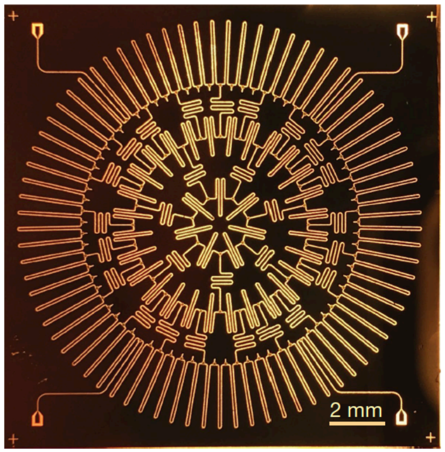
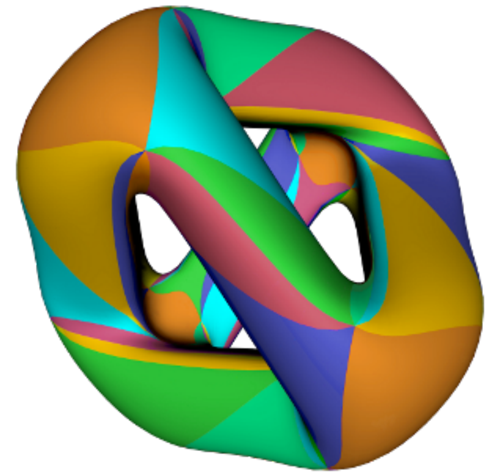
Hyperbolic band theory



Joseph Maciejko
University of Alberta

Discrete LGTs – Emergence & Quantum Simulations
MPQ Garching
May 11, 2022

S. Rayan
(Saskatchewan)

JM & S. Rayan, Sci. Adv. 7, eabe9170 (2021)
JM & S. Rayan, PNAS 119, e2116869119 (2022)

I. Boettcher, A. V. Gorshkov, A. J. Kollár, JM, S. Rayan, R. Thomale, PRB 105, 125118 (2022)

I. Boettcher
(Alberta)

A. Gorshkov
(JQI/UMD)

A. Kollár
(JQI/UMD)

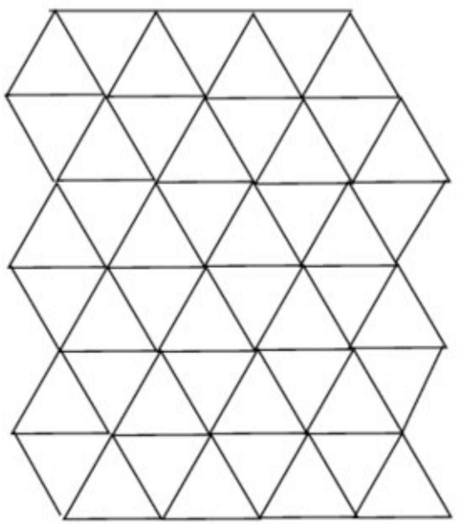
R. Thomale
(Würzburg)

T. Bzdušek
(PSI/Zürich)

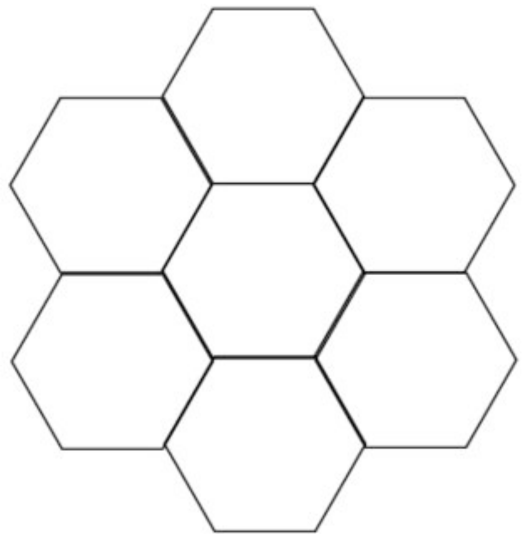
T. Bzdušek & JM, in preparation

Euclidean lattices

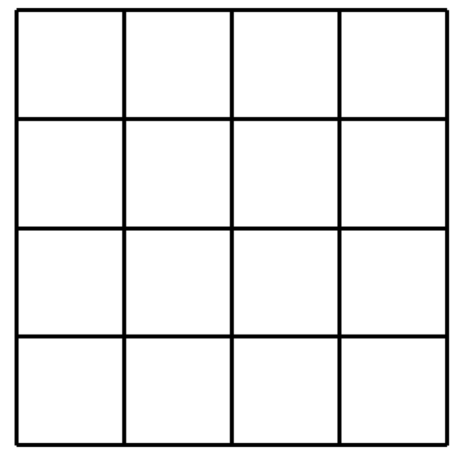
$\{3, 6\}$



$\{6, 3\}$

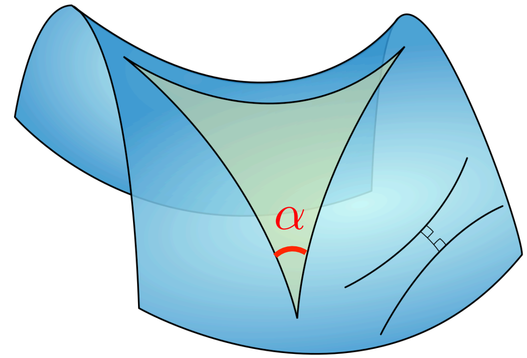


$\{4, 4\}$

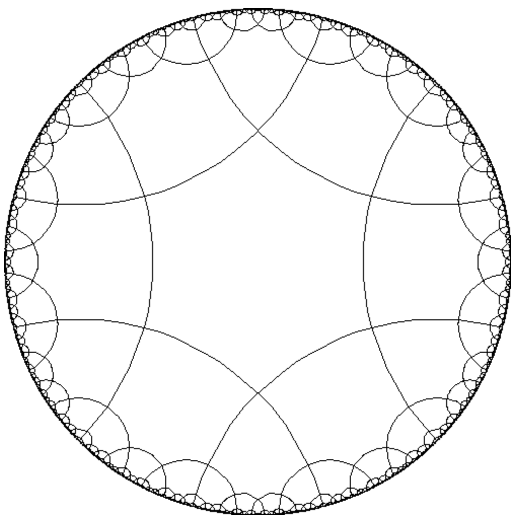


$$(p-2)(q-2) = 4$$

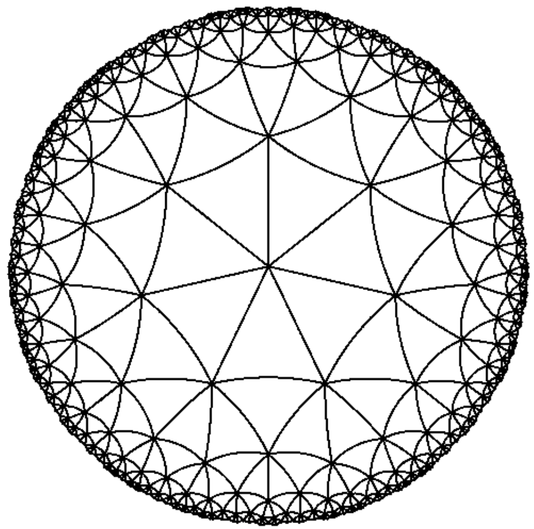
Hyperbolic lattices



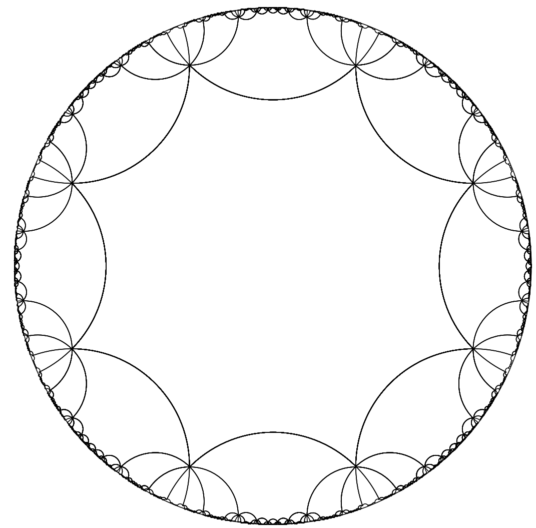
$\{6,4\}$



$\{3,7\}$



$\{8,8\}$



$\cdots$

$$(p-2)(q-2) > 4$$

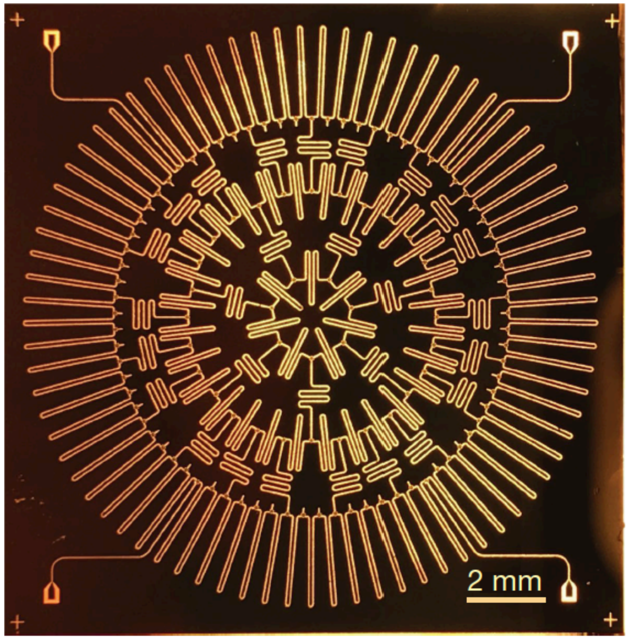
ARTICLE

https://doi.org/10.1038/s41586-019-1348-3

Hyperbolic lattices in circuit quantum electrodynamics

Alicia J. Kollár[1,2,3]*, Mattias Fitzpatrick[1] & Andrew A. Houck[1]

4 JULY 2019 | VOL 571 | NATURE | 45



ARTICLE

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Hyperbolic lattices in circuit quantum electrodynamics

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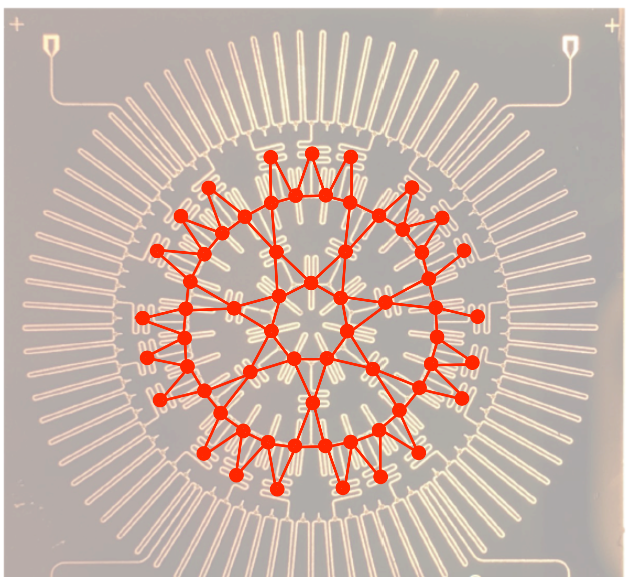
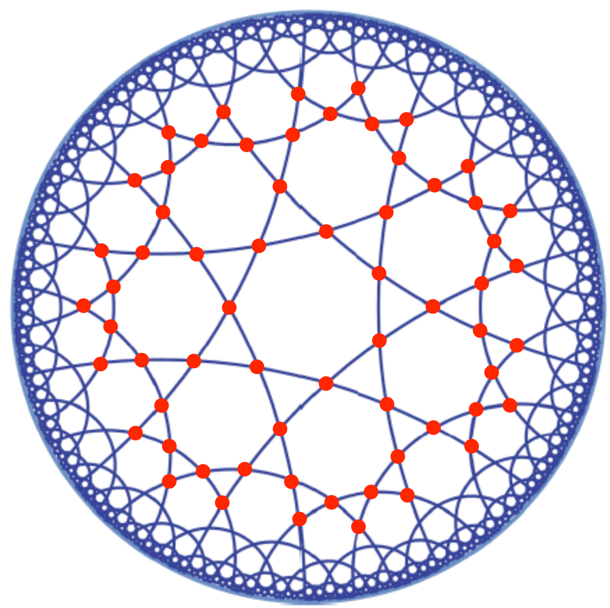


=

“hyperbolic kagome lattice”

