Fifty years of Ising lattice gauge theory

Franz J. Wegner, Institute for Theoretical Physics,
Ruprecht-Karls University Heidelberg

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Duality in Generalized Ising Models and Phase Transitions without Local Order Parameters*

Franz J. Wegner†
Department of Physics, Brown University, Providence, Rhode Island 02912
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It is shown that any Ising model with positive coupling constants is related to another Ising model by a duality transformation. We define a class of Ising models M_{dn} on d-dimensional lattices characterized by a number $n = 1, 2, \ldots, d$ ($n = 1$ corresponds to the Ising model with two-spin interaction). These models are related by two duality transformations. The models with $1 < n < d$ exhibit a phase transition without local order parameter. A nonanalyticity in the specific heat and a different qualitative behavior of certain spin correlation functions in the low and the high temperature phases indicate the existence of a phase transition. The Hamiltonian of the simple cubic dual model contains products of four Ising spin operators. Applying a star square transformation, one obtains an Ising model with competing interactions exhibiting a singularity in the specific heat but no long-range order of the spins in the low temperature phase.

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Motivation and Outline

Around 1971 I worked in critical phenomena. I was fascinated by the fact that Kramers and Wannier (1941) were able to determine the critical temperature of the two-dimensional Ising model on the square lattice exactly before Onsager (1944) gave the exact solution for this model. The basic idea was that this model was self-dual. Therefore I thought whether someting similar could be done for the three dimensional model. I realized that the dual model is a gauge-invariant model, but it is not self-dual. Increasing the number of dimensions from three to four the gauge-invariant model is self-dual and its critical temperature is the same as for the two-dimensional conventional Ising model.

The gauge-invariant model has the property that it has no local order parameter. Non-vanishing correlations are given by products of spins along a loop, called Wilson-loop. The expectation value obeys at high temperatures an area law and at low temperatures a perimeter law. If time permits I will mention some related work.

Outline

- Duality in the two-dimensional Ising model
- Duality in three dimensions
- Ising models $\mathcal{M}_{d,n}$
- Correlations
- Dislocations and Correlations
- Self-dual models in three dimensions

  What was missing?

  Lattice gauge theories

  Electromagnetic field
- Summary

Duality in two-dimensional Ising models

Kramers and Wannier 1941

Prediction of exact critical temperature of the two dimensional Ising model on a square lattice before exact solution by Onsager in 1944. They compared the high- and the low-temperature expansion for the partition function of the model.

Square lattice with $N_s = N_1 \times N_2$ lattice points and periodic boundary conditions. Ising spin $S_{i,j} = \pm 1$ at each lattice site (i,j). The Hamiltonian reads

$$H = -J \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (S_{i,j} S_{i,j+1} + S_{i,j} S_{i+1,j}). \tag{1}$$

High temperature expansion (*HTE*) The Boltzmann factor

$$\begin{aligned} e^{-\beta H} &= \prod_{i,j} (\cosh K + \sinh K S_{i,j} S_{i,j+1})(\cosh K + \sinh K S_{i,j} S_{i+1,j}) \\ &= (\cosh K)^{N_b} \prod_{i,j} (1 + \tanh K S_{i,j} S_{i,j+1})(1 + \tanh K S_{i,j} S_{i+1,j}), \end{aligned} \quad (2)$$

N_b number of bonds, $K = \beta J$. Partition function: expand in powers of $\tanh K S S'$, sum over all spin configurations: zero unless all spins appear with even powers, then the sum is 2^{N_s} and the interaction bonds form closed loops. Closed: An even number of bonds meets at each lattice site.
Expansion

$$Z(K) = 2^{N_s} (\cosh K)^{N_b} f(\tanh K), \quad (3)$$

$$f(a) = \sum_l c_l a^l, \quad (4)$$

Coefficients c_l count the number of closed loops of length l, $c_0 = 1$, $c_2 = 0$, $c_4 = N_s$, $c_6 = 2N_s$, $c_8 = N_s(N_s + 9)/2$, etc. and $c_l = 0$ for odd l.

