Initializing the ground state of lattice gauge theories with the quantum approximate optimization algorithm

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May 9, 2022

Work in collaboration with:

L. Lumia, P. Torta, G. Mbeng, G. Santoro, E. Ercolessi, M.B., M. Wauters, arXiv:2112.11787, PRX Quantum 2022

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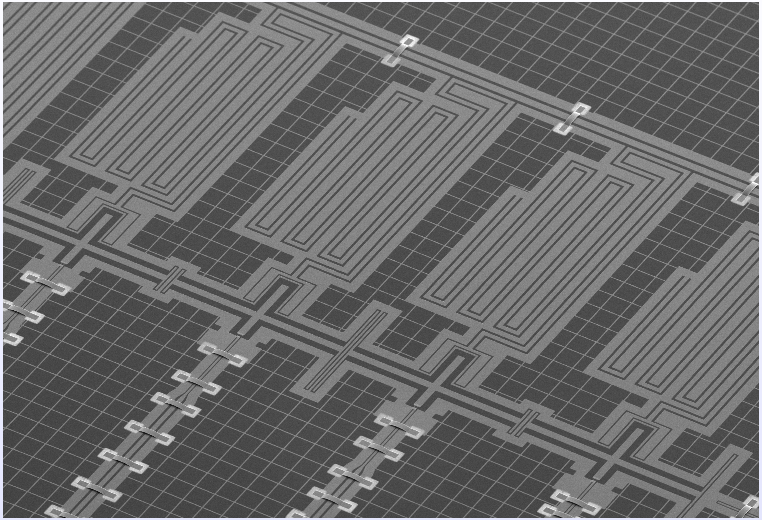
L. Lumia, P. Torta, G. Mbeng, G. Santoro, E. Ercolessi, M.B., M. Wauters, PRX Quantum 2022

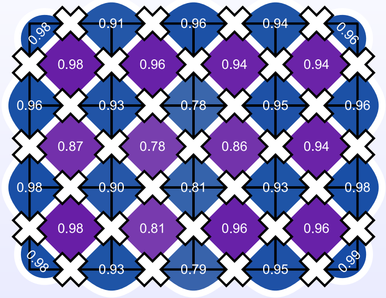
Superconducting NISQ processors



Superconducting NISQ processors

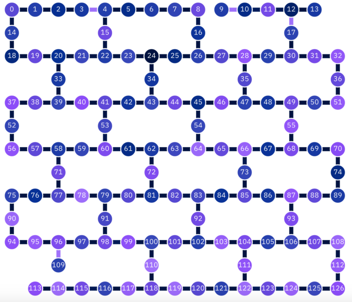
2D arrays

Google sycamore



Satzinger et al., Science 2021

IBM platforms



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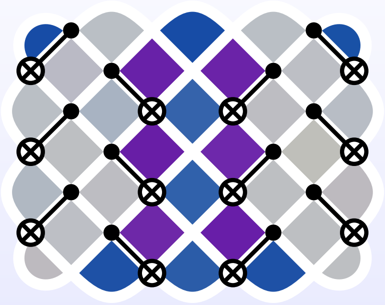

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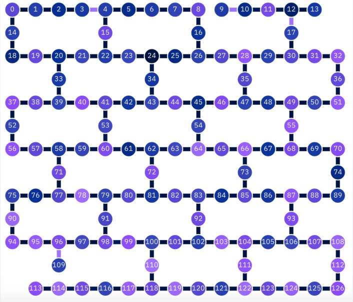

2D platforms with ~ 50 qubits and nearest neighbor CNOTs are available

Quantum simulation of LGT in 2+1D

LGTs constitute an intriguing playground to test **quantum simulation** techniques

- Most simulating platforms offer limited degrees of freedom
- Discretization or truncation of the gauge groups are typically required