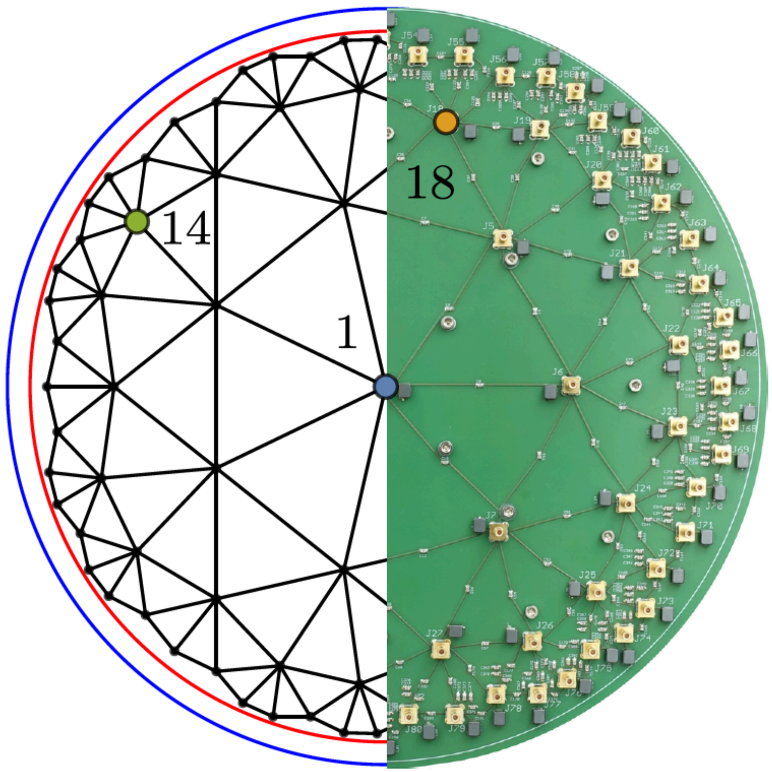


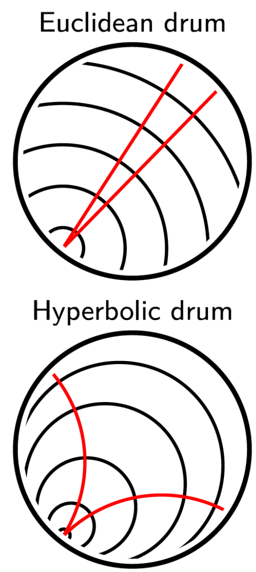
$$H_{\mathrm{TB}} = \omega_0 \sum_i a_i^\dagger a_i - t \sum_{<i,j>} (a_i^\dagger a_j + a_j^\dagger a_i)$$

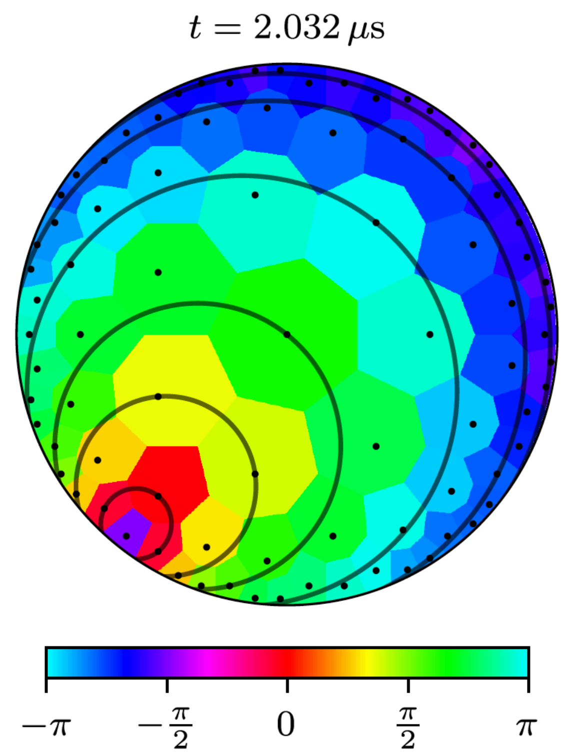
Electric-circuit realization of a hyperbolic drum

Patrick M. Lenggenhager,[1,2,3,*] Alexander Stegmaier,[4,*] Lavi K. Upreti,[4] Tobias Hofmann,[4] Tobias Helbig,[4] Achim Vollhardt,[2] Martin Greiter,[4] Ching Hua Lee,[5] Stefan Imhof,[6] Hauke Brand,[6] Tobias Kießling,[6] Igor Boettcher,[7,8] Titus Neupert,[2,†] Ronny Thomale,[4,†] and Tomáš Bzdušek[1,2,†]

$\{3,7\}$ “hyperbolic triangular lattice”





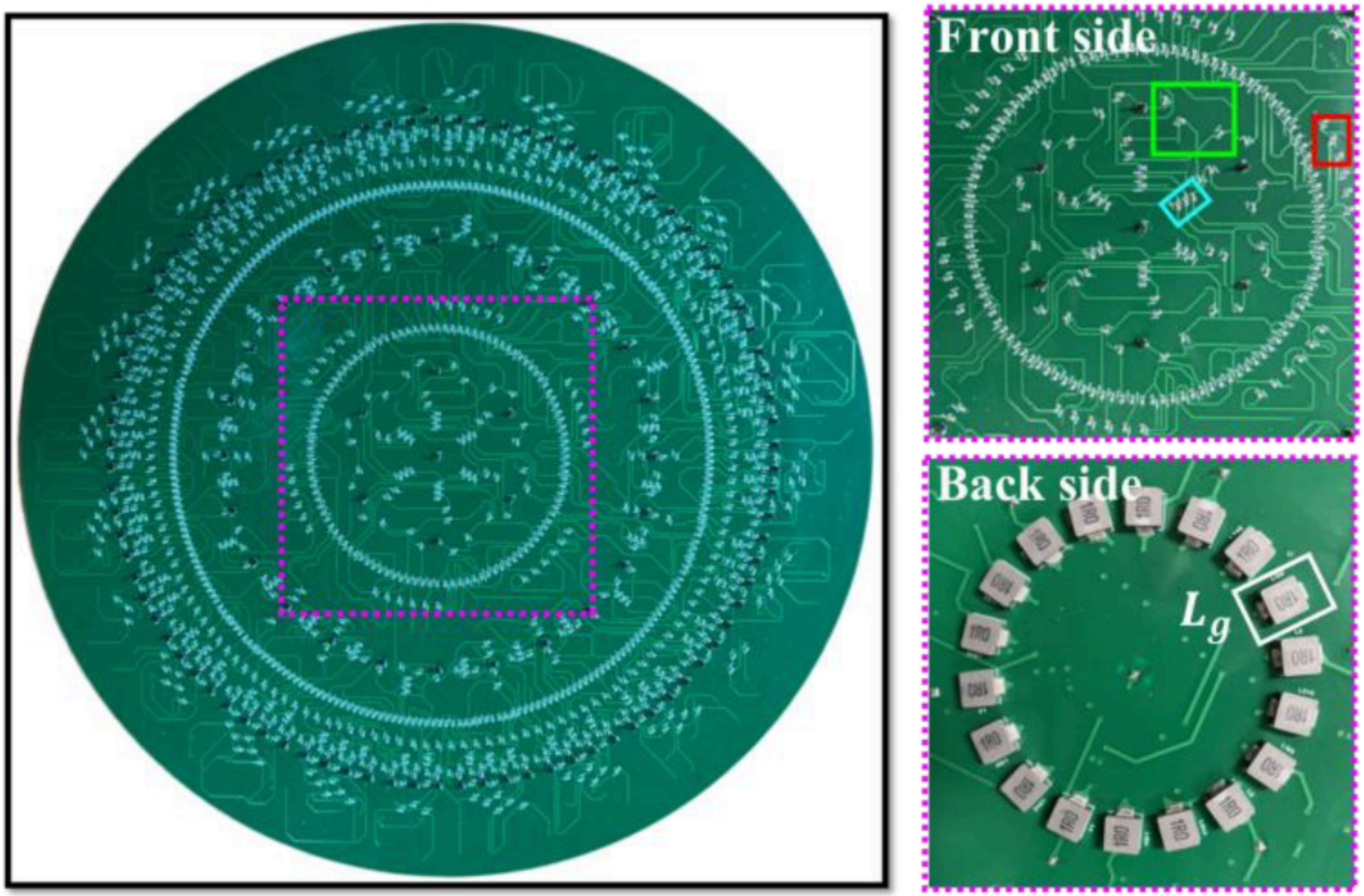


arXiv:2109.01148

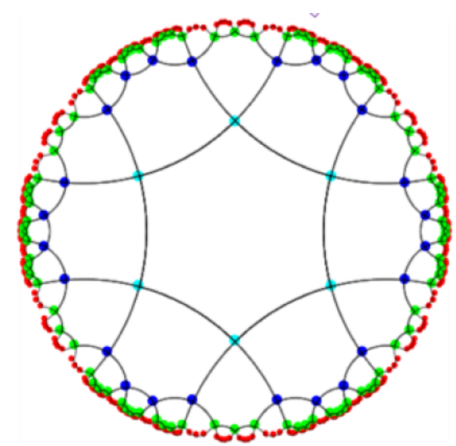
Observation of novel topological states in hyperbolic lattices

Weixuan Zhang[1*], Hao Yuan[1*], Na Sun[1], Houjun Sun[2], and Xiangdong Zhang[1$]

a

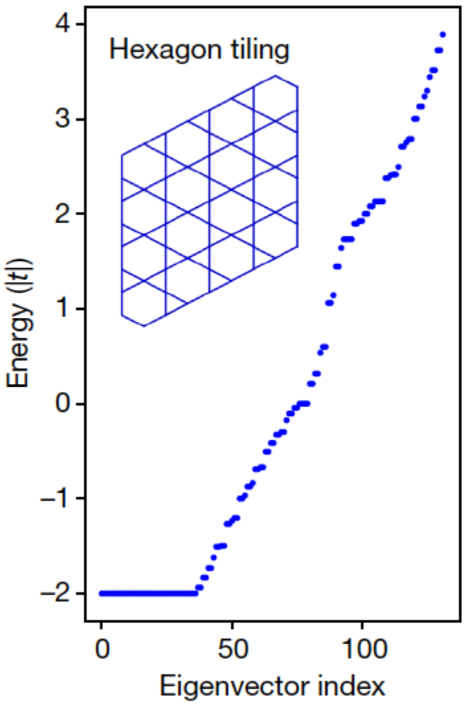


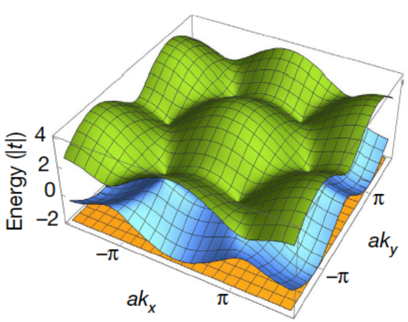
$\{6, 4\}$



"hyperbolic honeycomb lattice"

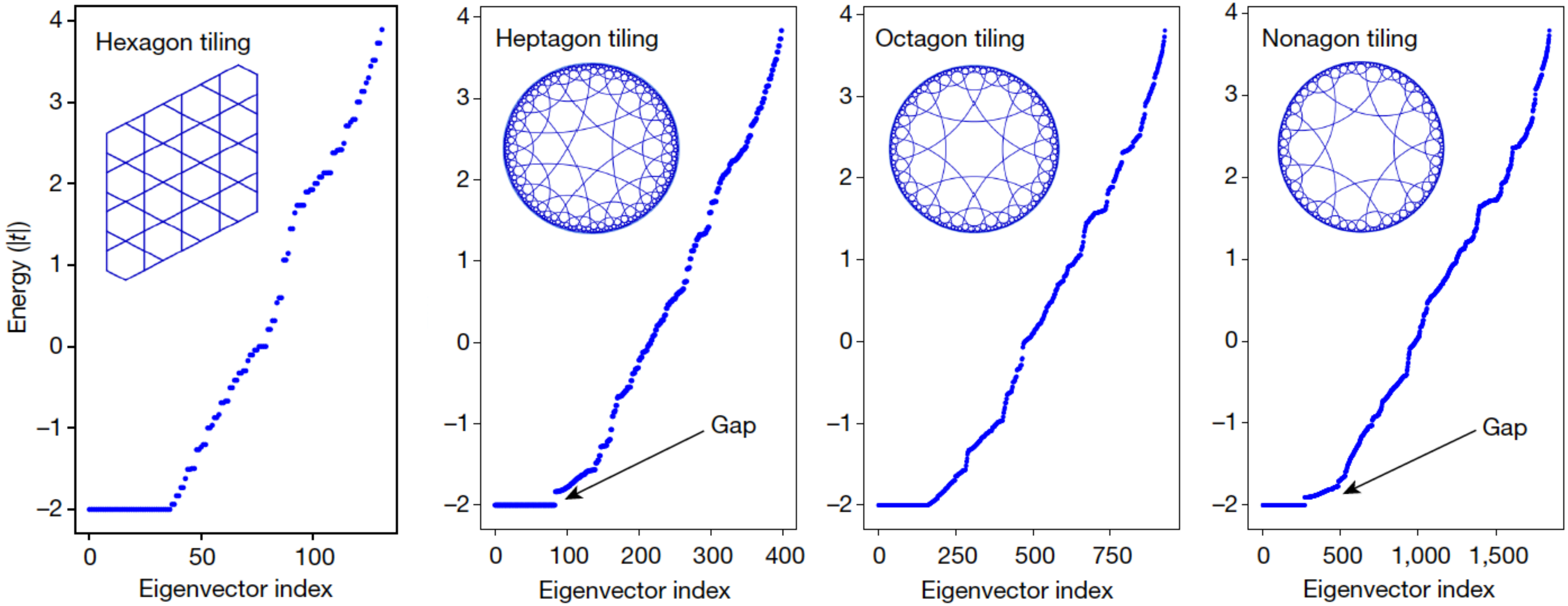
arXiv:2203.03214

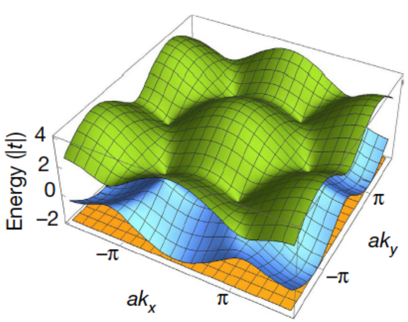




E(k)

Kollár, Fitzpatrick, Houck, Nature 571, 45 (2019)





E(k)

?

