News on the anti-D3-brane uplift

KPV decay at higher order in α'

Simon Schreyer

Based on 2402.13311
and older work 2208.02826, 2212.07437 w/ Arthur Hebecker and Gerben Venken

Swamplandia May 27 2024

Motivation

- Key challenge: Accelerated expanding universes in String Theory?
- Simplest explanation: positive cc, e.g. Anti-D3-brane uplift

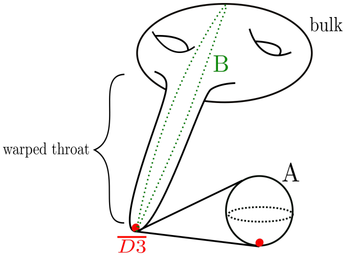

Figure: Illustration of Calabi-Yau geometry admitting a warped throat region.

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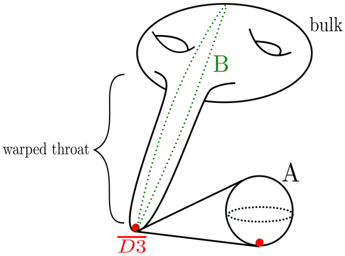
- Simplest explanation: positive cc, e.g. Anti-D3-brane uplift
- Put p $\overline{D3}$-branes at tip of KS throat
- KS throat: M units of F_3 (K units of H_3) flux on A-cycle (B-cycle), Tip topologically S^3 w/ $R_{S^3} \sim \sqrt{g_s M}\, l_s$



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Motivation (continued)

- Uplifting contribution: $V_{\overline{D3}} \sim \exp\left(-N/g_s M^2\right)/\mathcal{V}^{4/3}$, $N = KM$
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- Can quantum corrections be neglected in the KPV decay at small $g_s M^2$?
- No! Take as proxy curvature correction to $\overline{D3}$ action:

$$\alpha'^2 \int \mathrm{d}^4 x \mathcal{R}_{ab}\mathcal{R}^{ab} \sim \alpha'^2 \int \mathrm{d}^4 x 1/R_{S^3}^4 \,, \qquad R_{S^3} \sim \sqrt{g_s M}\, l_s$$

Goal: Study the KPV process including quantum corrections

Outline

KPV decay at leading order [KPV '01]

Classical instability: brane-flux annihilation

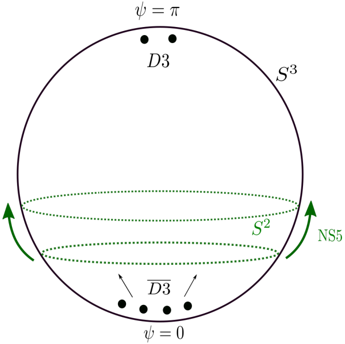

Figure: The tip of the throat.

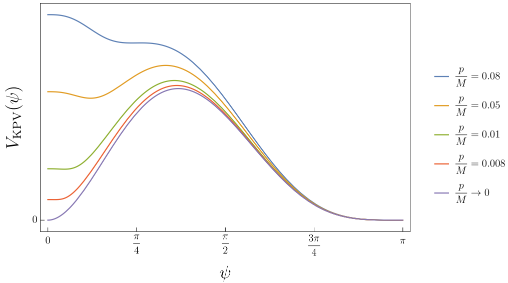

Figure: KPV potential.

$\Rightarrow$ Metastable minimum if $p/M < 0.08$, leading order in α' analysis!

Two perspectives to study the KPV decay

1. NS5-brane perspective
    - Single NS5 w/ p units of worldvolume flux
    - NS5-brane action controlled if $R_{S^2} = \sqrt{g_s M} \sin\psi \, l_s \approx \sqrt{g_s M} \frac{p}{M} \, l_s \gg l_s$

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2. Nonabelian stack of $\overline{D3}$-branes
   - Brane stack is nonabelian: Transverse displacement of branes described by noncommutative scalars Φ^i (live in adjoint rep of $U(p)$ gauge group of stack)
   - Nonabelian brane stack dynamics described by Myers action, controlled if distance of branes inside stack is substringy $\Rightarrow R_{S^2} \approx \sqrt{g_s M} \frac{p}{M} \, l_s \ll \sqrt{g_s p} \, l_s$

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$\Rightarrow$ at $\sqrt{g_s p} \lesssim \mathcal{O}(1)$: complementary regimes of validity, at $\sqrt{g_s p} \gtrsim \mathcal{O}(1)$: overlapping regimes of validity