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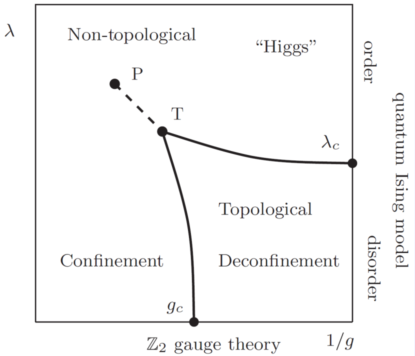
- The $\mathbb{Z}_2$ gauge theory is the simplest toy model

$$H = -\frac{1}{g}\sum_{\text{plaq.}} \sigma^z_{p_1}\sigma^z_{p_2}\sigma^z_{p_3}\sigma^z_{p_4} - g\sum_{\text{links}} \sigma^x_l$$
$$-\frac{1}{\lambda}\sum_{\text{vert.}} \tau^x_v - \lambda\sum_{\text{links}} \tau^z_v \sigma^z_{v,v'} \tau^z_{v'}$$



Fradkin and Shenker (1979)

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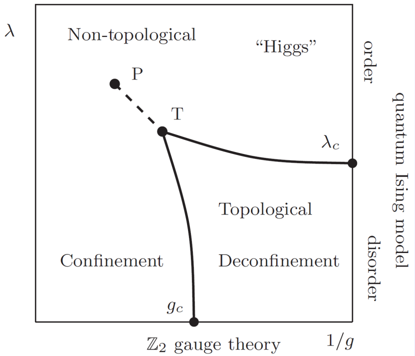
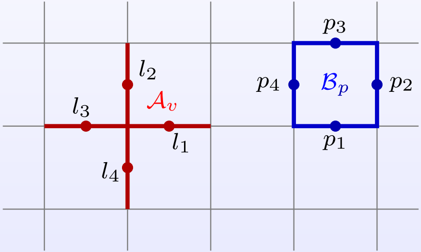


Fradkin and Shenker (1979)

How can we efficiently initialize the **ground states** of this model?
How does **topological order** affect quantum simulations?

Pure $\mathbb{Z}_2$ LGT

$$H = \underbrace{\sum_l (1 - \sigma_l^x)}_{H_E} \underbrace{-h \sum_p \sigma_{p_1}^z \sigma_{p_2}^z \sigma_{p_3}^z \sigma_{p_4}^z}_{H_B}$$



Two phases:

- **Confined** phase $(h < h_c)$:
  1. Trivial
  2. String tension
  3. Area law of the Wilson loop

Local gauge symmetry:

$$\mathcal{A}_v = \sigma_{l_1}^x \sigma_{l_2}^x \sigma_{l_3}^x \sigma_{l_4}^x$$

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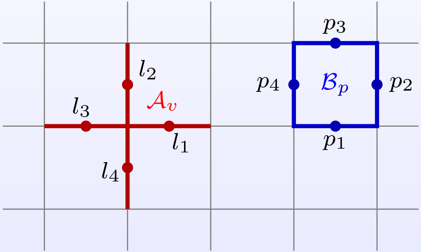

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Two phases:

- **Confined** phase $(h < h_c)$:
  1. Trivial
  2. String tension
  3. Area law of the Wilson loop
- **Deconfined** phase $(h > h_c)$:
  1. Topological order
  2. Static charges and magnetic excitations are anyons
  3. Perimeter law of the Wilson loop

Given a value of h, how can we prepare the ground state?
Basic attempt

- Trivial initial state: $|\Omega_E\rangle = \otimes_l |+\rangle_l$
- The initial state corresponds to the GS at $h = 0$
- Basic attempt: quantum **annealing** and Trotterization

$$H(m) = H_E - h\frac{m}{P} H_B \,, \qquad |\Psi_P\rangle = \prod_{m=1}^{\leftarrow P} \left[e^{-i\delta t H_E} e^{ih\frac{m}{P}\delta t H_B} \right] |\Omega_E\rangle$$