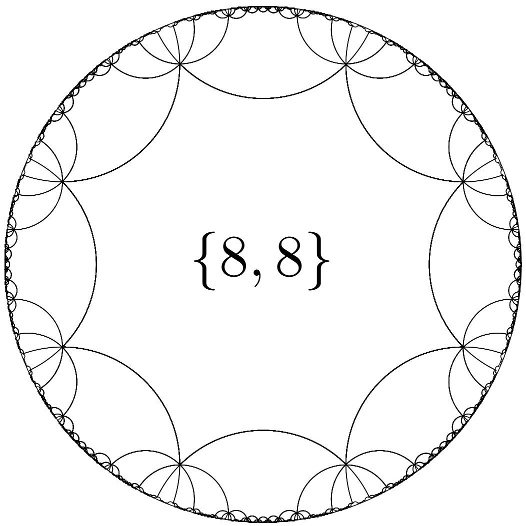
Kollár, Fitzpatrick, Houck, Nature 571, 45 (2019)

Outline

- Hyperbolic geometry & Fuchsian groups
- Hyperbolic Bloch ansatz
- Periodic boundary conditions & automorphic Bloch theorems
- Flat bands & topological bands
- Summary & outlook

Poincaré disk

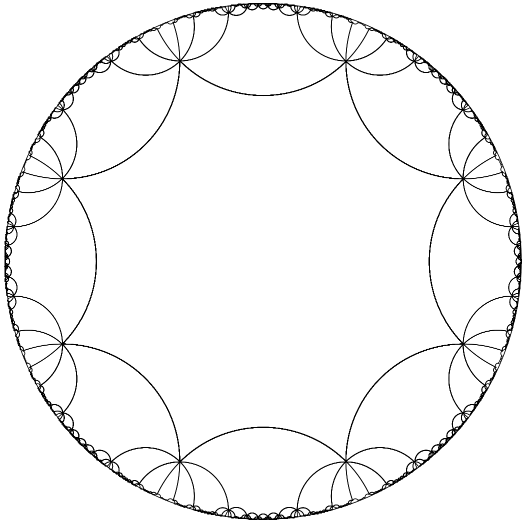
$$\mathbb{H} = \{|z| < 1\}$$



$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - |z|^2)^2}$$

Poincaré disk

$$\mathbb{H} = \{|z| < 1\}$$



$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - |z|^2)^2}$$

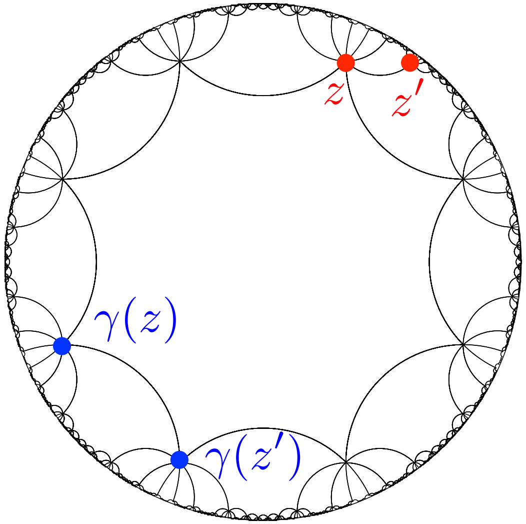
PSU(1,1) ≅ PSL(2,R):

$$\gamma = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \det\gamma = 1$$

$$z \to \gamma(z) = \frac{\alpha z + \beta}{\beta^* z + \alpha^*}$$

Poincaré disk

$$\mathbb{H} = \{|z| < 1\}$$



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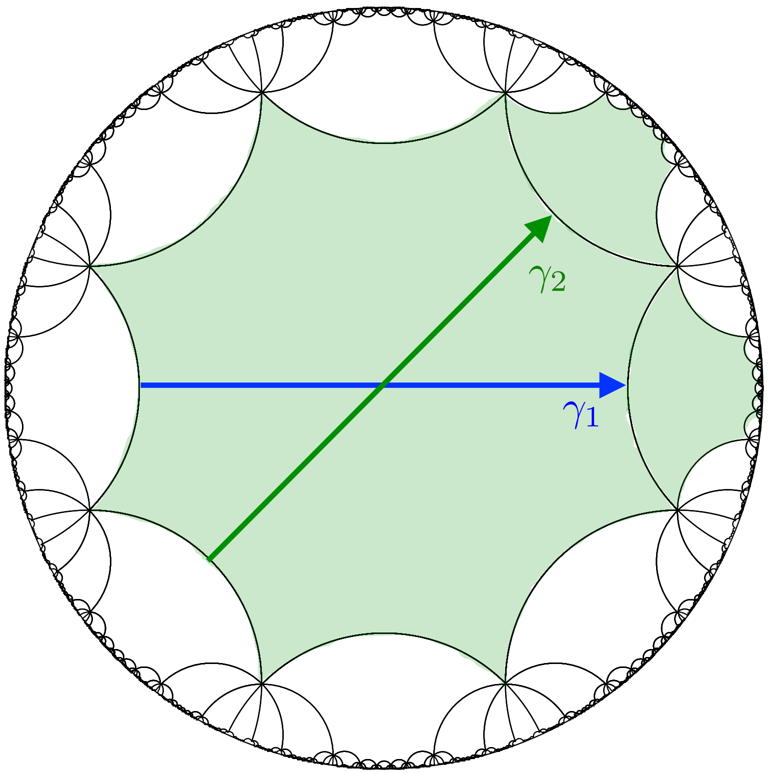
$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - |z|^2)^2}$$

$$d(z, z') = d(\gamma(z), \gamma(z'))$$

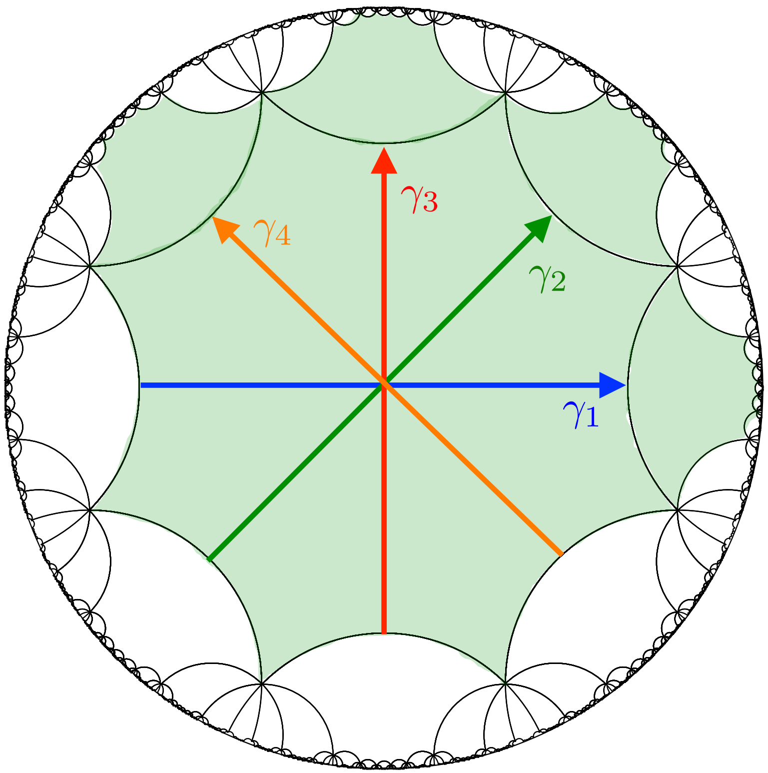
Nonabelian translation group



$$\Gamma \subset PSU(1,1)$$

Balazs & Voros, Phys. Rep. 143, 109 (1986)

Nonabelian translation group

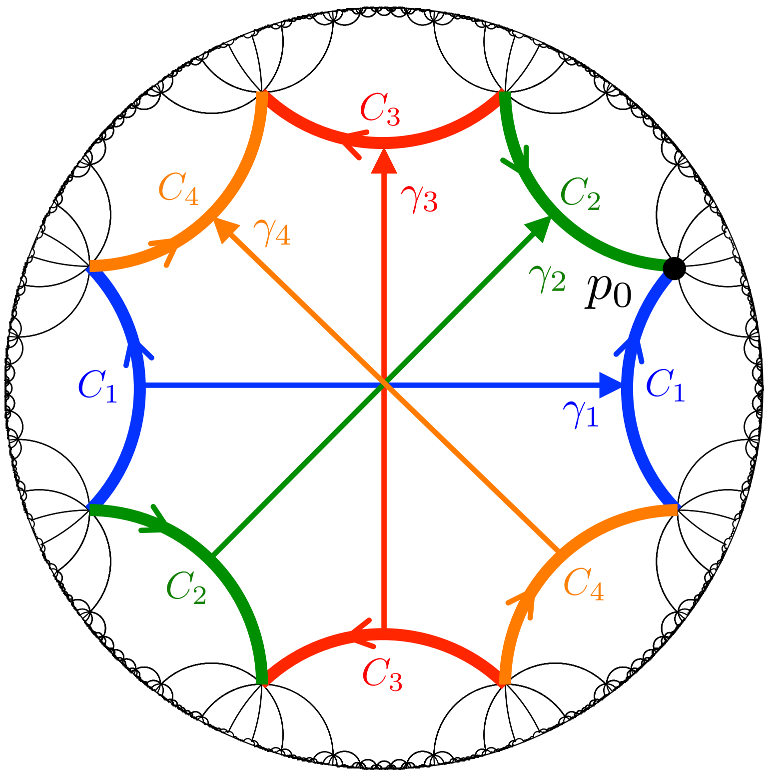


$$\Gamma \subset PSU(1,1)$$

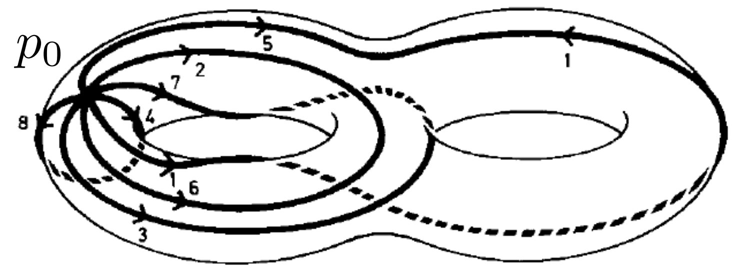
$$\Gamma = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 : \gamma_1 \gamma_2^{-1} \gamma_3 \gamma_4^{-1} \gamma_1^{-1} \gamma_2 \gamma_3^{-1} \gamma_4 = 1 \rangle$$

Balazs & Voros, Phys. Rep. 143, 109 (1986)

Compactified unit cell



$\cong$



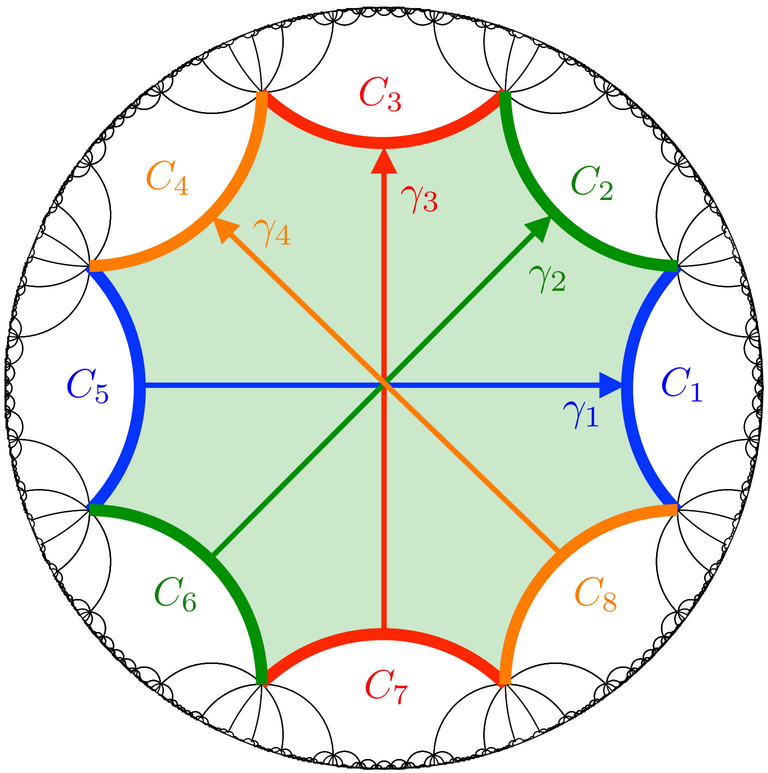
$$\mathbb{H}/\Gamma \cong \Sigma_2$$

$$\Gamma \cong \pi_1(\Sigma_2)$$

$$\gamma_1\gamma_2^{-1}\gamma_3\gamma_4^{-1}\gamma_1^{-1}\gamma_2\gamma_3^{-1}\gamma_4 = 1$$

Balazs & Voros, Phys. Rep. 143, 109 (1986)