The NS5-brane picture [Hebecker, SJS, Venken '22, SJS, Venken '22]

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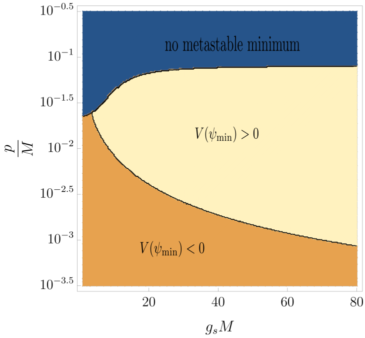
- Minimal bound on $g_s M^2$: $g_s M^2 > 144$ (compare to $g_s M^2 > 12$ from KPV!) $\Rightarrow$ need much more flux in throat for consistent uplift
- New way of uplifting w/o large warping by tuning tree-level against α' corrections?



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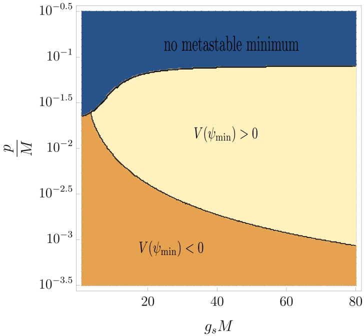
- Find for $V(\psi_{\text{min}}) = 0$: $R_{S^2} \sim \mathcal{O}(1)\, l_s$ $\Rightarrow$ very boundary of control



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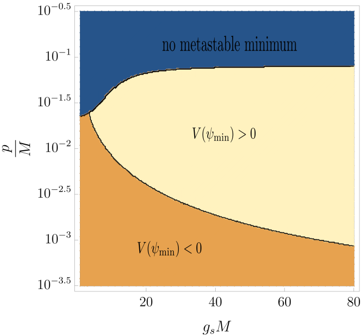
- New bound and new uplift work at boundary of control



$\Rightarrow$Advance into region not controlled from NS5 perspective by studying stack of $\overline{D3}$-branes (valid for $R_{S^2} \ll \sqrt{g_s p}\, l_s$)

The nonabelian brane stack picture

- Nonabelian $\overline{D3}$-brane stack dynamics in S-dual frame determined by [Myers '99]

$$S_{\mathrm{DBI}} = -\frac{T_3}{g_s}\int \mathrm{d}^4\sigma\, \mathrm{STr}\left(\sqrt{\det(Q^i{}_j)}\sqrt{-\det\left(P[E_{ab}+E_{ai}(Q^{-1}-\delta)^{ij}E_{jb}]+\lambda F_{ab}\right)}\right)$$

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- Nonabelian features: Nonabelian pullback P, interior product i_Φ, symmetric trace STr, Φ noncommutative scalars

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- Nonabelian features: Nonabelian pullback P, interior product i_Φ, symmetric trace STr, Φ noncommutative scalars
- Questions: Does stack expand? If yes, how large is S^2?
  $\Rightarrow$ Only need stationary points of potential for Φ

The leading order result [KPV '01]

- $V_{\mathrm{KPV}} = \frac{T_3}{g_s}\left(p + \frac{\lambda^2}{4g_s^2}\mathrm{tr}\left([\Phi^i,\Phi^j][\Phi^j,\Phi^i]\right) - \frac{i\lambda^2}{3}F_{ikl}\mathrm{tr}\left([\Phi^i,\Phi^k]\Phi^l\right)\right)$
- Stationary point: ansatz $[\Phi^i,\Phi^j] = A\,\varepsilon_{ijk}\Phi^k$ and $\Phi^i = -iA\alpha^i/2$, α^i: $(p\times p)$-dim irr. rep of SU(2) $\Rightarrow \Phi^i$ describe fuzzy sphere w/ $A = -ig_s^2 f$

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- Condition for minimum: $R_{S^2} \ll R_{S^3} \Leftrightarrow p/M \ll 0.138$ (compare to NS5: $p/M < 0.08$)

Nonabelian $\overline{D3}$-brane stack at higher order [SJS '24]

- Do same analysis including higher order corrections:
  - Higher commutator corrections ($(p/M)^2$ expansion), expand to next-to-leading order
  - Higher derivative corrections at order α'^2 ($1/(g_s M)^2$ expansion)