Low temperature expansion (LTE) on the dual lattice. Spins $S^*(r^*)$ inside each of the squares of the original lattice. Dual Hamitonian

$$H^* = -J^* \sum_{i,j} (S^*_{i-1/2,j-1/2} S^*_{i-1/2,j+1/2} + S^*_{i-1/2,j-1/2} S^*_{i+1/2,j-1/2}). \qquad (5)$$

Positive J^*: All spins are parallel in the ground state: $E^*_{min} = -N_b J^*$, $N_b = 2N^*_s$ number of bonds.

Excited states by turning some spins. Reversing one spin costs an excitation energy $8J$, since the spin interacts with 4 other spins. Quite generally the excitation energy is given by $2lJ$, if the overturned spins are surrounded by Bloch walls of a total number of l edges. In the case of the square lattice one obtains

$$Z^*(K^*) = 2\mathrm{e}^{N_b K^*} f(\mathrm{e}^{-2K^*}) \qquad (6)$$

with f defined in (4). Both expansions are governed by the same function f.

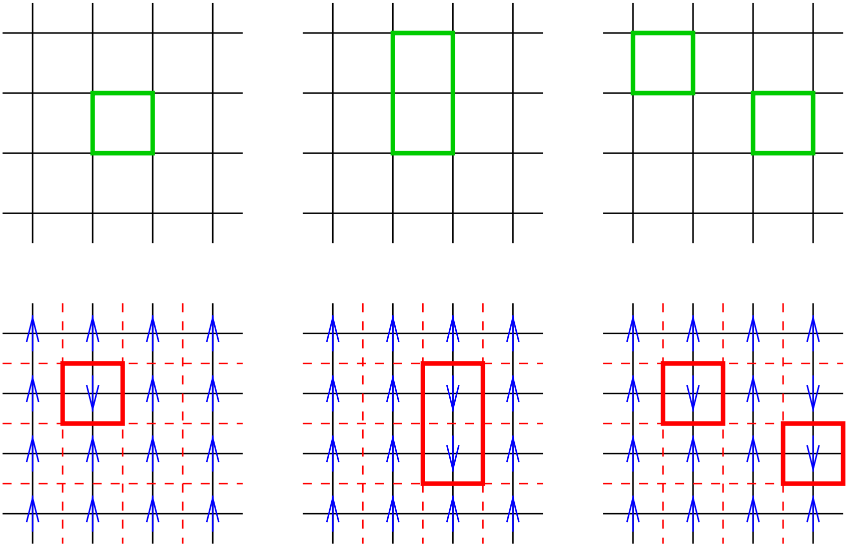

Fig. 1. Examples for closed loops in the *HTE* and Bloch walls in the *LTE* on the dual lattice

Comparison Kramers and Wannier argued: If the partition function or equivalently the free energy has a singularity at the critical point and no other singularity, then it must be determined by

$$\mathrm{e}^{-2K_c} = \tanh K_c, \tag{7}$$

which yields

$$K_c = \frac{1}{2}\ln(1+\sqrt{2}) = 0.4407, \tag{8}$$

which indeed turned out to be correct from Onsager's exact solution 1944. Thus there is a relation between the partition function and similarly the free energy at high ($K < K_c$) and low ($K^* > K_c$) temperatures for

$$\tanh K = \mathrm{e}^{-2K^*} \quad \leftrightarrow \quad \tanh K^* = \mathrm{e}^{-2K} \quad \rightarrow \quad \sinh(2K)\sinh(2K^*) = 1. \tag{9}$$

The square lattice is called self-dual, since the *HTE* and the *LTE* are performed on the same lattice.

Duality in 3 dimensions

Does there exist a dual model to the three-dimensional Ising model?
There is such a model, but of a different kind of interaction. *LTE* of the 3d-Ising model on a cubic lattice. I start out from the ordered state and then change single spins. These single spins are surrounded by closed Bloch walls. The expansion of the partition function is again of the form (4,6), but now with $c_2 = 0$, $c_4 = 0$, $c_6 = N_s$, $c_8 = 0$, $c_{10} = 3N_s$, $c_{12} = N_s(N_s - 7)/2$, etc.
The *HTE* of the dual model must be given by an interaction such that only closed surfaces yield a contribution. Thus locate a spin at each edge and introduce the interaction as a product of the spins surrounding an elementary square called plaquette. The interaction of the dual model $\mathcal{M}_{3,2}$