Hyperbolic Bloch ansatz

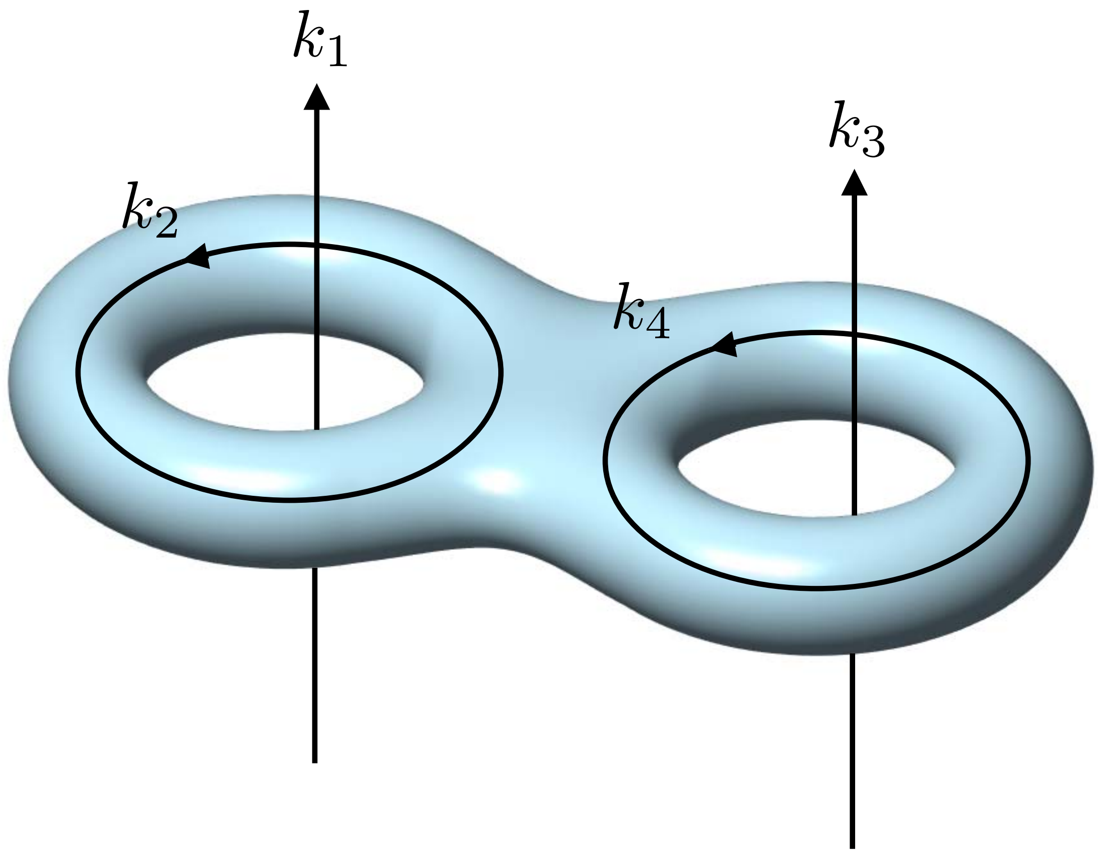


$$H\psi = E\psi$$

$$\psi(\gamma_j(z)) = e^{ik_j}\psi(z),$$
$$j = 1, 2, 3, 4$$

JM & Rayan, Sci. Adv. 7, eabe9170 (2021)

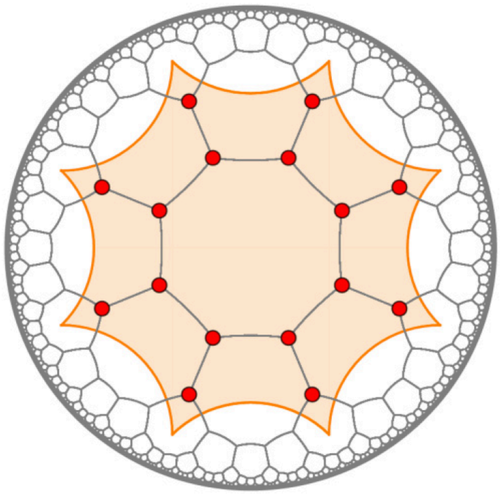
Hyperbolic crystal momentum



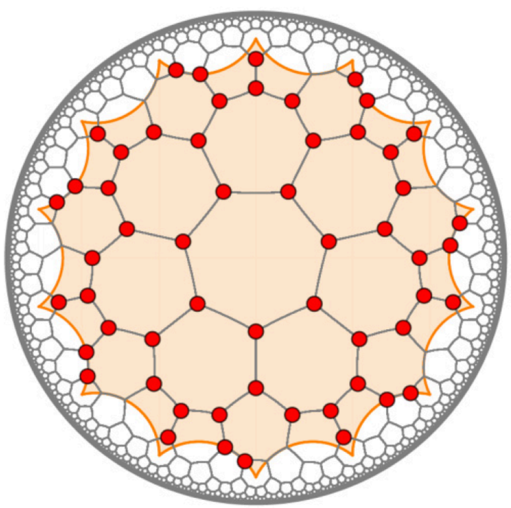
$$\boldsymbol{k} \equiv (k_1, k_2, \ldots, k_{2g-1}, k_{2g}) \in (-\pi, \pi]^{2g} \cong T^{2g} \cong \mathrm{Jac}(\Sigma_g)$$

JM & Rayan, Sci. Adv. 7, eabe9170 (2021)

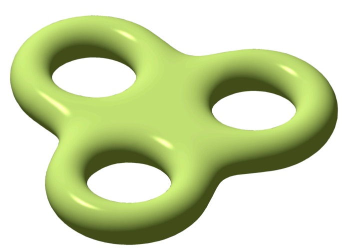
Bravais unit cell



{8,3} = {8,8} + 16-site basis

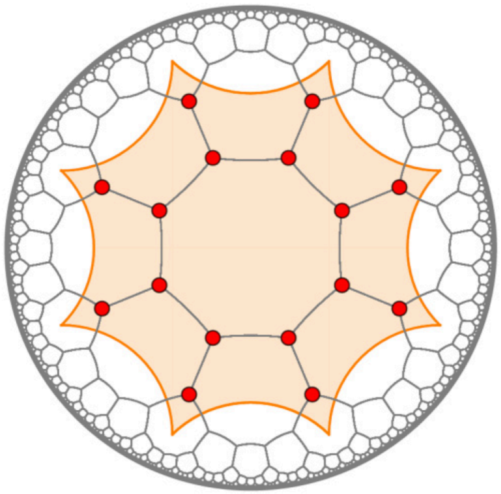


{7,3} = {14,7} + 56-site basis



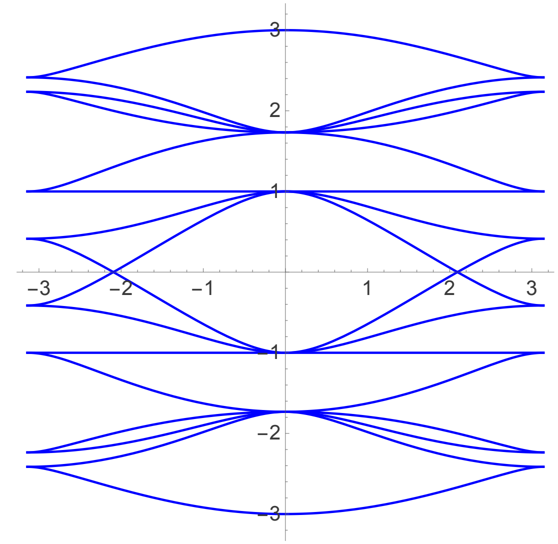
Boettcher, Gorshkov, Kollár, JM, Rayan, Thomale, PRB 105, 125118 (2022)

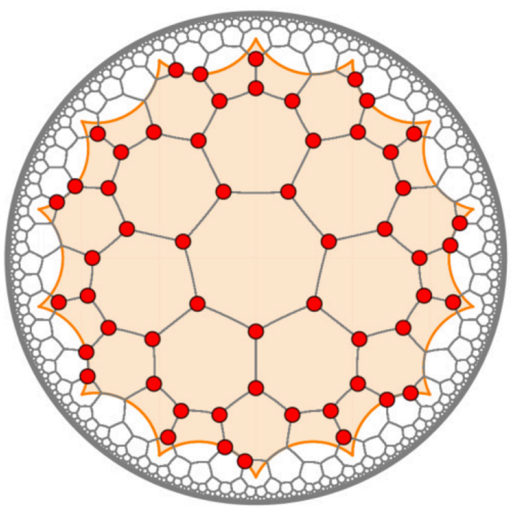
Bravais unit cell



{8,3} = {8,8} + 16-site basis

E/t vs (k,k,k,k)





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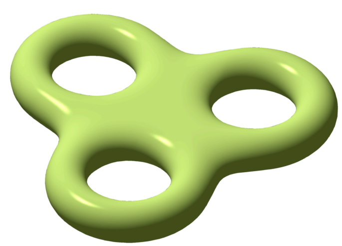
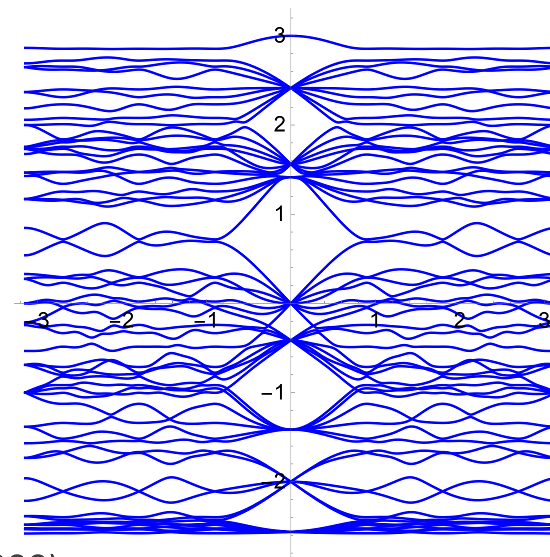


E/t vs (k,k,k,k,k,k)



Boettcher, Gorshkov, Kollár, JM, Rayan, Thomale, PRB 105, 125118 (2022)

Two issues

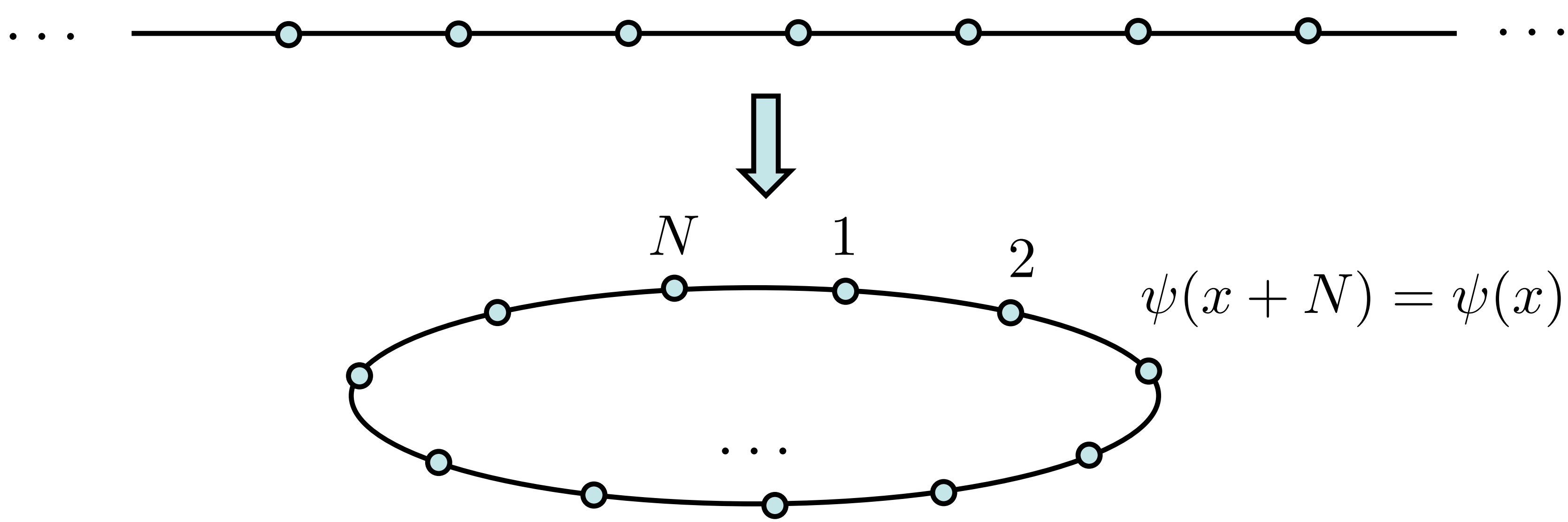
- Do hyperbolic Bloch states form a complete set (ansatz vs theorem)?
- What about finite lattices (experiment)?