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- **CONS:**
  1. Trotterization necessary for digital approaches but introduces considerable errors
  2. This kind of adiabatic evolution works as long as h is sufficiently far from h_c

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- **CONS:**
  1. Trotterization necessary for digital approaches but introduces considerable errors
  2. This kind of adiabatic evolution works as long as h is sufficiently far from h_c
- **PRO:** It preserves gauge invariance

Gauge-invariant circuit implementation of the digitized operators

See, for instance: Lamm, Laurence, Yamauchi, PRD (2019)

- **Electric:** $e^{-i\beta H_E} = e^{-i\sum_l \beta\sigma_l^x}$ $\Rightarrow$ Single-qubit rotations.

Gauge-invariant circuit implementation of the digitized operators

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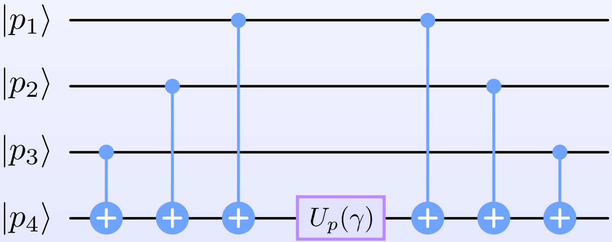
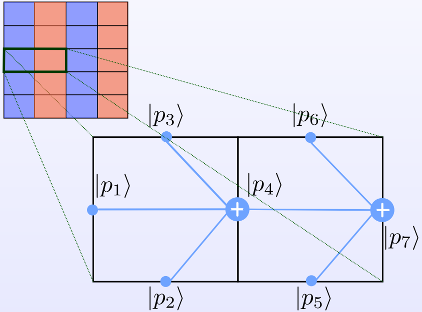
- **Electric:** $e^{-i\beta H_E} = e^{-i\sum_l \beta\sigma_l^x} \Rightarrow$ Single-qubit rotations.
- **Magnetic:** $e^{-i\gamma H_B} = \otimes_p e^{-i\gamma\sigma_{p1}^z\sigma_{p2}^z\sigma_{p3}^z\sigma_{p4}^z}$

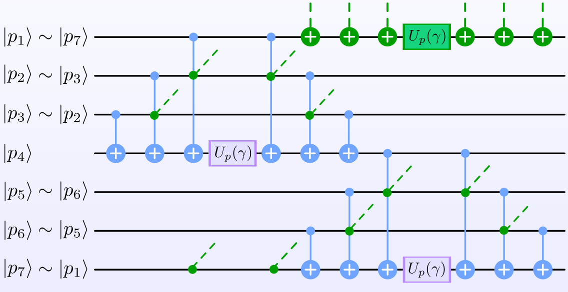
  We decompose it in single-qubit gates $U_p(\gamma) = e^{i\gamma\sigma^z}$ and CNOTs:



Parallelization of the magnetic operators

- Open boundaries
- Pairs of columns in parallel
- Each magnetic step: depth 12



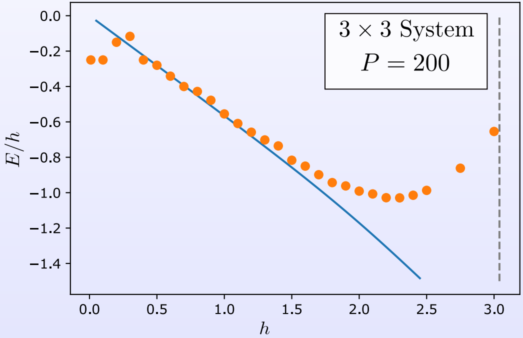


Total depth for each Trotterization step: **13**
(worst case scenario: 18 for PBC with odd columns and rows)

Preparation of the ground state

Quantum annealing and Trotterization:

$$H(m) = H_E - h\frac{m}{P}H_B\,, \qquad |\Psi_P\rangle = \prod_{m=1}^{\leftarrow P}\left[e^{-i\delta t H_E}e^{ih\frac{m}{P}\delta t H_B}\right]|\Omega_E\rangle$$



- Total depth: $13P$
- For $P = 200$ (!), the results are still quite disappointing.