$$\beta H_{3,2} = -K \sum_{i,j,k} (R(i+1/2,j,k) + R(i,j+1/2,k) + R(i,j,k+1/2)),$$

$$R(i+1/2,j,k) = S_{i+1/2,j,k+1/2} S_{i+1/2,j+1/2,k} S_{i+1/2,j,k-1/2} S_{i+1/2,j-1/2,k},$$

$$R(i,j+1/2,k) = S_{i+1/2,j+1/2,k} S_{i,j+1/2,k+1/2} S_{i-1/2,j+1/2,k} S_{i,j+1/2,k-1/2},$$

$$R(i,j,k+1/2) = S_{i+1/2,j,k+1/2} S_{i,j+1/2,k+1/2} S_{i-1/2,j,k+1/2} S_{i,j-1/2,k+1/2}.$$

is the sum of interactions R over three differently oriented plaquettes.

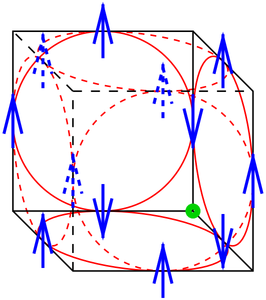

Fig. 2. Elementary cube with spins of the model $\mathcal{M}_{3,2}$. The red circles (ellipses) connect the four spins multiplied in the interaction.

Spin-independent products The product $R(i+1/2,j,k)R(i,j+1/2,k)$ $\times R(i,j,k+1/2)R(i-1/2,j,k)R(i,j-1/2,k)R(i,j,k-1/2)$ of the six Rs around the cube does not depend on the spin configuration, since each spin appears twice in the product.

Local gauge invariance Turning all six spins $S(i \pm 1/2, j, k)$, $S(i, j \pm 1/2, k)$, $S(i, j, k \pm 1/2)$ around the corner (i, j, k) does not change the energy of the configuration. As an example the three spins around the green corner are reversed from the state, in which all spins are aligned upwards.

Ising-models $\mathcal{M}_{d,n}$

The models considered up to now are generalized to models in arbitrary dimensions d.

Cut the d-dimensional hypervolume into d-dimensional hypercubes $B^{(d)}$ by $(d-1)$-dimensional hyperplains at integer Cartesian coordinates. Hypercubes $B^{(d)}$ are bounded by $(d-1)$-dimensional hypercubes $B^{(d-1)}$.

k-dimensional hypercubes $B^{(k)}$ are bounded by $(k-1)$-dimensional hypercubes $B^{(k-1)}$.

0-dimensional hypercubes $B^{(0)}$ are corners of the $B^{(d)}$.

Associate lattice points $r^{(k)}$ at the centers of the $B^{(k)}$.

$(d-k)$ coordinates are integers given by the $(d-1)$-dimensional hyperplains.

The other k coordinates are half-integer, that is integer+1/2.

Similarly for the dual lattice, but the d-dimensional hypercubes $B^{*(d)}$ are bounded by hyperplains at half-integer coordinates. The associated centers of the k-dimensional hypercubes $B^{*(k)}$ are denoted by $r^{*(k)}$. The $(d-k)$ coordinates are half-integer, the other k coordinates are integer.
The points $r^{(k)}$ coincide with the $r^{*(d-k)}$.
They are the intersections of $B^{(k)}$ with $B^{*(d-k)}$.
Incidence matrix with elements $\theta(r^{(k)}, r^{(k-1)})$ equals 1, if $B^{(k-1)}$ with center $r^{(k-1)}$ is on the boundary of the hypercube $B^{(k)}$ with center $r^{(k)}$, otherwise 0.
Similarly $\theta^*(r^{*(k)}, r^{*(k-1)})$ for the dual lattice.
Hamiltonian of model $\mathcal{M}_{d,n}$ and its dual

$$\beta H_{d,n} = -\sum_b K(b)R(b), \qquad R(b) = \prod\nolimits_{r^{(n-1)}} S(r^{n-1})^{\theta(b,r^{(n-1)})}, \tag{10}$$

$$\beta^* H^* = -\sum_b K^*(b)R^*(b), \quad R^*(b) = \prod\nolimits_{r^{*(d-n-1)}} S^*(r^{d-n-1})^{\theta^*(b,r^{(d-n-1)})}, \tag{11}$$

with b for $r^{(n)} = r^{*(d-n)}$.