Two issues

- Do hyperbolic Bloch states form a complete set (ansatz vs theorem)?
- What about finite lattices (experiment)?
- Solution: proper formulation of PBC

JM & Rayan, PNAS 119, e2116869119 (2022)

Euclidean PBC

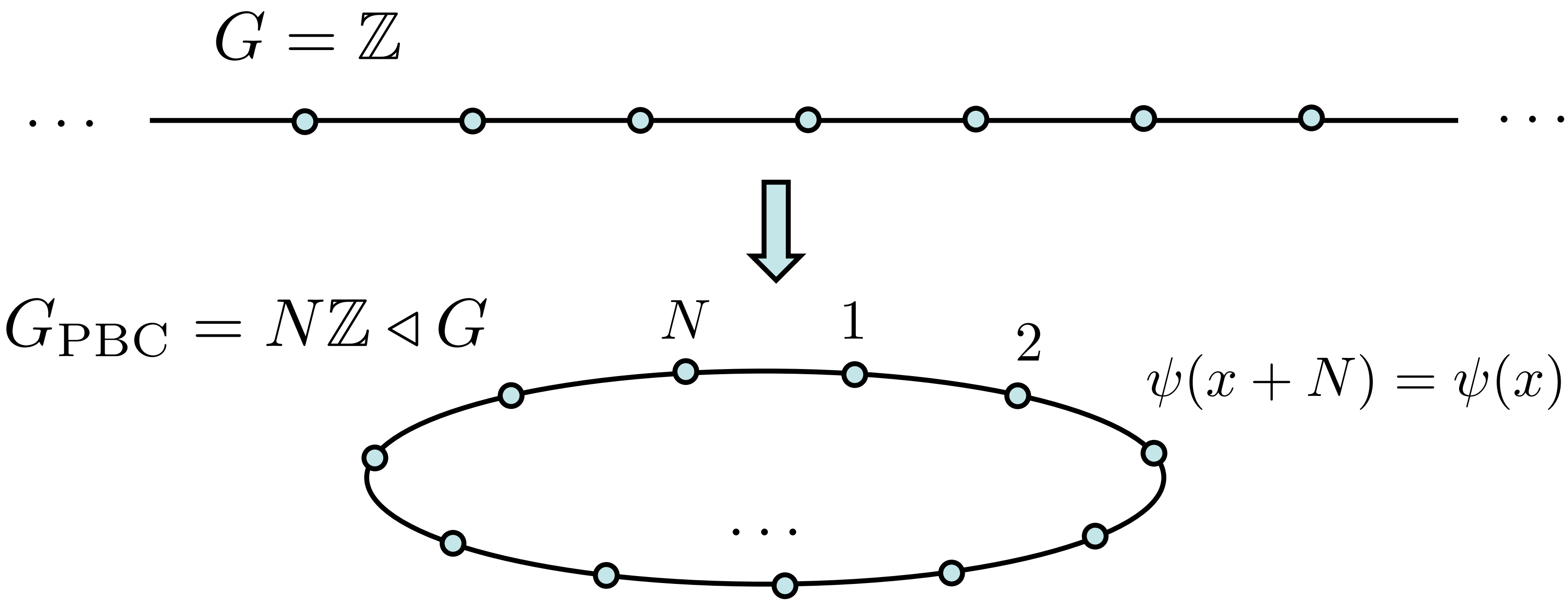


$$\psi(x+N) = \psi(x)$$

$$k = \frac{2\pi n}{N},\ n = 0, 1, \ldots, N-1$$

N sites = *N* Bloch states: complete set

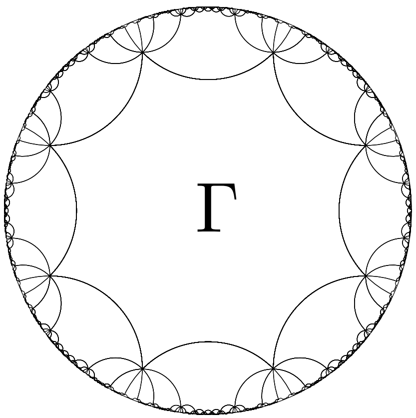
Euclidean PBC: algebraic viewpoint



$$G/G_{\mathrm{PBC}} = \mathbb{Z}_N$$

N allowed *k* values = *N* unitary irreps of G/G_{PBC}

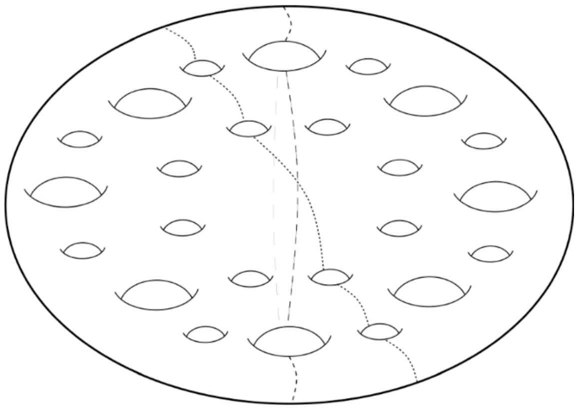
Hyperbolic PBC



infinite lattice

$$\Gamma_{\text{PBC}} \triangleleft \Gamma$$

normal subgroup of index N



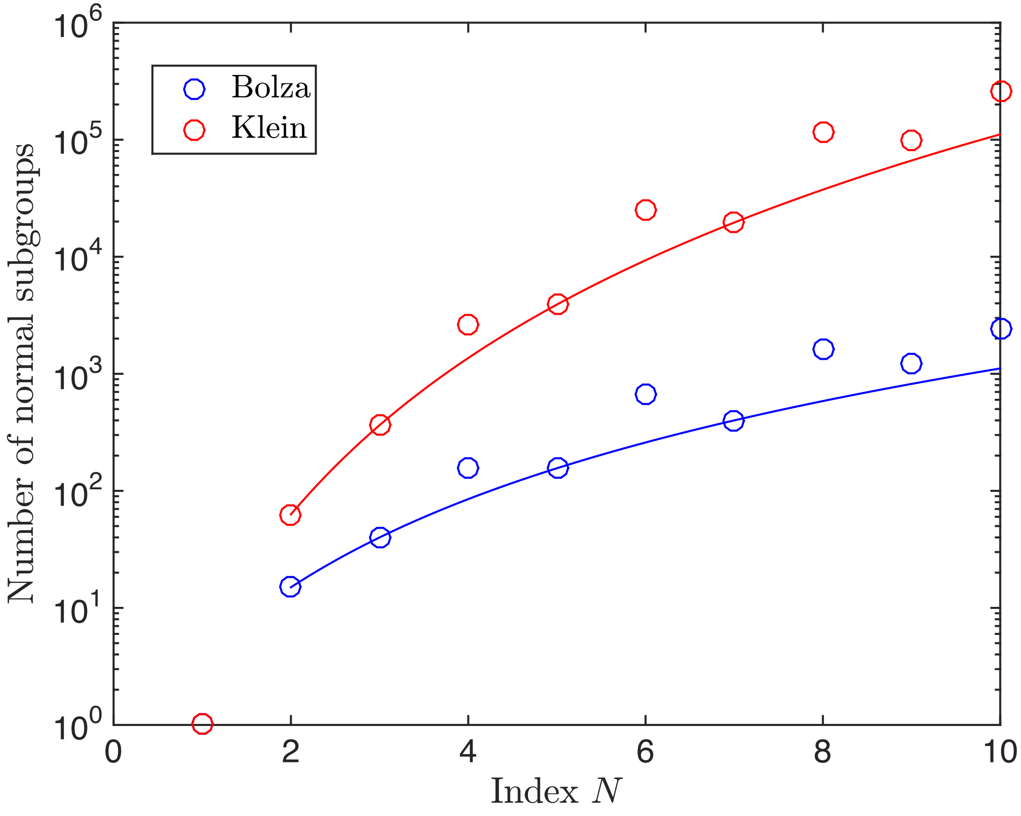
PBC cluster with N unit cells

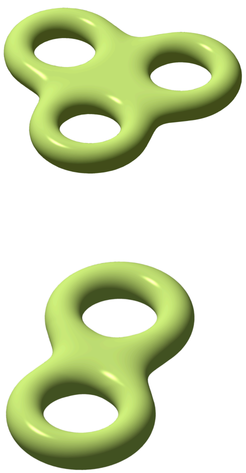
$$\psi(\gamma_{\text{PBC}}(z)) = \psi(z)$$

Bloch states = unitary irreps of $\Gamma/\Gamma_{\text{PBC}}$

Normal subgroups

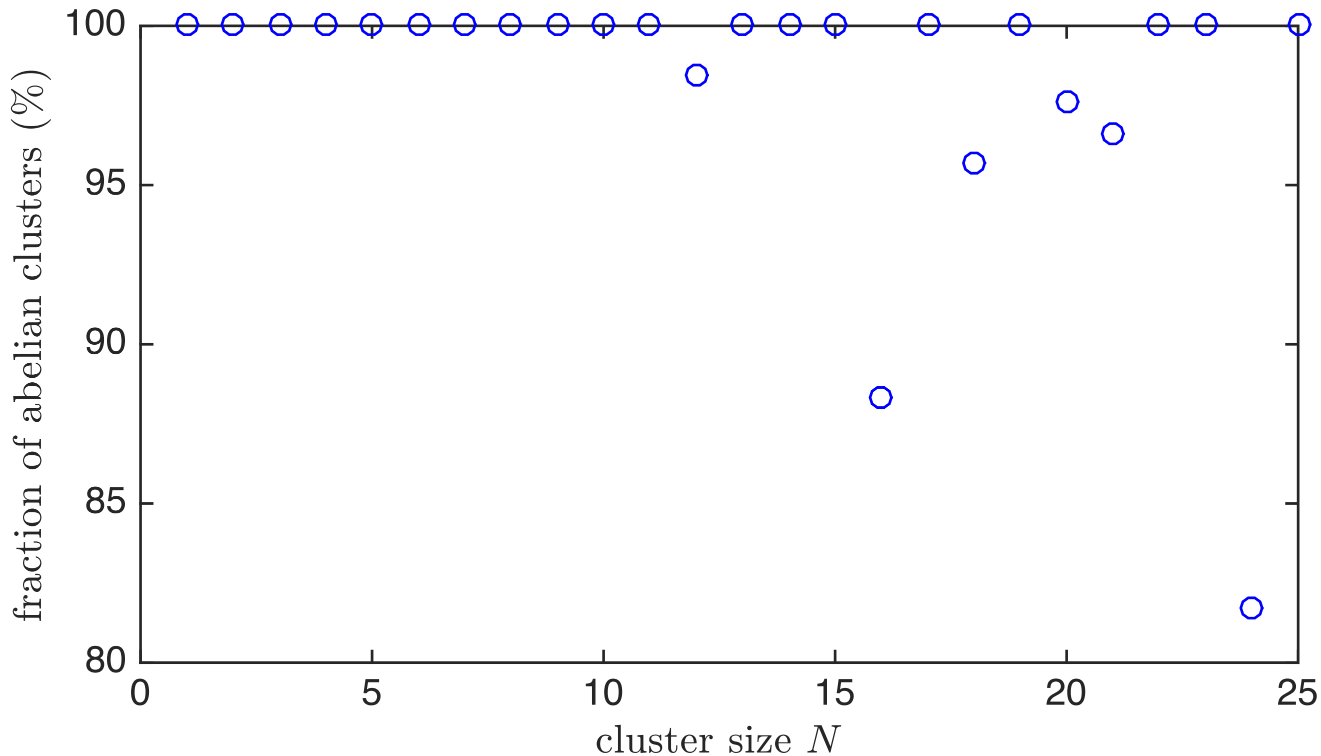
- For a given *N*, many distinct subgroups $\Gamma_{\mathrm{PBC}} \triangleleft \Gamma$
- Enumerate all normal subgroups of index up to N_{max} using computational group theory methods





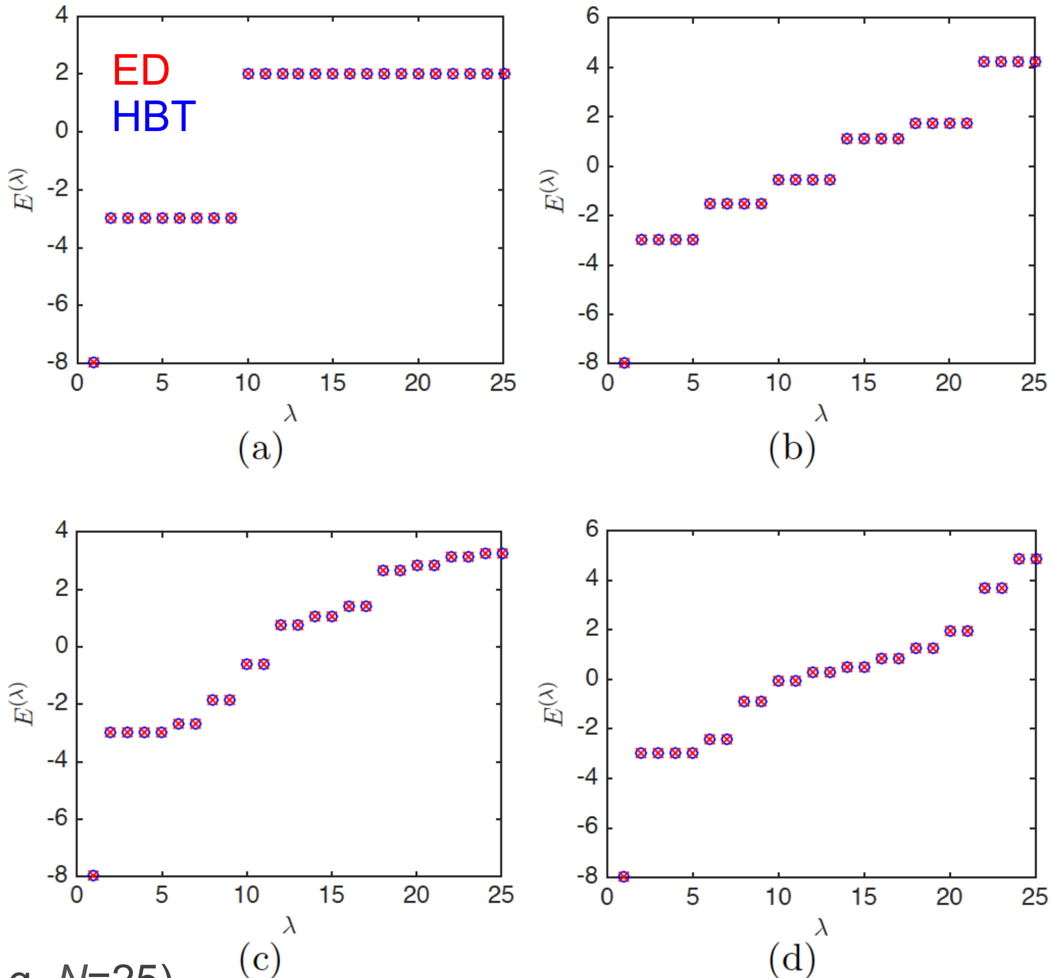
Abelian clusters

- Eigenstates of the clusters fall into irreps of the residual translation group $\Gamma/\Gamma_{\text{PBC}}$ = finite group of order *N*
- For many clusters (e.g., all prime *N*), this group is abelian!



