Exercise 1: OPE and correlation functions

Consider the Operator Product Expansion (OPE) $\mathcal{O}_1(x_1)\mathcal{O}_2(0)$ which takes the schematic form

$$\mathcal{O}_1(x_1) \times \mathcal{O}_2(0) = \frac{A}{|x_1|^k} \left(\mathcal{O}_3(0) + \alpha\, x_1^\mu \partial_\mu \mathcal{O}_3(0) + \dots\right). \tag{1}$$

Fix the coefficients k and α.

Hint: there are a few ways to do it. I suggest you to consider the correlator $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(0)\mathcal{O}_3(z)\rangle$

Exercise 2: Symmetric traceless

Argue that the only operators that can appear in the OPE of identical scalar operators are symmetric traceless operators of even spin.
To see it, you can use the fact that

$$\langle \phi_1(x_1)\phi_2(x_2)\mathcal{O}^{\mu_1\dots\mu_\ell}(x_3)\rangle = \frac{f_{\phi\phi\mathcal{O}_\ell}\left(Z^{\mu_1}\dots Z^{\mu_\ell} - \text{traces}\right)}{x_{12}^{\Delta_1+\Delta_2-\Delta_3+\ell} x_{13}^{\Delta_1+\Delta_3-\Delta_2+\ell} x_{23}^{\Delta_2+\Delta_3-\Delta_1+\ell}} \tag{2}$$

where $Z^\mu = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{12}^\mu}{x_{12}^2}$.

Exercise 3: Casimir of the conformal group

Conformal blocks are eigenfunctions of the quadratic conformal Casimir operator $C = -\frac{1}{2}L_{ab}L^{ab}$, where L_{ab} are the generators of the conformal algebra, which is isomorphic to $SO(d+1,1)$. Due to the fact that the Casimir operator acts with the same eigenvalue on all states in an irreducible representation, it is possible to write its action as a differential operator $\mathcal{D}_2$

$$\begin{aligned}\mathcal{D}_2 = 2\left(z^2(1-z)\,\partial_z^2 - z^2\partial_z\right) + 2\left(\bar{z}^2(1-\bar{z})\,\partial_{\bar{z}}^2 - \bar{z}^2\partial_{\bar{z}}\right)\\ + 2\,(d-2)\,\frac{z\bar{z}}{z-\bar{z}}\left((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\right)\end{aligned} \tag{3}$$

and

$$\mathcal{D}_2 g_{\Delta,\ell}(z,\bar{z}) = \mathcal{C}_{\Delta,\ell}\, g_{\Delta,\ell}(z,\bar{z}). \tag{4}$$

By using the following boundary conditions for the blocks

$$g_{\Delta,\ell}(u,v) \sim u^{\Delta/2}\,(1-v)^\ell \tag{5}$$

check that the conformal blocks in $d=2$ and $d=4$ satisfy the above equation and find the form of $\mathcal{C}_{\Delta,\ell}$. Notice that these coefficients depend on d as well.

Exercise 4: Symmetries of four point functions

The four point function of scalars ϕ_i with dimension Δ_i is constrained to be

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} \frac{g(u,v)}{x_{12}^{\Delta_{\phi_1}+\Delta_{\phi_2}} x_{34}^{\Delta_{\phi_3}+\Delta_{\phi_4}}}. \tag{6}$$

- Consider the case of identical scalars $\phi_i = \phi$ with $\Delta_i = \Delta$ and write the two crossing relations.
- Consider the case of pairwise identical scalars $\phi_1 = \phi_2 = \phi$ and $\phi_3 = \phi_4 = \psi$ and write the two crossing relations, also in terms of conformal blocks decomposition.
- Consider the generic case and write down the crossing relations.

Exercise 5: Generalised free theories

The four point function of identical scalars ϕ with dimension Δ_ϕ in generalised free field theories is given by

$$g(u,v) = 1 + u^{\Delta_\phi} + \left(\frac{u}{v}\right)^{\Delta_\phi}. \tag{7}$$

Use the conformal block decomposition

$$g(u,v) = \sum_{\Delta,\ell} a_{\Delta,\ell}\, g_{\Delta,\ell}(u,v) \tag{8}$$

with the explicit expression for the conformal blocks in $d = 2$ or $d = 4$ to find the OPE data of low lying operators. Can you infer something about the generic structure of operators?