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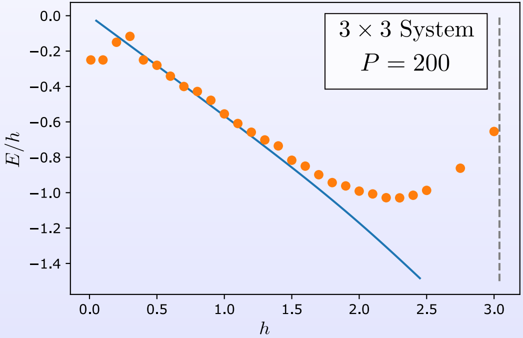
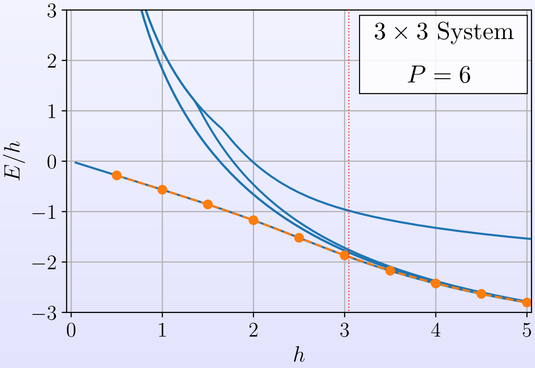


- Total depth: $13P$
- For $P = 200$ (!), the results are still quite disappointing.
- Large Trotterization errors
- Difficult to approach the phase transition.

A more refined approach: **QAOA**

$$|\Psi_P\rangle = \prod_{m=1}^{\leftarrow P} \left[e^{-i\beta_m H_E} e^{-i\gamma_m H_B} \right] |\Omega_E\rangle$$



- We introduce and optimize $2P$ variational parameters $\{\beta_m, \gamma_m\}$

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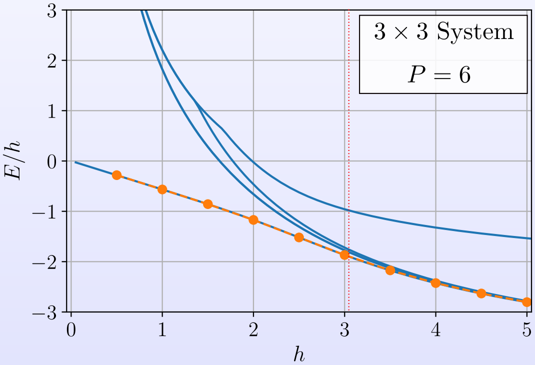
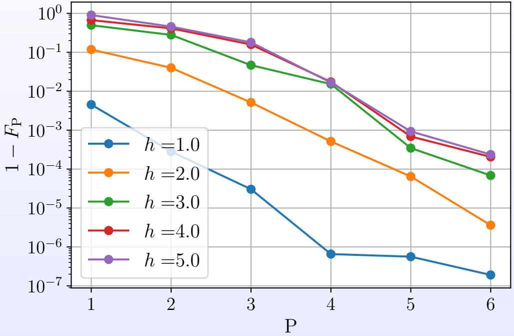


- We introduce and optimize $2P$ variational parameters $\{\beta_m, \gamma_m\}$
- To which extent can QAOA work across a topological phase transition?
- How can we efficiently optimize the variational parameters?
- Can the variational parameters calculated for small system sizes be transferred to larger systems?