Closure relation An important property of the lattices is the closure relation: Consider a pair $r^{(k+1)}$ and $r^{(k-1)}$. They lie in cells $B^{(k+1)}$ and $B^{(k-1)}$. Then

$$\sum_{r^{(k)}} \theta(r^{(k+1)}, r^{(k)})\theta(r^{(k)}, r^{(k-1)}) \equiv 0 \bmod 2. \tag{12}$$

Proof: If $B^{(k-1)}$ is on the boundary of $B^{(k+1)}$, then two cells $B^{(k)}$ on the boundary of $B^{(k+1)}$ have $B^{(k-1)}$ as boundaries. If $B^{(k-1)}$ is not at the boundary of $B^{(k+1)}$, then none of the $B^{(k)}$ on the boundary of $B^{(k+1)}$ has $B^{(k-1)}$ as boundary. This proofs (12).

Gauge invariance Changing all spins close to a point $r^{(n-2)}$,

$$S(r^{(n-1)}) \to (-)^{\theta(r^{(n-1)}, r^{(n-2)})} S(r^{(n-1)}) \tag{13}$$

leaves $R(b)$ invariant, since it is multiplied by

$$(-)^{\sum_{r^{(n-1)}} \theta(r^{(n-1)}, r^{(n-2)}))\theta(r^{(n)}, r^{(n-1)})}, \tag{14}$$

which due to the closure relation (12) yields one.

Spin-independent products $R(b)$ The product over all $R(b)$ around a given $r^{(n+1)}$, that is

$$\prod_b R(b)^{\theta(r^{(n+1)},r^{(n)}(b))} = \prod_{r^{(n-1)}} S(r^{(n-1)})^{\sum_{r^{(n)}} \theta(r^{(n)},r^{(n-1)})\theta(r^{(n+1)},r^{(n)})} = 1. \quad (15)$$

does not depend on the spin configuration, since it yields one due to the closure relation (12). Of course also products of these products are spin-independent. The closure relation plays an important role in deriving the duality relation between $\mathcal{M}_{d,n}$ and $\mathcal{M}_{d,d-n}$ with

$$\sum_b \theta(b, r^{(n-1)})\theta^*(b, r^{*(d-n-1)}) = 0 \bmod 2, \quad (16)$$

since $\theta^*(b, r^{*(d-n-1)}) = \theta(r^{(n+1)}, b)$.
One uses **linear algebra on a ring with elements (0,1) with algebra modulo 2**.

Inequality One obtains

$$2^{-N_m/2} Y\{K\} < Y^*\{K^*\} < 2^{N_m/2} Y\{K\}. \tag{17}$$

for

$$Y\{K\} = \frac{2^{-(N_s+N_g)/2}}{\prod_b (\cosh(2K(b))^{1/2}} Z\{K\} \tag{18}$$

and similarly for $Y^*\{K^*\}$. N_b number of bonds, N_s and N_s^* number of spins of the lattices and N_g and N_g^* number of independent gauge transformations. $(N_s - N_g = \text{rank}(\theta))$. Then

$$N_m = N_b - N_s + N_g - N_s^* + N_g^* \tag{19}$$

depends on the boundary conditions and is normally negligible in the thermodynamic limit.
One obtains $N_m = \binom{d}{n}$ for periodic boundary conditions. Then N_m does not depend on the size of the model.

Self-duality The model $\mathcal{M}_{d,n}$ on the hypercubic lattice is self-dual, if $d = 2n$. This is the case for $\mathcal{M}_{2,1}$, which is the two-dimensional Ising model on the square lattice. But also the four-dimensional model $\mathcal{M}_{4,2}$ with the plaquette interaction is self-dual. Both have the phase transition at $K_c = 0.4407$, (8). The Ising model $\mathcal{M}_{2,1}$ shows a continuous transition. Creutz, Jacobs, and Rebbi have investigated the model $\mathcal{M}_{4,2}$ by Monte Carlo techniques. They determined $\langle R(b) \rangle$ as a function of K. They found a first order transition with hysteresis. By decreasing K the system showed superheating until ≈ 0.48 and by increasing K undercooling until ≈ 0.40. Starting from a mixed phase they located the transition between 0.43 and 0.45.

