Can you hear the Planck mass?

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Based on

2405.XXXXX with De Ponti, Mondino, Tomasiello
2104.12773 with Tomasiello
2109.11560, 2212.02511, 2306.05456, with De Ponti, Mondino, Tomasiello

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Spin 2 Kaluza-Klein modes

- A single higher-dimensional field gives rise to infinite towers of 4-dimensional fields: Kaluza-Klein modes

$$ds_D^2 = e^{2A(y)}(g_4^{\Lambda}(x) + g_n(y))$$

$$\varphi(x, y) = \varphi^0(x) + \sum_i \delta\phi(x)\xi_i(y)$$

$$g_{4\ \mu\nu}(x) = g_{\mu\nu}^{(\Lambda)}(x) + \sum_i h_{\mu\nu}^i(x)\psi_i(y)$$

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- The masses of these fields, are given by eigenvalues of certain differential operators in the internal space. For example

[Duff, Nilsson, Pope, '85]

Table 5
Mass operators from the Freund–Rubin ansatz

Spin	Mass operator
2^+	Δ_0
$(3/2)^{(1),(2)}$	$\not{D}_{1/2} + 7m/2$
$1^{-(1),(2)}$	$\Delta_1 + 12m^2 \pm 6m(\Delta_1 + 4m^2)^{1/2}$
1^+	Δ_2
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- But when $A(y)$ is non-zero, the general form is not known, except for the universal spin 2 sector

$$f \equiv (D-2)A \qquad \Delta_f \psi_i \equiv \Delta\psi_i - \nabla f \cdot \nabla \psi_i = m_i^2 \psi_i$$

[Csaki, Erich, Hollowood, Shirman, '00, Bachas,Estes, '11]

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- In particular, $f \neq 0$ for compactifications with sources, where it is generically singular.

- We want to find general results on the masses m_i^2 , for *any* compactification solving its equations of motion:

$$\boxed{\frac{1}{D-2} e^{-f}\Delta(e^f) = \frac{1}{d}\hat{T}^{(d)} - \Lambda} \qquad \boxed{R_{mn} - \nabla_m \nabla_n f + \frac{1}{n-(2-d)} \nabla_m f \nabla_n f = \Lambda g_{mn} + \tilde{T}_{mn}}$$

Curvature bounds and the spectrum

$$R_{mn} - \nabla_m \nabla_n f + \frac{1}{D-2} \nabla_m f \nabla_n f = \Lambda g_{mn} + \tilde{T}_{mn}$$

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- Synthetic Ricci curvature in effective dimension $N = 2 - d$
- Studied in the Optimal Transport literature, controls the spectrum of Δ_f

[Sturm '06, Lott, Villani '07, Villani '09, Ambrosio, Gigli, Savaré '14 , …]

  - Allows to derive rigorous bounds on $m_{KK}^2 / |\Lambda|$ in general
  - Stronger bounds if only classical sources are allowed. But not enough to prove absence of separation of scales even with just classical sources

[GBDL, Tomasiello, '21, GBDL, De Ponti, Mondino, Tomasiello, '21, '22, '23]

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- Are there universal results, independent of the curvature bounds?
  - Yes: Bounds in terms of the Cheeger constants [GBDL, De Ponti, Mondino, Tomasiello, '21]

$$m_1^2 \geqslant \frac{1}{4} h_1^2 \qquad m_k^2 \geqslant C k^{-6} h_k^2$$

  - Allow to rigorously bound and estimate the sprectrum of the warped Laplacian even in presence of orientifold planes
    - Can be used to check sep. of scale in explicit proposed examples (e.g. DGKT) without needing to compute the spectrum

[**D**eWolfe, **G**iryavets, **K**achru, **T**aylor, '05 Acharya, Benini, Valandro '06, Junghans '20, Marchesano, Palti, Quirant, Tomasiello '20]

$$h_1^2 \sim N^{-1/2} \, , |\Lambda| \sim N^{-3/2}$$

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$$h_1^2 \sim N^{-1/2}, |\Lambda| \sim N^{-3/2}$$

- This talk: another class of universal properties

Weyl law

- The eigenvalues of the standard Laplacian on smooth, compact, n-dimensional Riemannian manifolds satisfy

$$\Delta_0 \psi_i = \lambda_i \psi_i \qquad \lambda_k \sim 4\pi\Gamma(1+n/2)^{2/n}\mathrm{Vol}^{-2/n}k^{2/n} \qquad k \gg 1$$

[Weyl, 1912
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- For a compactification without warping ($f = 0$): $m_k^2 = \lambda_k \qquad m_{Pl,4}^2 = m_{Pl,D}^{D-2}\text{Vol}$

  - Knowing the asymptotics of the spectrum allows to reconstruct the $4d$ Planck mass

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- With warping: $\Delta_f \psi_k = (\Delta - \nabla f \cdot \nabla)\psi_k = m_k^2 \psi_k$ $\qquad m_{Pl,4}^2 = m_{Pl,D}^{D-2} \text{Vol}_f$ $\qquad \text{Vol}_f \equiv \int_{M_n} \sqrt{g_n} e^f$

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  - But this is false: for weighted Laplacians the Weyl law still depends on the ordinary volume

- Next: we will obtain the Weyl law for the warped case by using consistency of the gravitational theory, end extend it to spaces with singularities

Weyl law from the gravitational potential

- Main idea: The $4d$ theory at distances ρ much smaller than the compactification scale has to reproduce the higher-D behavior. For this to happen, the very massive KK modes have to behave in an appropriate way.