Fidelity across the topological phase transition

3×3 system



Initial state: $|\psi_0\rangle = |\Omega_E\rangle$

- Fidelity improves exponentially with P

Fidelity across the topological phase transition

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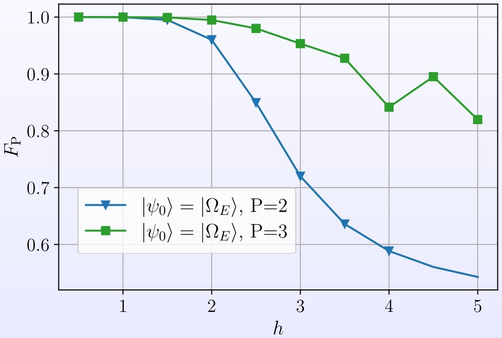

Initial state: $|\psi_0\rangle = |\Omega_E\rangle$

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Fidelity across the topological phase transition

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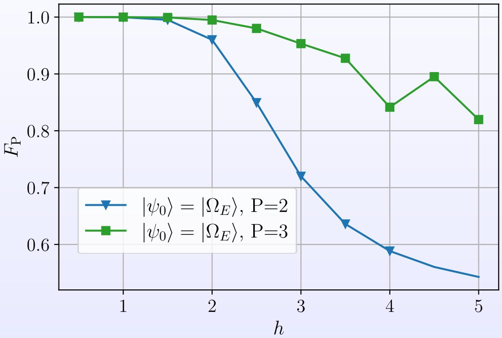

Initial state: $|\psi_0\rangle = |\Omega_E\rangle$

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- Creating topological order with local gates requires a circuit of depth $\sim L$

Bravyi, Hastings, Verstraete PRL 2006;
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Fidelity across the topological phase transition

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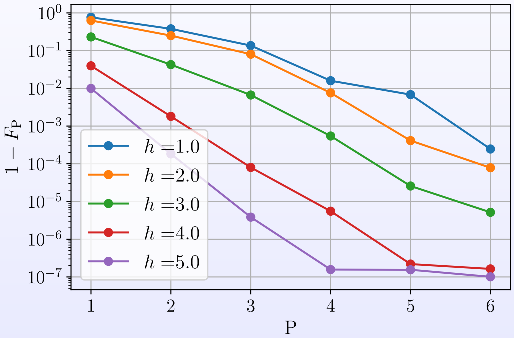

Initial state: $|\psi_0\rangle = |\Omega_B\rangle$

- Fidelity improves exponentially with P
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- Creating topological order with local gates requires a circuit of depth $\sim L$

  Bravyi, Hastings, Verstraete PRL 2006;

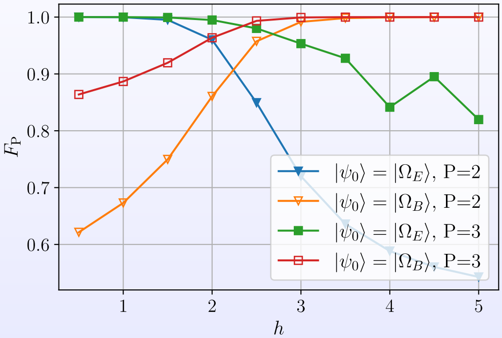
  Chen, Gu, Wen PRB 2010
- We include an overhead $\sim L$ to initialize the system in the toric code ground state $|\Omega_B\rangle$

  Satzinger *et al.*, Science 2021;

  Liu, Shtengel, Smith and Pollmann 2021

Fidelity across the topological phase transition

3×3 system



Selection of the initial state:

$$|\psi_0\rangle = |\Omega_E\rangle \text{ for } h < h_c$$
$$|\psi_0\rangle = |\Omega_B\rangle \text{ for } h > h_c$$