Correlations

Non-vanishing correlations are only obtained for gauge-invariant products. These are products of $R(b)$. In particular we consider the product of spins on the boundary of an n-dimensional hypercube of $\mathcal{M}_{d,n}$. The *HTE* yields

$$\langle\prod_r S(r)\rangle = (\tanh K + 2(d-n)(\tanh K)^{1+2n} + ...)^{\upsilon}, \quad n > 1, \tag{20}$$

$$\begin{aligned} &= \frac{1}{2}[\tanh K + (2(d-1))^{1/2}(\tanh K)^2 + ...]^{\upsilon} \\ &+ \frac{1}{2}[\tanh K - (2(d-1))^{1/2}(\tanh K)^2 + ...]^{\upsilon}, \quad n = 1. \end{aligned} \tag{21}$$

where υ is the volume of the hypercube. For $n = 1$ this is the distance between the two spins; for $n = 2$ it is the area spanned by the spins. The *LTE* yields

$$\begin{aligned} \langle\prod_r S(r)\rangle &= (1 - \mathrm{e}^{4(d-n+1)K} + ...)^{f}, \quad n < d, \\ \langle\prod_r S(r)\rangle &= (1 - 2\mathrm{e}^{-2K} + ...)^{\upsilon}, \quad n = d, \end{aligned} \tag{22}$$

where f is the hyperarea of the boundary of the hypercube (for $n = 1$ it is the number $f = 2$ of ends of the line; for $n = 2$, f is the perimeter of the square).

Thus the behavior of the correlation functions of large hypercubes differs in the high and low temperature phases, and we expect

$$\langle\prod_r S(r)\rangle \propto \begin{cases} \mathrm{e}^{-v/v_0(T)} & T > T_c,\, n < d \\ \mathrm{e}^{-f/f_0(T)} & T < T_c,\, n < d \end{cases} \tag{23}$$

We attribute the qualitatively different asymptotic behavior in both temperature regions to different states of the system above and below a critical temperature T_c.

Dislocations and Correlations

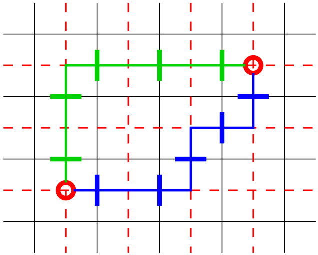

Fig. 3. Red dots: Location of disorder variable in the original lattice and of spins in the dual lattice

Kadanoff and Ceva for $\mathcal{M}_{2,1}$: Disorder variables at red points in Fig. 3. The couplings K are changed along a line to $-K$ between these two points. The change of the partition function is independent of choice of line (e.g. blue or green). Denote the operator of change of sign by $M(b)$ and let $\phi^*(b) = 0, 1$ for bonds with unchanged, changed sign. Then

$$\langle \prod_b M(b)^{\phi^*(b)} \rangle = \langle \prod_b \mathrm{e}^{-2\phi^*(b)K(b)R(b)} \rangle = \frac{Z\{(-)^{\phi^*} K\}}{Z\{K\}} = \frac{Y\{(-)^{\phi^*} K\}}{Y\{K\}}. \quad (24)$$

From (9) we obtain $\tanh((-)^{\phi^*} K) = \mathrm{e}^{-2K^* - \mathrm{i}\pi\phi^*}$ and thus

$$\langle\prod_b M(b)^{\phi^*(b)}\rangle = Y\{K^* + \mathrm{i}\pi\phi^*/2\}/Y\{K^*\}$$
$$= \quad \mathrm{i}^{-\sum_b \phi^*(b)}\langle\prod_b \mathrm{e}^{\mathrm{i}\pi\phi^*(b)R^*(b)/2}\rangle\{K^*\} = \langle\prod_b R^*(b)^{\phi^*(b)}\rangle\{K^*\}. \tag{25}$$

$\langle\prod_b M(b)^{\phi^*(b)}\rangle$ equals the ratio of the partition functions with the changed bonds and the unchanged bonds, thus the exponential of the difference $\beta\Delta F$ of the free energy without and with the changed bonds at K,

$$\langle S(r^*)S(r^{*'})\rangle(K^*) = \mathrm{e}^{-\beta\Delta F(K)}. \tag{26}$$

If K is in the paramagnetic region, then the disturbance of the bonds yields a contribution to $\beta\Delta F$ only close to the points, where this line of bonds ends. Thus for large separation of the two spins it approaches a finite value, which corresponds to the square of the magnetization at K^*. On the other hand if K is in the ferromagnetic region, then the disturbance will change the free energy proportional to the distance between the two spins $S(r^*)$ and $S(r^{*'})$, which yields an exponential decay of the correlation function.