Abelian Bloch theorem

$$\psi^{(\lambda)}(g_k^{-1}(z_i)) = \chi^{(\lambda)}([g_k])\psi^{(\lambda)}(z_i), \qquad [g_k] \in \Gamma/\Gamma_{\rm PBC}$$



$$E^{(\lambda)} = -2\sum_{j=1}^{4} \cos k_j^{(\lambda)},$$

$$\lambda = 1, \ldots, N$$

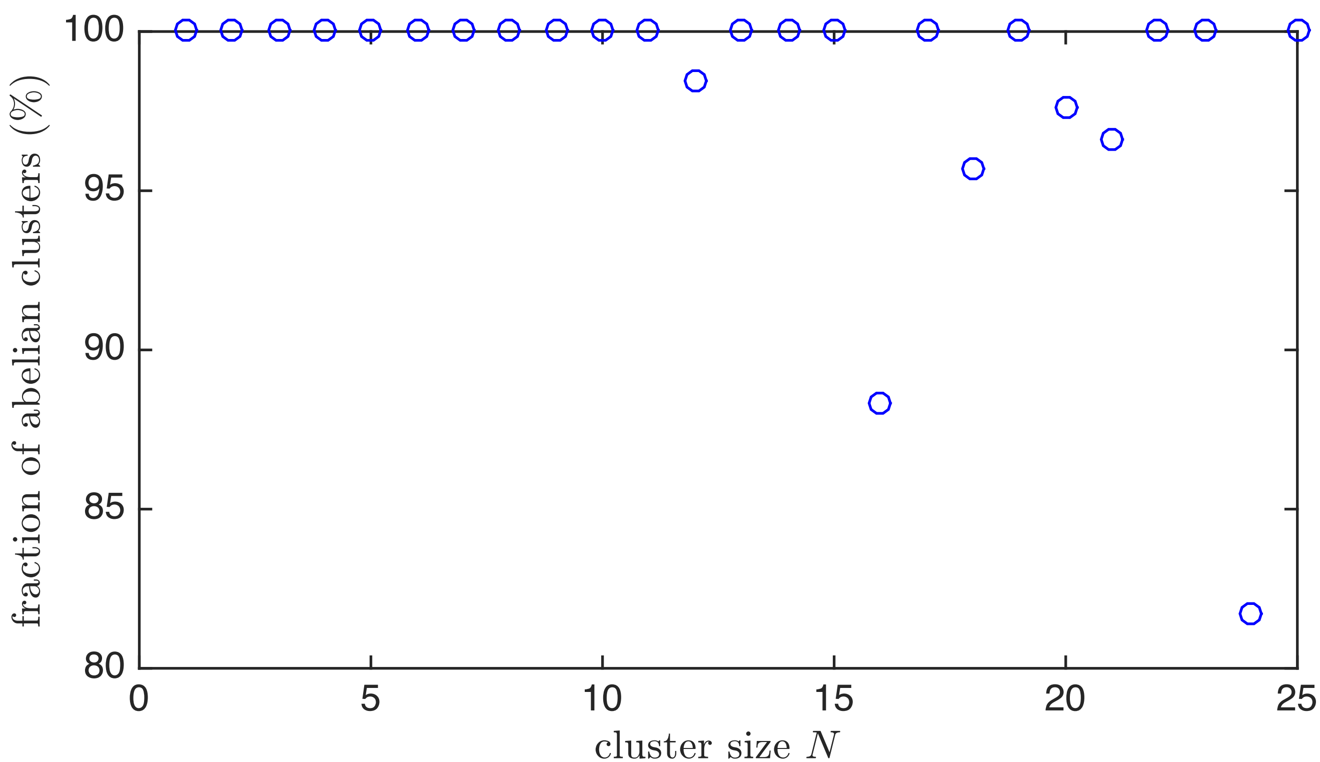
$$k_j^{(\lambda)} \in 2\pi\mathbb{Q}$$

4D BZ is discretized

(e.g. N=25)

Nonabelian clusters

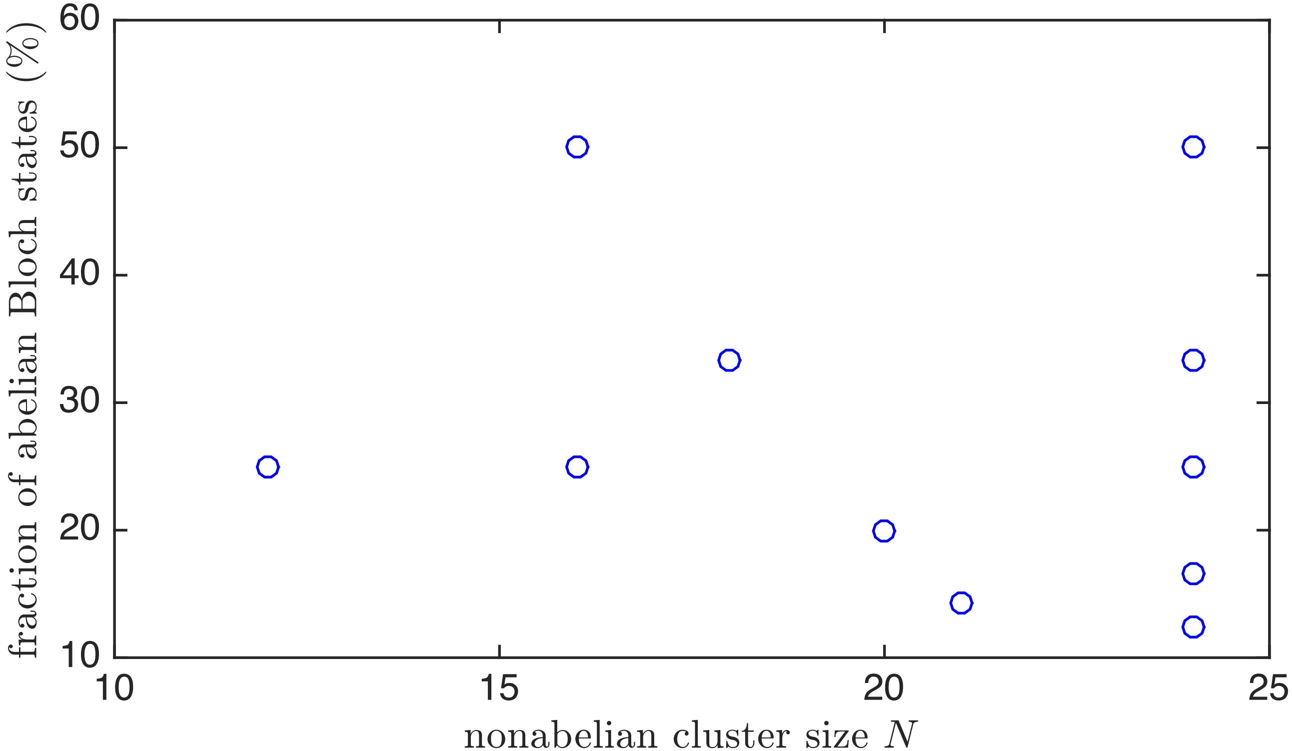
- For $N < 25$, nonabelian $\Gamma/\Gamma_{\text{PBC}}$ found only at $N = 12,16,18,20,21,24$



Nonabelian Bloch theorem

- Nonabelian $\Gamma/\Gamma_{\mathrm{PBC}}$ possesses higher-dimensional unitary irreps:

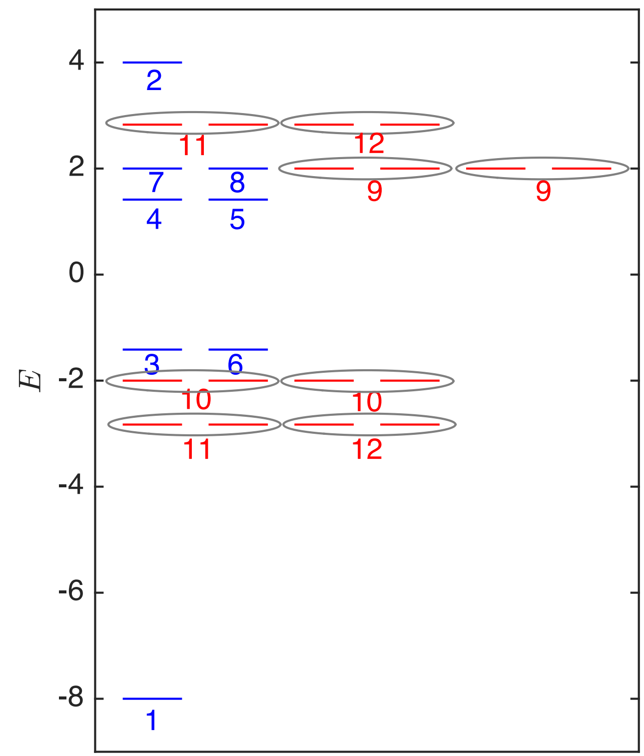
$$\psi_\nu^{(\lambda)}(g_k^{-1}(z_i)) = \sum_{\mu=1}^{r_\lambda} \psi_\mu^{(\lambda)}(z_i) D_{\mu\nu}^{(\lambda)}([g_k]), \qquad [g_k] \in \Gamma/\Gamma_{\mathrm{PBC}}$$



Nonabelian Bloch theorem

- Example (N=24): 8 abelian irreps, 4 nonabelian (2D) irreps

C	1	2	3	4	5	6	7	8	9	10	11	12
n_C	1	1	1	1	2	2	2	2	3	3	3	3
$D^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	1
$D^{(2)}$	1	1	1	1	1	1	1	1	-1	-1	-1	-1
$D^{(3)}$	1	-1	a	$-a$	1	-1	a	$-a$	c	$-c$	$-1/c$	$1/c$
$D^{(4)}$	1	-1	a	$-a$	1	-1	a	$-a$	$-c$	c	$1/c$	$-1/c$
$D^{(5)}$	1	-1	$-a$	a	1	-1	$-a$	a	$-1/c$	$1/c$	c	$-c$
$D^{(6)}$	1	-1	$-a$	a	1	-1	$-a$	a	$1/c$	$-1/c$	$-c$	c
$D^{(7)}$	1	1	-1	-1	1	1	-1	-1	a	a	$-a$	$-a$
$D^{(8)}$	1	1	-1	-1	1	1	-1	-1	$-a$	$-a$	a	a
$D^{(9)}$	2	2	-2	-2	-1	-1	1	1	0	0	0	0
$D^{(10)}$	2	2	2	2	-1	-1	-1	-1	0	0	0	0
$D^{(11)}$	2	-2	b	$-b$	-1	1	$-a$	a	0	0	0	0
$D^{(12)}$	2	-2	$-b$	b	-1	1	a	$-a$	0	0	0	0

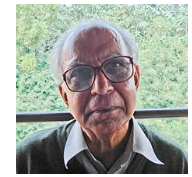


Nonabelian Brillouin zones

$$\mathrm{Irrep}(\Gamma, U(r))/U(r)$$

(topological)

Riemann-Hilbert



Narasimhan-Seshadri (1965)

$$\frac{\text{flat } U(r) \text{ connections on } \Sigma_g}{\text{gauge group}}$$

(smooth)

