbol{p},t)$, $v_{(\pm)}(\boldsymbol{p},t)$ and $z_{(\pm)}(\boldsymbol{p},t)$ is obtained

$$\frac{d}{dt} f_{(\pm)}(\boldsymbol{p},t) = \frac{1}{2}\mathcal{W}_{(\pm)}(\boldsymbol{p},t')\, v_{(\pm)}(\boldsymbol{p},t)\,, \tag{12}$$

$$\frac{d}{dt} v_{(\pm)}(\boldsymbol{p},t) = \mathcal{W}_{(\pm)}(\boldsymbol{p},t')\, F_{(\pm)}(\boldsymbol{p},t) - 2\omega(\boldsymbol{p},t)\, z_{(\pm)}(\boldsymbol{p},t)\,, \tag{13}$$

$$\frac{d}{dt} z_{(\pm)}(\boldsymbol{p},t) = 2\omega(\boldsymbol{p},t)\, v_{(\pm)}(\boldsymbol{p},t)\,. \tag{14}$$

Renormalisation Procedure

The charged particles accelerated by the external electric field generate a current that is capable of modifying the latter due to Maxwell's equation

$$\dot{E}(t) \;=\; -\, j_{ex} - j_{in} \;, \tag{15}$$

where the effect of the feedback is described by the internal current (16). The internal current consists of the conduction current $j_{cond}(t)$, which is generated by the newly created particles and a polarisation current $j_{pol}(t)$

$$\begin{aligned} j_{in}(t) \;&=\; j_{cond}(t) + j_{pol}(t) \\ &=\; e g_{(\pm)} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p_3(t)}{\omega(\boldsymbol{p},t)} \left(f_{(\pm)}(\boldsymbol{p},t) + \underbrace{\frac{1}{2} v_{(\pm)}(\boldsymbol{p},t) \left(\frac{\epsilon_\perp}{p_3(t)}\right)^{g_{(\pm)}-1}}_{\text{divergent polarisation current}} \right) . \end{aligned} \tag{16}$$

The polarisation current is logarithmically divergent and will be renormalised. To do this it is helpful to explore the asymptotic behaviour of the latter. With the help of power counting in (16) it is obvious that

$$f(\boldsymbol{p},t),\; v(\boldsymbol{p},t),\; z(\boldsymbol{p},t) \;\overset{|\boldsymbol{p}|\to\infty}{\leq}\; \frac{1}{|\boldsymbol{p}|^4} \;. \tag{17}$$

Renormalisation Procedure

With the separable ansatz

$$f(\boldsymbol{p},t)=\sum_{k=0}^{\infty}\frac{f_k(t)}{|\boldsymbol{p}|^k},\ v(\boldsymbol{p},t)=\sum_{k=0}^{\infty}\frac{v_k(t)}{|\boldsymbol{p}|^k},\ z(\boldsymbol{p},t)=\sum_{k=0}^{\infty}\frac{z_k(t)}{|\boldsymbol{p}|^k} \tag{18}$$

and the approximation $p_3 \approx \omega(\boldsymbol{p},t) \approx \epsilon_\perp$ for large $|\boldsymbol{p}|$ substituted into the transport equations (12)-(14) the leading order terms

$$f_4=\frac{1}{16}e^2E^2(t),\ v_3=\frac{1}{4}e\dot{E}(t),\ z_2=\frac{1}{2}eE(t) \tag{19}$$

are obtained. By inserting (19) into the internal current (16) the total current (15) takes the form

$$\begin{aligned}\dot{E}(t) \;=\; & -j_{ex}(t)\\ & -eg_{(\pm)}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p_3(t)}{\omega(\boldsymbol{p})}\Bigg[f_{(\pm)}(\boldsymbol{p},t)+\\ & \qquad +\frac{1}{2}\left(v_{(\pm)}(\boldsymbol{p},t)-\frac{e\dot{E}(t)p_3(t)}{4\omega^4(\boldsymbol{p})}\right)\left(\frac{\epsilon_\perp}{p_3(t)}\right)^{g_{(\pm)}-1}\Bigg]\\ & -e^2\dot{E}_{(\pm)}(t)I_{(\pm)}(\Lambda)\ ,\end{aligned} \tag{20}$$

Renormalisation Procedure

where a cutoff Λ for $|\boldsymbol{p}|$ is obtained

$$I_{(\pm)}(\Lambda) \; = \; \frac{g_{(\pm)}}{4} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p_3^2(t)}{\omega^5(\boldsymbol{p})} \left(\frac{\epsilon_\perp}{p_3(t)} \right)^{g_{(\pm)}-1} \tag{21}$$

$$\overset{\Lambda \to \infty}{=} \; \frac{g_{(\pm)}}{8\pi^2} \ln\left(\frac{\Lambda^2}{m^2} \right) . \tag{22}$$