Consider $\mathcal{M}_{3,1}$ and $\mathcal{M}_{3,2}$. Change the sign of the interaction $\sum_{ij} S_{i,j,k}S_{i,j,k+1}$ over a whole region (area) in the plane spanned by ij. Analogous to the two-dimensional Ising model, the change $\Delta F(K)$ will be proportional to the perimeter f for paramagnetic K and proportional to the area v for ferromagnetic K. The product $\prod_b R^*(b)$ is now the product of the Ising spins along the perimeter of the dislocations. Consequently the expectation value decays proportional to $\mathrm{e}^{-f/f_0(T^*)}$ at low temperatures T^* and proportional to $\mathrm{e}^{-v/v_0(T^*)}$ at high temperatures T^* in accordance with (23).

Local order parameter If all states are taken into account, then the correlations different from zero are only obtained from products of R. For $n = 1$ the product of two spins $S(0)S(r)$ can be written as product of Rs. For $n > 1$ products of spins $\prod_k S(a_k) \prod_l S(r + a_l)$ with with a_k and a_l restricted to some finite region $|a_k| < c$, $|a_l| < c$ yield only non-vanishing correlations for distances $r > 2c$, if both $\prod_k S(a_k)$ and $\prod_l S(r + a_l)$ are separately gauge invariant, that is, they are expressed as finite products of R. However, with (24, 25) expectations of products of R in one phase can be expressed by correlations in the other phase

$$\langle \prod_{\text{some } b} R(b) \rangle \{K\} = \langle \prod_{\text{same } b} (\cosh(2K^*(b)) - R^*(b) \sinh(2K^*(b))) \rangle.$$

Thus since there is no long range order in the high temperature phase, there can be none in the dual low temperature phase,

$$\lim_{r \to \infty} (\langle \prod_k S(a_k) \prod_l S(r + a_l) \rangle - \langle \prod_k S(a_k) \rangle \langle \prod_l S(a_l) \rangle) = 0.$$

Thus there is no local order parameter for models $\mathcal{M}_{d,n}$ with $n > 1$. This argument does not apply for $n = 1$, since in this case the number of Rs in the product increases with $|r|$.

Self-dual models in 3 dimensions

If we add a magnetic field to the model $\mathcal{M}_{3,2}$,

$$-\beta H'_{3,2} = -\beta H_{3,2} + h \sum_{i,j,k} (S_{i,j+1/2,k+1/2} + S_{i+1/2,j,k+1/2} + S_{i+1/2,j+1/2,k}), \quad (27)$$

then the model is self-dual in the two couplings K and h,

$$\tanh K = \mathrm{e}^{-2h^*}, \quad \tanh h = \mathrm{e}^{-2K^*}. \quad (28)$$

Another self-dual model in three dimensions: D.W. Wood, J. Phys. C5, L181 (1972); see also P.A. Pearce and R.J. Baxter, Phys. Rev. B24 (1981) 5295.

The Ising spins are placed on a face-centered cubic lattice. The interaction is given by the sum of the products over the four spins at the corners of the elementary tetrahedra of the lattice

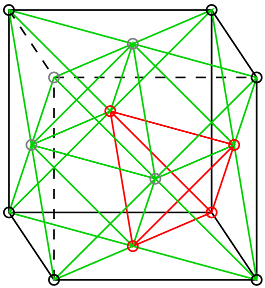

Fig. 4. Cube of fcc-lattice with eight elementary tetrahedra, one in red

What was missing?

The word **gauge**

The word **loop**. Would have been useful for the models $\mathcal{M}_{d,2}$.

The type of transition. I simply did not / could not know it and was surprised, when I learned, it is of first order.

Generalization from $Z(p = 2)$ Ising to larger p by Korthals Altes and by Yoneya. For $p > 4$ there are two transitions.

Elitzur, Pearson, and Shigemitsu have discussed the properties of this phase.