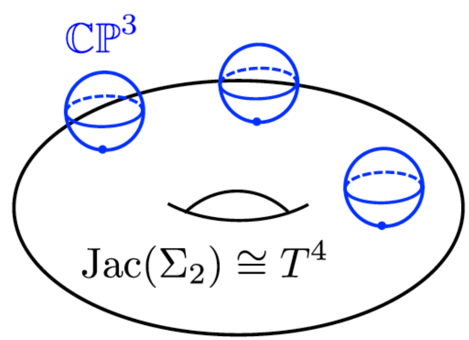
Atiyah-Bott, Witten…

$$\mathcal{M}(\Sigma_g, U(r))$$

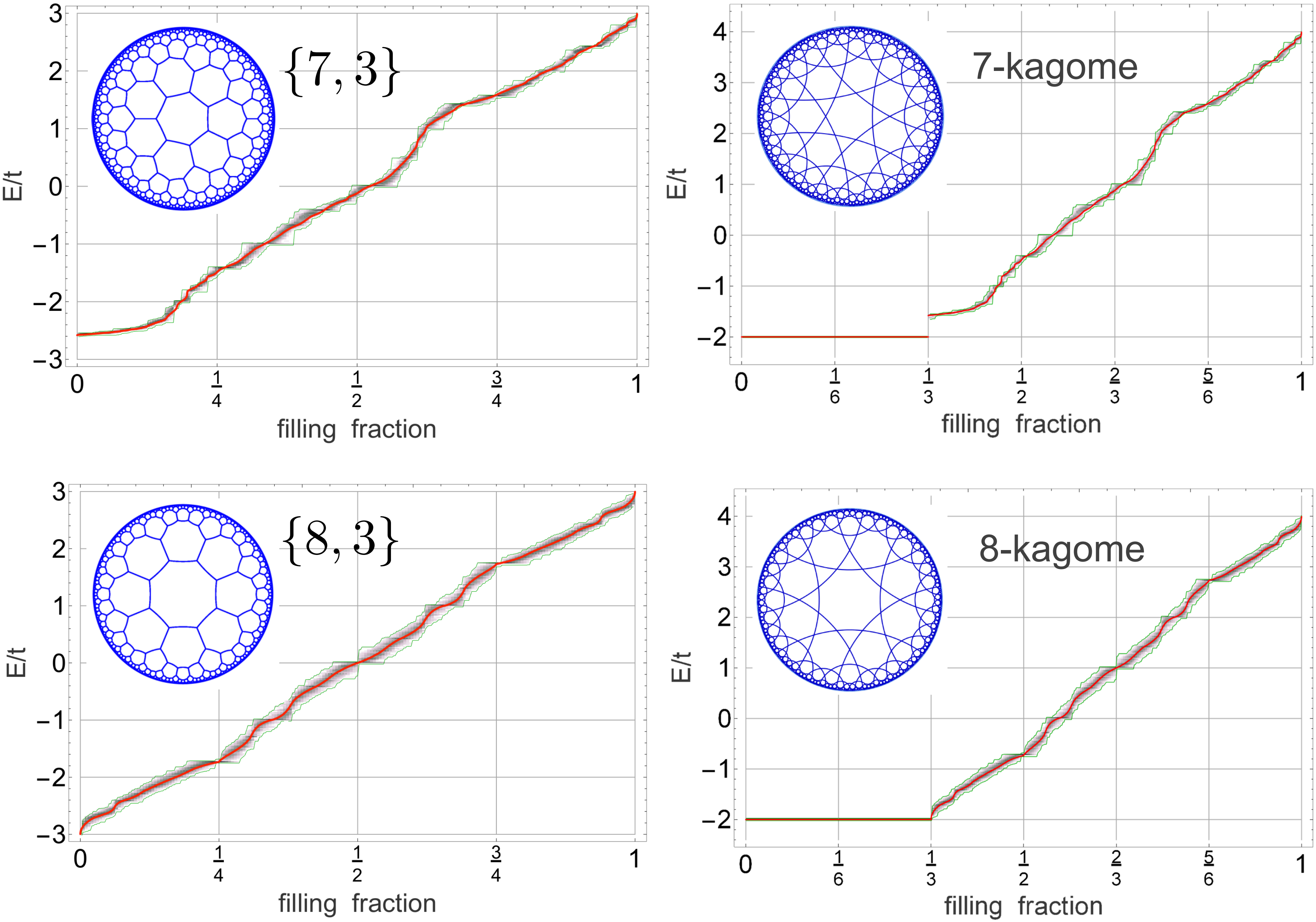
moduli space of stable holomorphic vector bundles of rank *r*

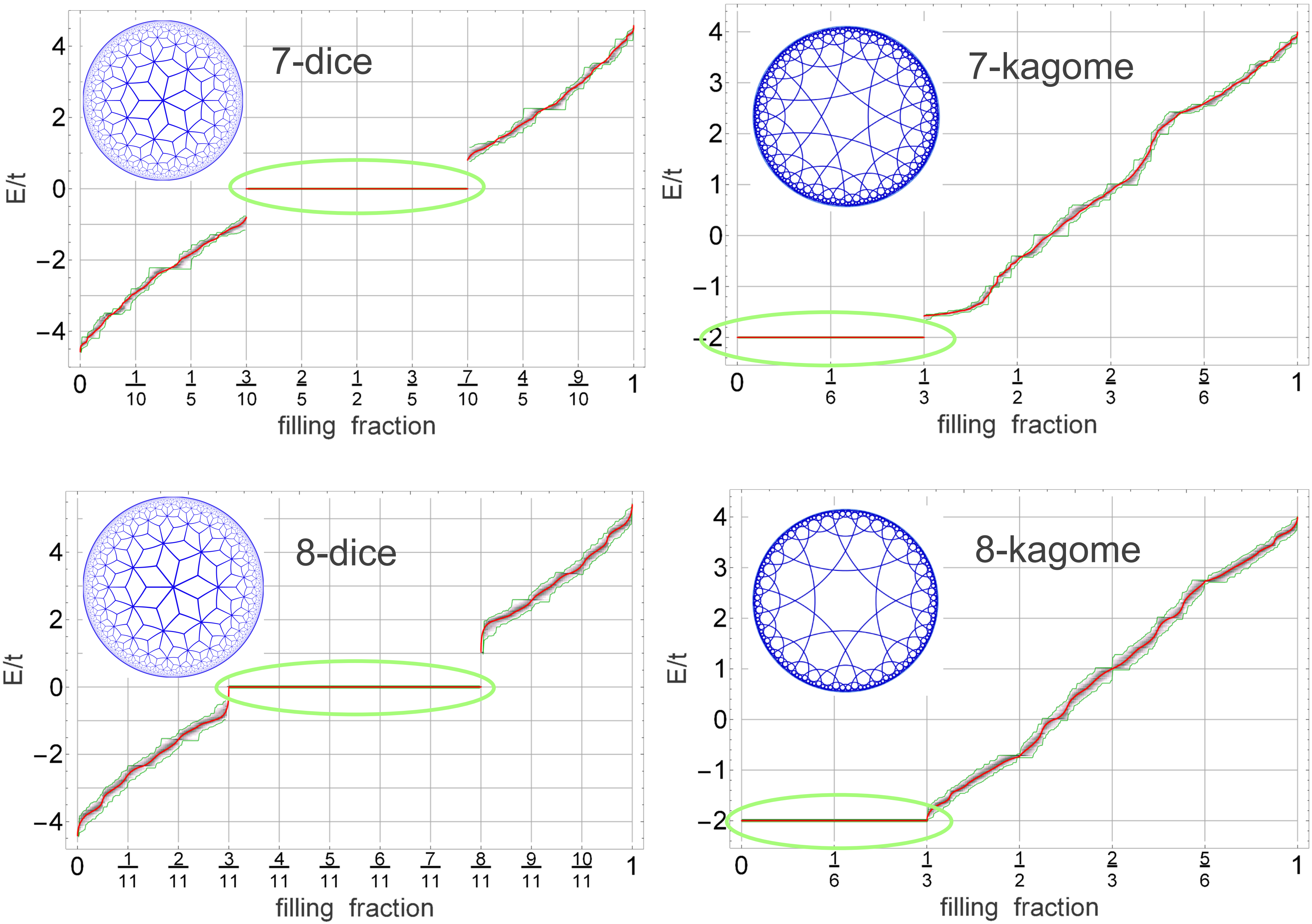
(holomorphic)

{8,8} lattice, 2D irreps: $\mathcal{M} \approx T^4 \times \mathbb{CP}^3$

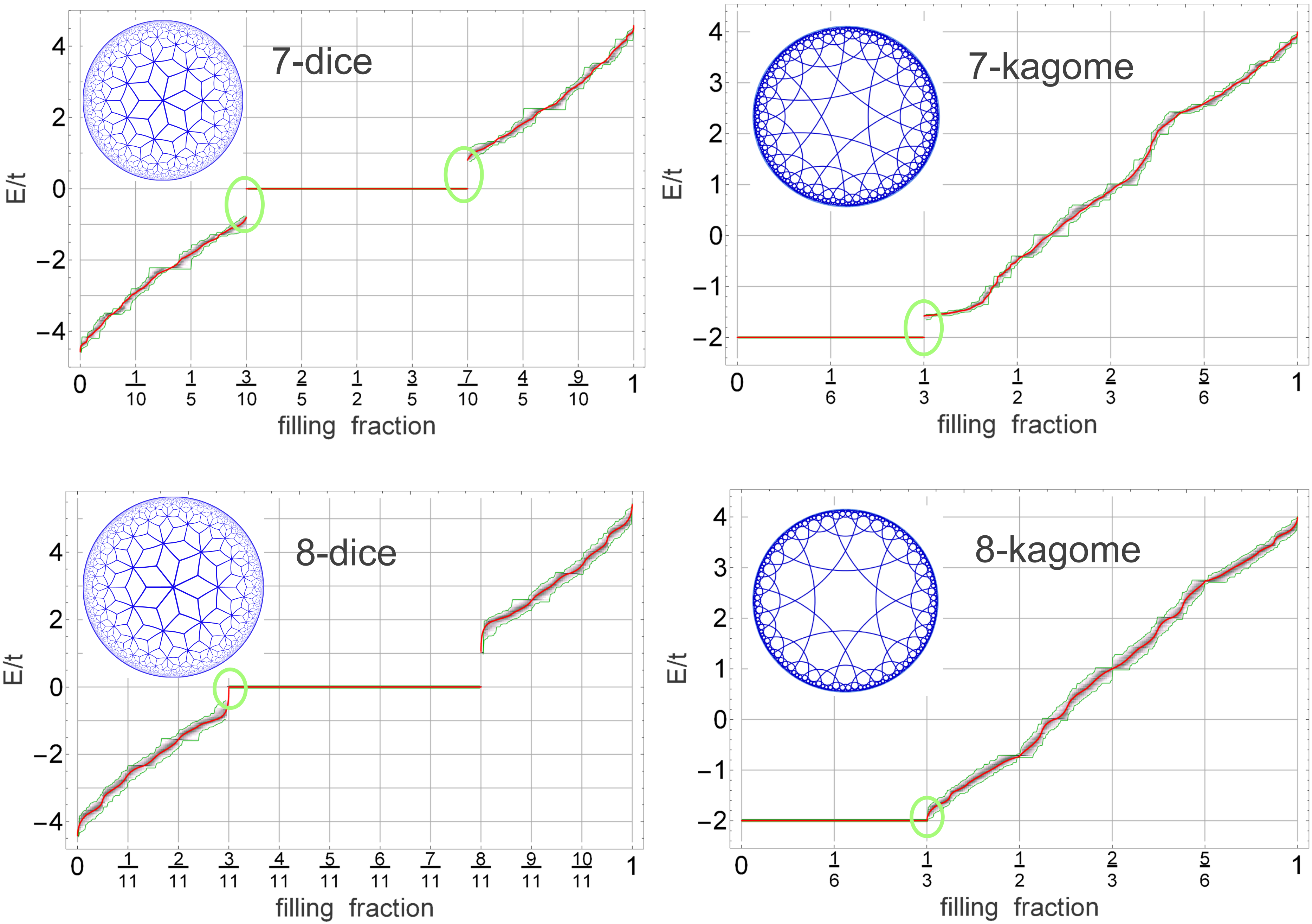


Are higher-dimensional irreps important?


{7, 3}
7-kagome
{8, 3}
8-kagome
E/t
filling fraction
0
1/4
1/2
3/4
1
1/6
1/3
2/3
5/6



T. Bzdušek & JM, in preparation



T. Bzdušek & JM, in preparation

Hyperbolic band topology?

Chern insulator in a hyperbolic lattice

Zheng-Rong Liu,[1] Chun-Bo Hua,[1] Tan Peng,[1] and Bin Zhou[1,*]

arXiv:2203.02101 (th)

Observation of novel topological states in hyperbolic lattices

Weixuan Zhang[1*], Hao Yuan[1*], Na Sun[1], Houjun Sun[2], and Xiangdong Zhang[1$]

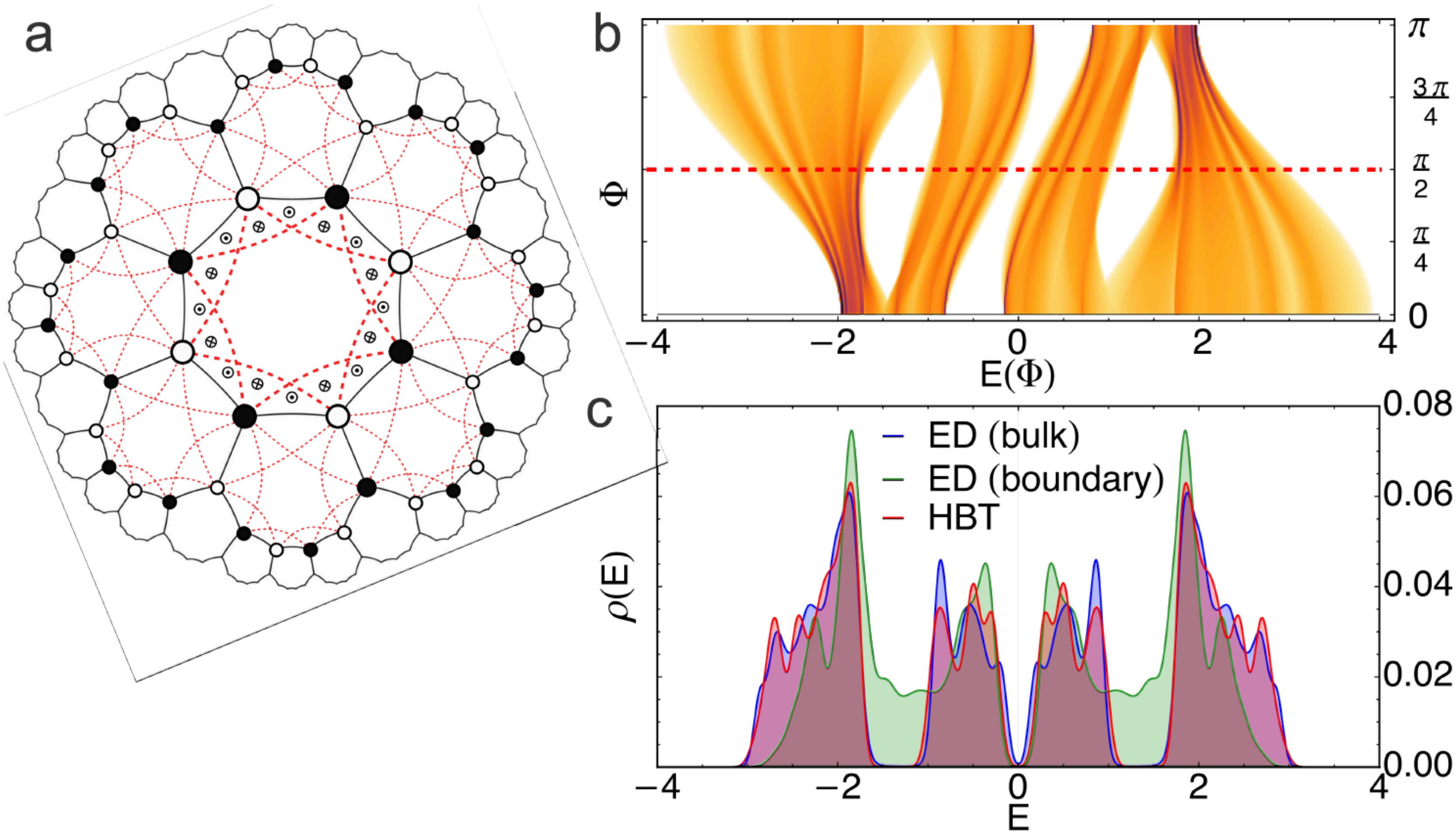
arXiv:2203.03214 (th+exp)