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- Goal: Calculate next-to-lo corrections to V_{KPV}, A, R_{S^2} based on

$$S = -\frac{T_3}{g_s}\int d^4\sigma\, \text{STr}\left(\sqrt{\det(G_{ab})\det(Q^i{}_j)}\,(1+\alpha'^2\mathcal{L}_{\alpha'^2})\right) + S_{\text{CS},\alpha'^2}$$
$$-\,T_3\int \text{STr}\left(P\left[i\lambda \mathrm{i}_\Phi \mathrm{i}_\Phi C_2\wedge C_4 - \frac{\lambda^2}{4}(\mathrm{i}_\Phi \mathrm{i}_\Phi)^2 C_2\wedge C_2\wedge C_4 + \dots\right]\right)$$

$\Rightarrow$ Derive next-to leading order potential $V_{\text{KPV,nlo}}$

Next-to-leading order solution

Solving perturbatively for stationary points of $V_{\text{KPV,nlo}}$:

- $A_{\text{nlo}} = -ig_s^2 f\gamma(1+\delta), \gamma = 1 + \frac{c}{(g_s M)^2} + \cdots,$
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Where do we find minima? Where do we trust solution?

Conditions for Control and Minima

1. Control over α' expansion: $(g_s M)^2 > c = 4.91$ (or $\gamma - 1 \ll 1$)
2. Control over commutator expansion: $\frac{R_{S^2}^2}{g_s p \lambda} \equiv f_p \ll 1$

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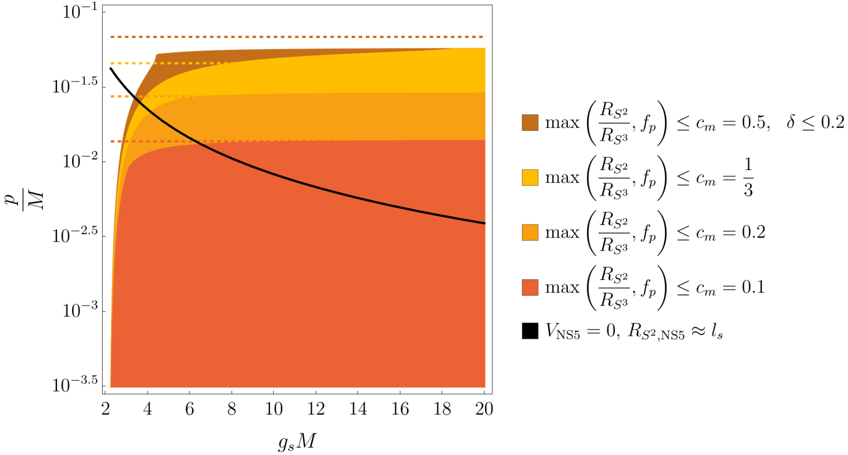
Where in $(g_s M, p/M)$ space how well satisfied?

Pheno Implications I



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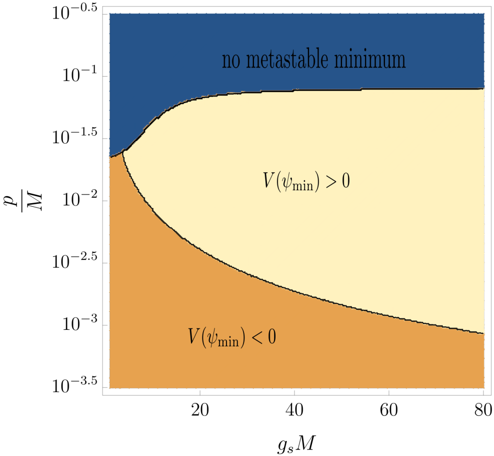

Pheno Implications II

- Can access new region in $(g_s M, p/M)$ space
- Compare $(g_s M^2)_{\text{min}}$ in Table w/ NS5 for $p = 1$: $(g_s M^2)_{\text{min,NS5}} = 144$

Table: Minimal values of $g_s M^2$.

c_m	$(g_s M^2)_{\text{min}}$	$(g_s M)_{\text{min}}$
0.1	$302p$	3.23
0.2	$165p$	3.559
$\frac{1}{3}$	$106p$	3.714
0.5	$83p$	4.369