Lattice gauge theories

Models $\mathcal{M}_{d,n}$ with $n > 1$ show local gauge invariance. Such models are related to quantum chromodynamics. The basic idea first formulated by Wilson is to start from the lattice $\mathcal{M}_{4,2}$. (Many reprints on this subject are compiled in Rebbi's book). The degrees of freedom are now denoted by U in place of S. These U are elements of a group. It may be a finite or a continuous group, it may be an Abelian or non-Abelian group. In the case of QCD one considers the 'colour'-group SU(3). $U_{i,j}$ is placed on the link between lattice sites i and j with $U_{j,i} = U_{i,j}^{-1}$. The action is

$$g^{-2} \sum_{\text{plaquettes}} (1 - \frac{1}{q} \Re \text{tr}(U_{ij} U_{jk} U_{kl} U_{li})), \tag{29}$$

where q is the dimension of U. One introduces quarks (fermions) with interaction

$$g'^{-2} \sum_{\text{links}} \psi_i^\dagger U_{ij} \psi_j. \tag{30}$$

These interaction terms are invariant under local gauge transformations

$$\psi_j \to G_j \psi_j, \quad \psi_j^\dagger \to \psi_j^\dagger G_j^\dagger, \quad U_{ij} \to G_i U_{ij} G_j^\dagger. \tag{31}$$

The couplings depend on temperature and pressure of the hadron system. At low temperature and pressure the correlations fall of with an area law. Since the action is an integral over time, this behaviour corresponds to an increase of the effective potential between quarks proportional to the distance between them. The gradient of the potential is called string tension and given by $1/v_0(T)$ in (23). This potential binds three quarks, which constitute a hadron. Or one quark and one antiquark are bound and constitute a meson. Generally the difference between the number of quarks and antiquarks has to be a multiple of three. At high temperature and high pressure the system forms a quark-gluon plasma. This corresponds to the phase in which the correlation increases proportional to the perimeter of the loop. Then the effective potential between the quarks stays finite at large distances and the quarks are rather free to move in this plasma.

Electromagnetic field

The electromagnetic field in QED and its coupling to charged particles can be described similarly with the group $U(1)$, $U_{ij} = \exp(\mathrm{i} \int_j^i A_\mu \mathrm{d}x^\mu)$ Then

$$\begin{aligned} &\mathrm{tr}(U_{r,r+a^\mu e_\mu} U_{r+a^\mu e_\mu, r+a^\mu e_\mu + a^\nu e_\nu} U_{r+a^\mu e_\mu + a^\nu e_\nu, r+a^\nu e_\nu} U_{r+a^\nu e_\nu}) \\ &\approx \exp(\mathrm{i} a^\mu a^\nu F_{\mu\nu}(r + (a^\mu e_\mu + a^\nu e_\nu)/2)) \end{aligned} \tag{32}$$

with the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
Since only the real part of $\mathrm{tr}(\prod U)$ contributes, one obtains in leading order the well-known action of the electromagnetic field proportional to $F_{\mu\nu}F^{\mu\nu}$.
If one performs the continuum limit ($a \to 0$) then only these terms survive.
The discretized Maxwell equations can be solved on such a lattice (Weiland 1977). One places the components A_μ on sites $r^{(1)}$, the six electromagnetic field components $F_{\mu\nu}$ on sites $r^{(2)}$, the components of the charge and current densities on sites $r^{(1)}$. Lorenz gauge and charge conservation can be put on sites $r^{(0)}$.

Summary

- I reported my motivation and the main ideas of my 1971 paper.
- I started with the duality arguments by Kramers and Wannier: High- and low-temperature expansions of Z of $\mathcal{M}_{2,1}$ are expressed by the same functions of different arguments.
- Requiring the same functions in three dimensions leads to duality between $\mathcal{M}_{3,1}$ and the local gauge-invariant model $\mathcal{M}_{3,2}$.
- Generalization to duality of d-dimensional models $\mathcal{M}_{d,n}$ and $\mathcal{M}_{d,d-n}$. Of particular interest is the self-dual model $\mathcal{M}_{4,2}$.
- The correlation functions at high and low temperatures show different behavior (area and perimeter law) and suggest different phases. They are related by duality to dislocations.

Thank you very much.

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