Hyperbolic topological band insulators

David M. Urwyler,[1] Patrick M. Lenggenhager,[1,2,3] Igor Boettcher,[4,5] Ronny Thomale,[6] Titus Neupert,[1] and Tomáš Bzdušek[2,1,*]

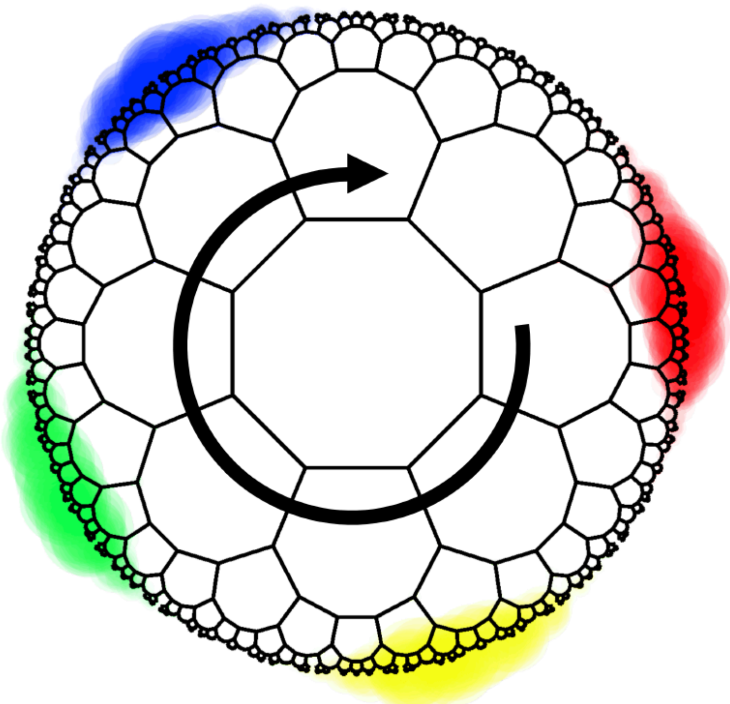
arXiv:2203.07292 (th)

Hyperbolic Haldane model

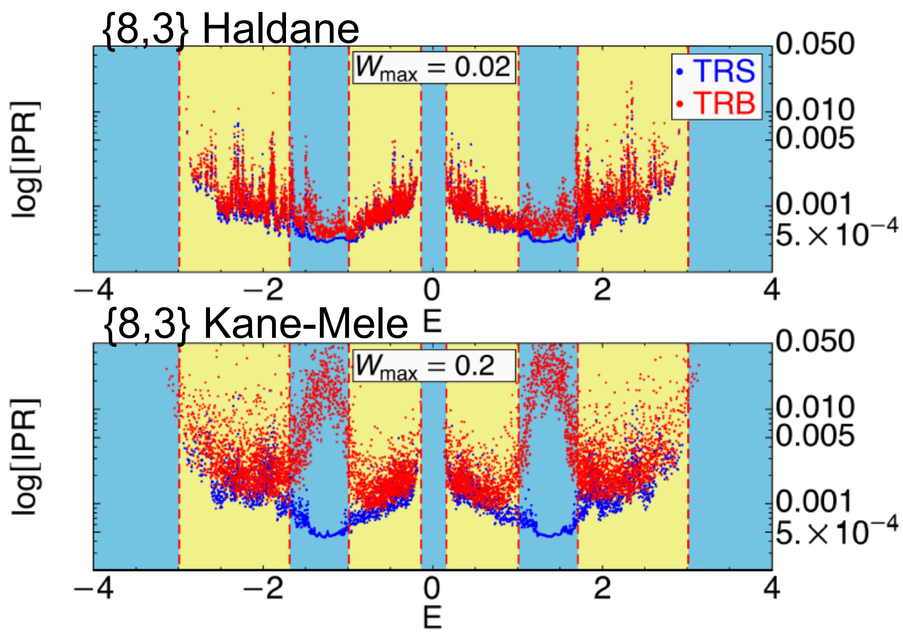


D. M. Urwyler et al., arXiv:2203.07292

Hyperbolic topological insulators



chiral/helical edge propagation



topological protection

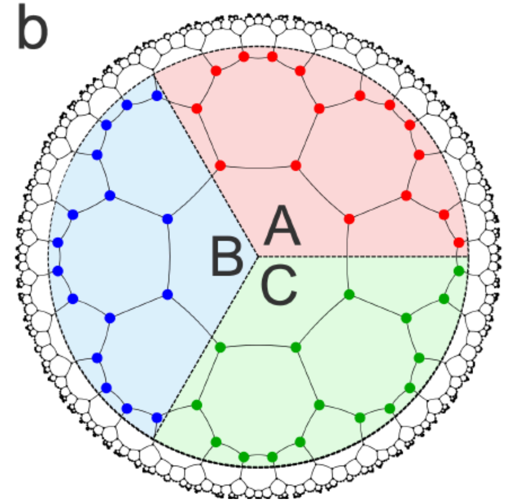
D. M. Urwyler et al., arXiv:2203.07292

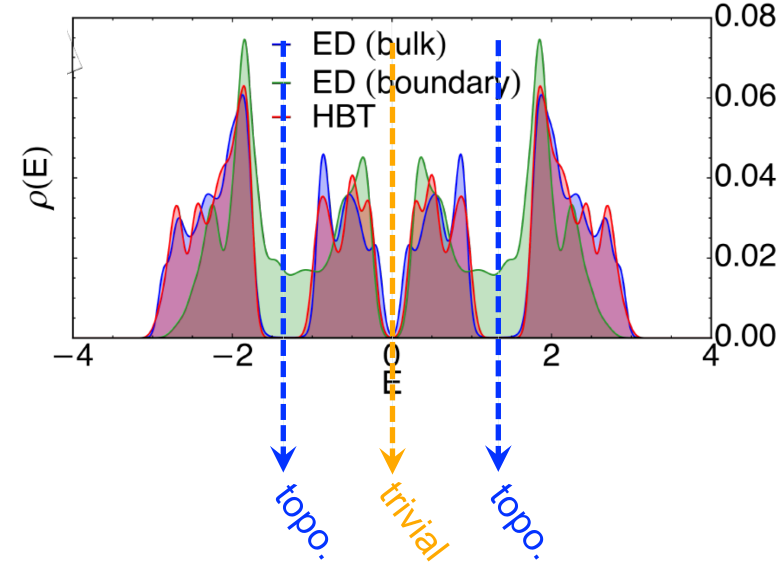
Hyperbolic band topology

a

		Haldane		Kane-Mele	
f	μ		$\mathcal{C}_{RS}$		ν_{RS}
$5/16$	-1.3		-0.986		-0.971
$8/16$	0		0		0
$11/16$	+1.3		-0.986		-0.971

b





D. M. Urwyler et al., arXiv:2203.07292

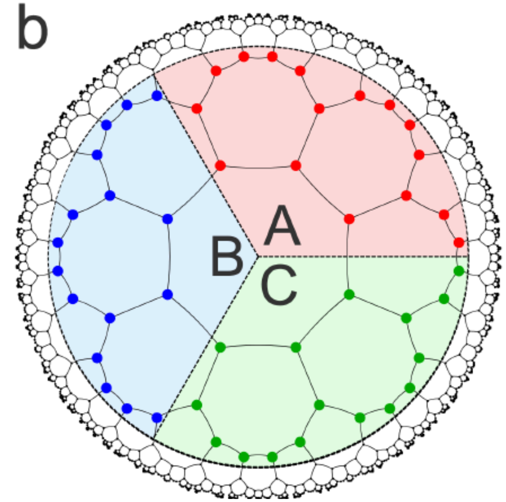
Hyperbolic band topology

a

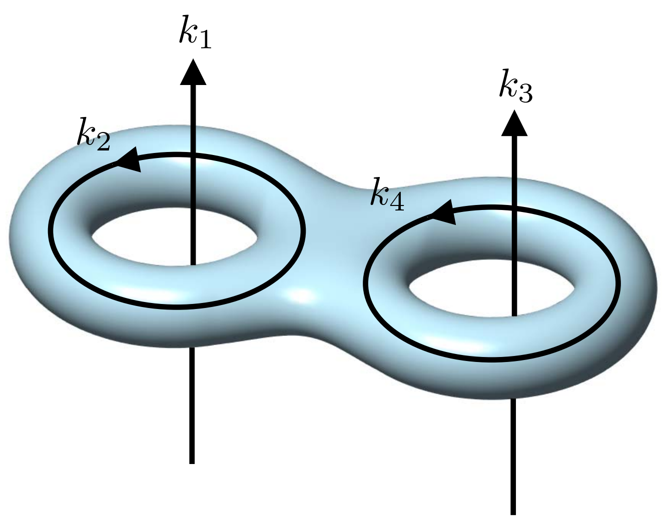
		Haldane			Kane-Mele		
f	μ	$\mathcal{C}_a$	$\mathcal{C}_b$	$\mathcal{C}_{RS}$	ν_a	ν_b	ν_{RS}
5/16	-1.3	-1	1	-0.986	1	1	-0.971
8/16	0	0	0	0	0	0	0
11/16	+1.3	-1	1	-0.986	1	1	-0.971

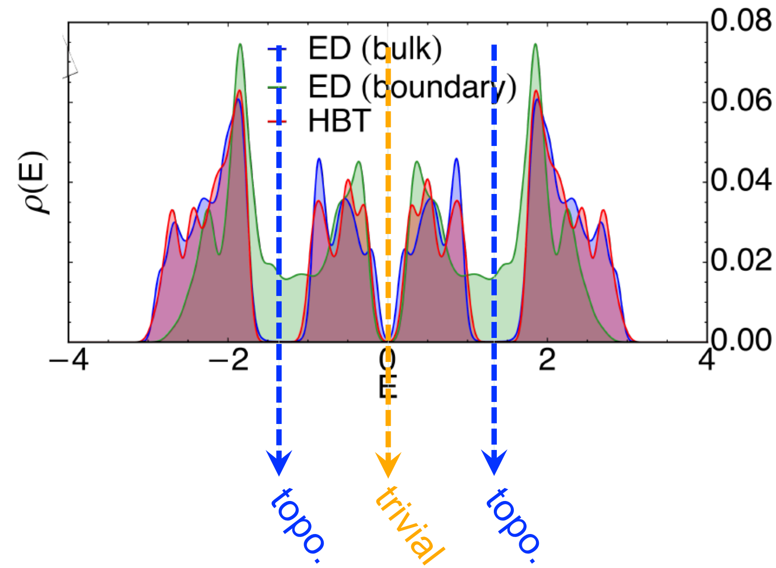
topological invariants in 2D planes of 4D Brillouin zone

b



$C_a = C_{12}$
$C_b = C_{13}$





D. M. Urwyler et al., arXiv:2203.07292

Summary

- Hyperbolic lattices have a nonabelian translation group: $\Gamma \cong \pi_1(\Sigma_g)$
- Hyperbolic (automorphic) Bloch states: $\psi_{\boldsymbol{k}}(\gamma(z)) = \chi_{\boldsymbol{k}}(\gamma)\psi_{\boldsymbol{k}}(z)$
- Hyperbolic Brillouin zones: $\mathrm{Jac}(\Sigma_g) \cong T^{2g} \cong \mathcal{M}(\Sigma_g, U(1))$

  $\mathcal{M}(\Sigma_g, U(r)),\ r > 1$
- Finite clusters with PBC correspond to $\Gamma_{\mathrm{PBC}} \triangleleft \Gamma$
- Automorphic Bloch theorems (abelian/nonabelian)
- Hyperbolic flat bands
- Hyperbolic topological bands

Outlook

- Why does abelian HBT work so well?
- Parametrization of nonabelian Brillouin zones
- Physics of nonabelian Bloch states?
- Bulk-boundary correspondence ($C_{\text{real-space}} \leftrightarrow C_{k\text{-space}}$)?
- Implement PBC in experiment (e.g. electrical circuits)

JM & S. Rayan, Sci. Adv. 7, eabe9170 (2021)
JM & S. Rayan, PNAS 119, e2116869119 (2022)
I. Boettcher *et al.*, PRB 105, 125118 (2022)

Thank you!

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