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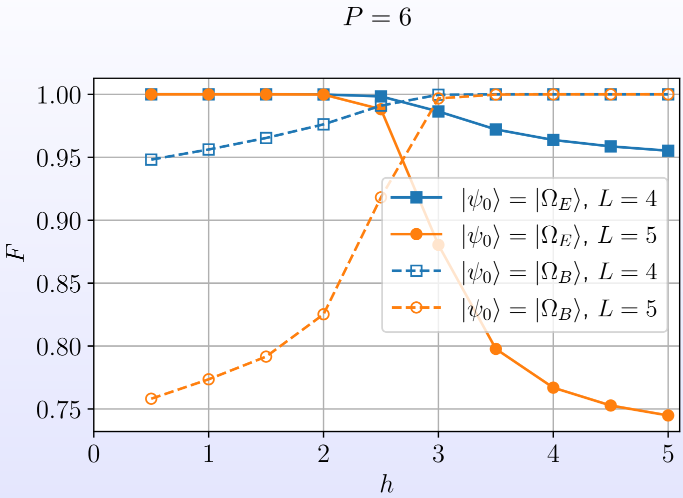
Fidelity across the topological phase transition

4×4 and 5×5 systems



Wilson loops and Creutz ratio

We can characterize the phases based on the Wilson loops $\mathcal{W}$:

- Deconfined / topological phase:

$$\langle \mathcal{W} \rangle \propto e^{\alpha \text{ Perimeter}}$$

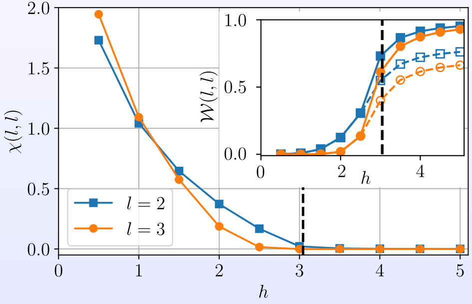
- Confined / trivial phase:

$$\langle \mathcal{W} \rangle \propto e^{\chi \text{ Area}}$$

- **Creutz ratio**:

$$\chi(l,l) = -\log \frac{\langle \mathcal{W}_{l,l} \rangle \langle \mathcal{W}_{l-1,l-1} \rangle}{\langle \mathcal{W}_{l,l-1} \rangle \langle \mathcal{W}_{l-1,l} \rangle}$$

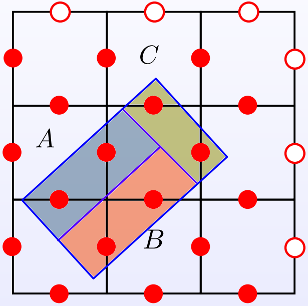
$L = 5\,, \qquad P = 6$



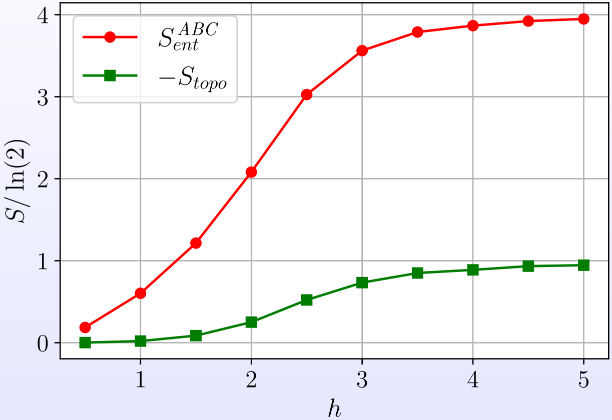
(Empty symbols: $|\Omega_E\rangle$ only)

Topological entropy

3×3 system



$$S_{topo} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}$$



Different topological sectors on the torus

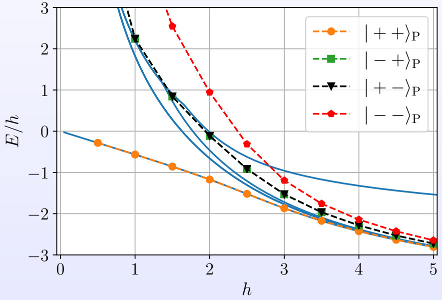
- Degenerate GSs with topological order: $\{|a_v, a_h\rangle, a_{v,h} = 0, 1\}$
- The QAOA commutes with non-contractible 't Hooft loops
- Two strategies to get GS in different sectors:

$$|a_v, a_h\rangle_P = \mathcal{W}_v^{a_v} \mathcal{W}_h^{a_h} \mathcal{U}(\gamma^*, \beta^*) |\Omega_B\rangle$$

$$|a_v, a_h\rangle'_P = \mathcal{U}(\gamma^*, \beta^*) \mathcal{W}_v^{a_v} \mathcal{W}_h^{a_h} |\Omega_B\rangle$$