The limit (22) is used to define the renormalised current, charge, and field

$$e_R^2 = Ze^2, \; \mathcal{E}(t) = \frac{E(t)}{\sqrt{Z}}, \; \mathcal{A}(t) = \frac{A}{\sqrt{Z}}, \; \mathcal{J}_{ex}(t) = \sqrt{Z} j_{ex}(t) \; , \tag{23}$$

where

$$Z = \frac{1}{1 + e^2 I_{(\pm)}(\Lambda)} \; . \tag{24}$$

Renormalisation Procedure

Inserting (23) into the total current (20) yields the now renormalised total current

$$\begin{aligned}\dot{\mathcal{E}}(t) \;=\; & -\mathcal{J}_{ex}(t) \\ & - e_R g_{(\pm)} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p_3(t)}{\omega(\boldsymbol{p})} \Bigg[f_{(\pm)}(\boldsymbol{p}, t) + \\ & + \frac{1}{2} \left(v_{(\pm)}(\boldsymbol{p}, t) - \frac{e_R \dot{\mathcal{E}}(t) p_3(t)}{4\omega^4(\boldsymbol{p})} \right) \left(\frac{\epsilon_\perp}{p_3(t)} \right)^{g_{(\pm)}-1} \Bigg] \,. \end{aligned} \tag{25}$$

Limitations of the approach:

- A time-dependent magnetic field is neglected.
- The approach gives the vacuum expectation value of the one-particle correlator in a purely time-dependent electric field.
- The approach is purely analytical. A numerical renormalisation procedure has yet not been defined.

General One-Particle Correlators

A more general approach as done in the paper by Vasak et al.[3] that includes the magnetic field is obtained using the Lagrangian

$$\mathcal{L} \;=\; \bar{\psi}(x)\big(i\partial\!\!\!/ - m\big)\psi(x) - e\bar{\psi}(x)A\!\!\!/(x)\psi(x) - \frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x)\;. \tag{26}$$

The gauge invariant Wigner operator takes the form

$$\hat{W}_{\alpha\beta}(x,p) \;\equiv\; \int \frac{d^y}{(2\pi)^4} e^{-ip\cdot y}\bar{\psi}_\beta(x)e^{\frac{1}{2}y\cdot\overleftarrow{D}}e^{-\frac{1}{2}y\cdot D}\psi_\alpha(x)\;. \tag{27}$$

This allows the derivation of the quantum Vlasov equation that includes the spin corrections

$$\begin{aligned}
&\big(p\cdot\partial_x - ep_\mu\bar{F}^{\mu\nu}\partial^p_\nu\big)W^H(x,p) + \frac{1}{4}ie\bar{F}^{\mu\nu}\left[\sigma_{\mu\nu}, W^H(x,p)\right] \\
&= \; -\frac{1}{12}\hbar^2 e\Delta\bar{F}_{\mu\nu}\left[\partial^\nu_x - e\bar{F}^{\nu\lambda}\partial^p_\lambda\right]\partial^\mu_p W^H(x,p) \\
&\quad\; -\frac{1}{8}\hbar e\Delta\bar{F}^{\mu\nu}\left\{\sigma_{\mu\nu}, W^H(x,p)\right\}\,,
\end{aligned} \tag{28}$$

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[3] Vasak et al., **Quantum Transport Theory for Abelian Plasmas**, Annals of Physics vol. 173, Issue 2, Pages 462-492

General One-Particle Correlators

and the constraint equation in the Hartree aproximation,

$$
\begin{aligned}
(p^2 - m^2)\, W^H(x,p) \;=\;& \\
& \frac{1}{4}\hbar e \bar{F}^{\mu\nu}\left\{\sigma_{\mu\nu}, W^H(x,p)\right\} \\
& - \frac{1}{8} i\hbar^2 e \Delta \bar{F}^{\mu\nu}\left[\sigma_{\mu\nu}, W^H(x,p)\right] \\
& + \hbar^2 \Big(\frac{1}{6} ep \cdot \Delta F \cdot \partial_p + \frac{1}{12} e(\partial_x^\mu \bar{F}_{\mu\nu})\partial_p^\nu + \frac{1}{4}(\partial_x - e\bar{F}\cdot\partial_p)^2\Big) W^H(x,p)\;. \qquad (29)
\end{aligned}
$$

With the help of the general correlators (28) and (29) we aim to derive a numerical renormalisation procedure that includes time-depenendent electromagnetic fields.