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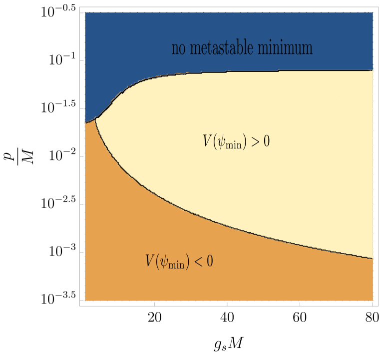
- $V_{\text{KPV,nlo}}(A = A_{\text{nlo}}) > 0 \;\forall g_s M > 2.25$
  $\Rightarrow$ No new way of uplifting for large $g_s M$



Limitations

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- Not all relevant α'^2 corrections on branes known, but crucial for value c in $\gamma = 1 + c/(g_s M)^2 + \cdots$
- Worked in flat space approx. ($R_{S^3} \gg R_{S^2}$) $\Rightarrow$ cannot see KPV decay explicitly + additional corrections
- Do not control large p/M from $\overline{D3}$ perspective $\Rightarrow$ need sum up all commutator corrections

Summary

- KPV setup can (and should!) be studied from 2 perspectives:
  1. NS5-brane: when $R_{S^2} \gg l_s (\Leftrightarrow$ large $g_s M$, $p/M)$, and to explicitly see decay
  2. $\overline{D3}$-brane stack: when $R_{S^2} \ll \sqrt{g_s p}\, l_s (\Leftrightarrow$ large $g_s M$, small $p/M)$, includes phenomenologically interesting regime of string size S^2 (smallish $g_s M^2$)

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- $(g_s M^2)_{\text{min}} \approx \mathcal{O}(100)$ from both perspectives (tree level: $g_s M^2 > 12$) $\Rightarrow$ requires much larger flux contribution to D3 tadpole than assumed from tree level
- New way of uplifting proposed in [Hebecker, SJS, Venken '22] can (if at all) work for small $g_s M$ where α' expansion breaks down

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- New way of uplifting proposed in [Hebecker, SJS, Venken '22] can (if at all) work for small $g_s M$ where α' expansion breaks down
- Future: Extend stack analysis to KPV background, sum up all order commutators to access regime of larger p/M

Thank you!
Any questions?

Higher order scalar potential

$$V_{\text{tot}} = V_{\text{DBI},\mathcal{O}(\lambda^4)}\left(1 - \frac{c}{(g_s M)^2}\right) + V_{\text{CS},\mathcal{O}(\lambda^4)} + \frac{T_3\, p}{g_s}\,\frac{c_2}{(g_s M)^2}\,,$$

$$V_{\text{CS},\mathcal{O}(\lambda^4)} = -\frac{T_3}{g_s}\left(\frac{i\lambda^2}{6}F_{ikl}\,\text{tr}\left([\Phi^i,\Phi^k]\Phi^l\right) + \frac{\lambda^4}{72}\text{STr}\left([\Phi^i,\Phi^j]\Phi^k[\Phi^l,\Phi^m]\Phi^n\right)F_{kji}F_{nml}\right),$$

$$V_{\text{DBI},\mathcal{O}(\lambda^4)} = \frac{T_3}{g_s}\Bigg(\text{tr}(\mathbb{1}) + \frac{\lambda^2}{4g_s^2}\text{tr}\left([\Phi^i,\Phi^j][\Phi^j,\Phi^i]\right) - \frac{i\lambda^2}{6}F_{ikl}\text{tr}\left([\Phi^i,\Phi^k]\Phi^l\right)$$

$$- \frac{\lambda^4}{72}\text{STr}\left([\Phi^i,\Phi^k]\Phi^l[\Phi^j,\Phi^m]\Phi^p\right)F_{lki}F_{pmj} + \frac{\lambda^4}{36}\text{STr}\left([\Phi^i,\Phi^k]\Phi^l[\Phi^j,\Phi^m]\Phi^p\right)F_{lkj}F_{pmi}$$

$$+ \frac{i\lambda^4}{24g_s^2}\text{STr}\left([\Phi^i,\Phi^k]\Phi^l[\Phi^j,\Phi^m][\Phi^m,\Phi^j]\right)F_{lki} - \frac{\lambda^4}{32g_s^4}\text{STr}\left(\left([\Phi^i,\Phi^j][\Phi^j,\Phi^i]\right)^2\right)$$

$$- \frac{i\lambda^4}{6g_s^2}\text{STr}\left([\Phi^i,\Phi^j][\Phi^j,\Phi^m][\Phi^m,\Phi^k]\Phi^l F_{lki}\right)\Bigg).$$