- They provide analogous results and the optimized variational parameters (γ^*, β^*) do not change
- In both cases, the non-contractible Wilson operators $\mathcal{W}$ introduce excitations

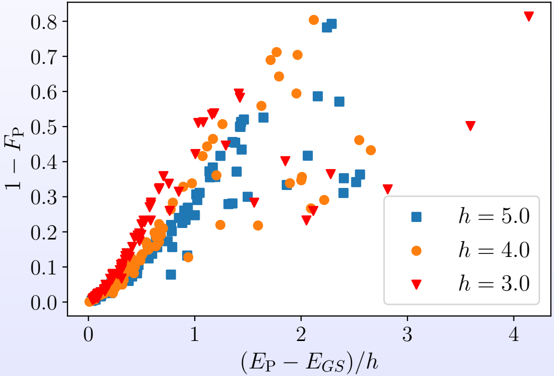
$L = 3 \qquad P = 6$



Some detail on the optimization of the parameters

Fidelity vs Residual energy in local optimizations

Results from **random local optimizations**:



- Good correlation between fidelity and residual energy
- Many local minima:
  local optimization is not viable!
- **Global optimization** is computationally expensive

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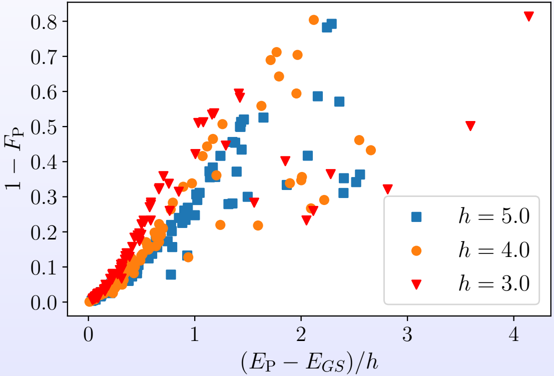

- Good correlation between fidelity and residual energy
- Many local minima: **local optimization is not viable**!
- **Global optimization** is computationally expensive
- We adopt an alternative **two-step optimization**

Two-step optimization

Inspired by Mbeng, Arceci, Santoro, PRB 2019

- **First step: annealing**. For fixed P:

$$|\Psi_P\rangle = \prod_{m=1}^{\leftarrow P} \left[e^{-i\delta t H_E} e^{ih\frac{m}{P}\delta t H_B}\right] |\Omega_E\rangle$$

We optimize δt

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- **Second step: QAOA**
  We **locally** optimize the $2P$ variational $\{\gamma_m, \beta_m\}$ from the annealing result (+ noise)

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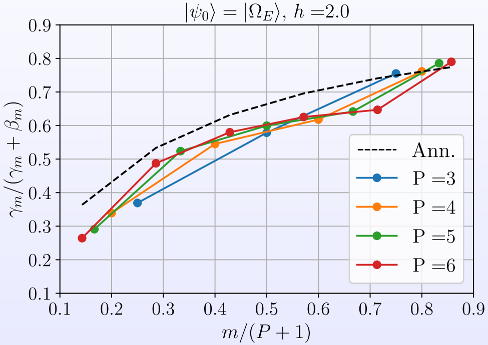
  We optimize δt

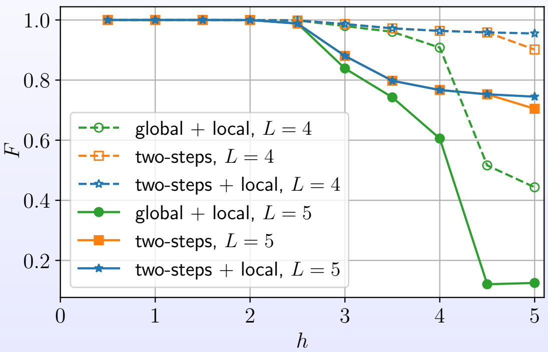
- **Second step: QAOA**
  We **locally** optimize the $2P$ variational $\{\gamma_m, \beta_m\}$ from the annealing result (+ noise)
- The obtained parameters are quite “regular”
- Smooth parameters: transferability to larger system sizes

Smoothness and scalability from $|\psi_0\rangle = |\Omega_E\rangle$

Smooth parameters

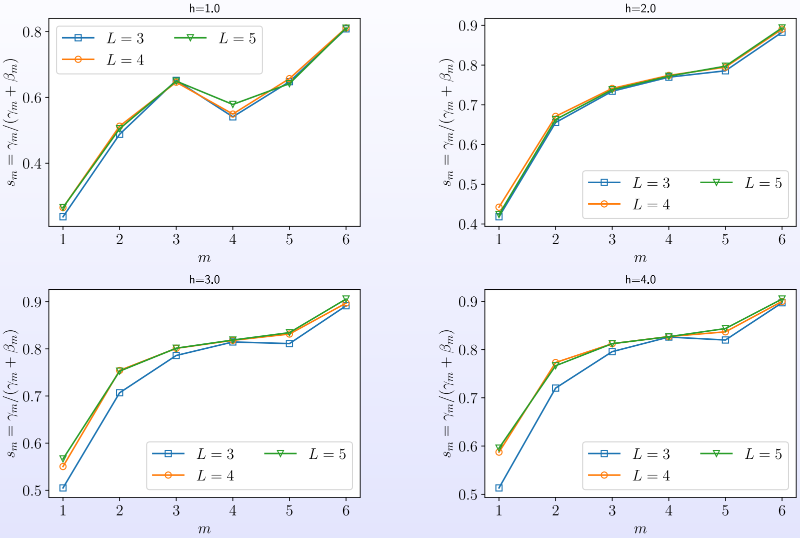


Transferability from $L = 3$



Global opt.:
$\sim$ 100 times more expensive than local opt.

Transferability of the variational parameters



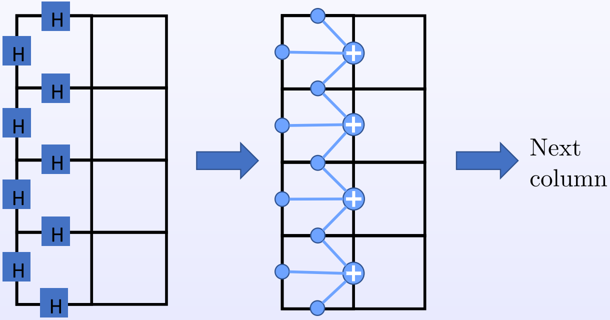
Conclusions

L. Lumia, P. Torta, G. Mbeng, G. Santoro, E. Ercolessi, M.B., M. Wauters, PRX Quantum 2022

- The quantum approximate optimization algorithm constitutes a practical technique to prepare the gauge-invariant GS of 2D LGT with **shallow circuits** in small systems
- Some care is required in crossing **topological phase transitions**
- **Observables and entanglement** features of the $\mathbb{Z}_2$ phase transitions are obtained already for small systems and circuits of depth $\lesssim 100$
- **Two-step optimization**: smooth parameters and **transferability** to larger systems
- This GS preparation can be used to initialize the system for the simulation of its dynamics

Overhead circuit for generating $|\Omega_B\rangle$

Satzinger et al., Science 2021



- $|\Omega_B\rangle$ is exactly prepared without variational parameters
- It requires a circuit of depth L
- Long range entanglement cannot be obtained with a fixed depth circuit