- We can check this explicitly in the universal spin 2 sector
  - Take a general higher dimensional theory of gravity
  - And compute the gravitational potential U between two mass sources with masses M_1 and M_2

$$S = m_D^{D-2} \int \sqrt{-g_D} R_D + \dots$$

$$+$$

$$S_{M_{1,2}} = -M_{1,2} \int_\Sigma \sqrt{-g_D|_\Sigma}$$

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- We compare two situations:

(1) D-dimensional space-time is Mink_D

(2) D-dimensional space-time is a warped product $\mathscr{M}_4 \times X$

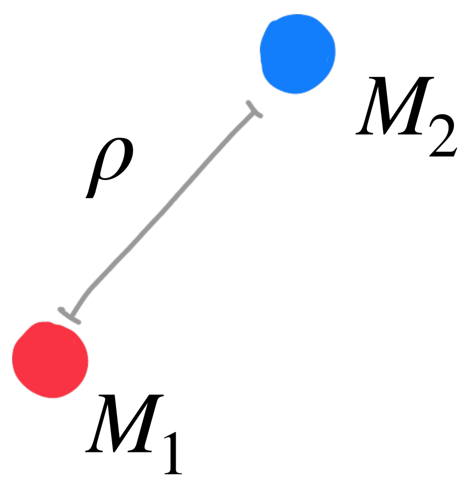
$$ds_D^2 = e^{2A(y)}(\bar{\eta}_{\mu\nu} dx^\mu dx^\nu + g_n(y))$$

- And require $U_{4d}(\rho) \to U_D(\rho) \qquad \rho \to 0$

[Readily generalizes to higher dimensions or $\Lambda \neq 0$]

Constraint on spectral data from gravity

1 Mink_D



$$\Delta_{D-1} h_{00} = \frac{M_1}{2} \delta_{D-1} \qquad h_{00} = -\frac{M_1}{2(D-3)\omega_{(D-2)}} \frac{1}{\rho^{D-3}}$$

$$U_D(\rho) = M_2 \left(1 - m_{Pl,D}^{2-D} h_{00}(\rho) \right)$$

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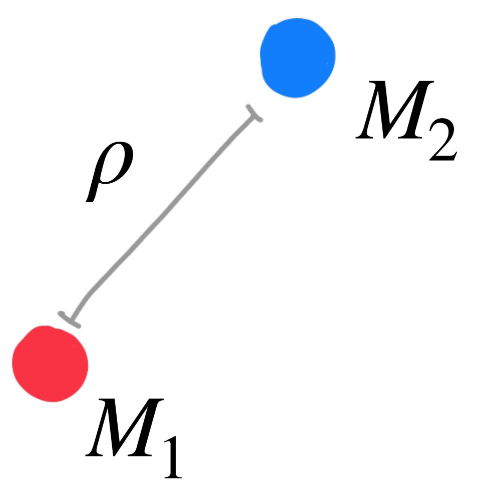

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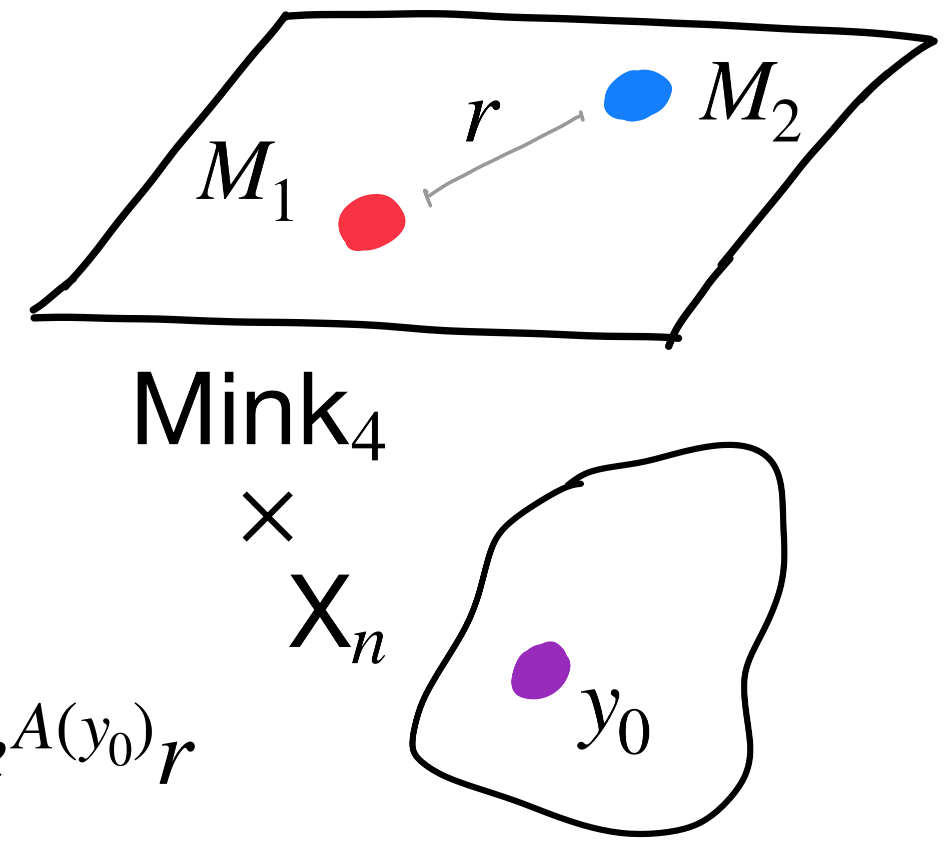

$\rho = e^{A(y_0)} r$

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$$U_{4d}(r) = M_2\left(1 - \frac{m_{Pl,4}^{-2}}{2}\sum_k e^{2A(y_0)}\bar{h}_{00}^k(r, y_0)\psi_k(y_0)\right)$$

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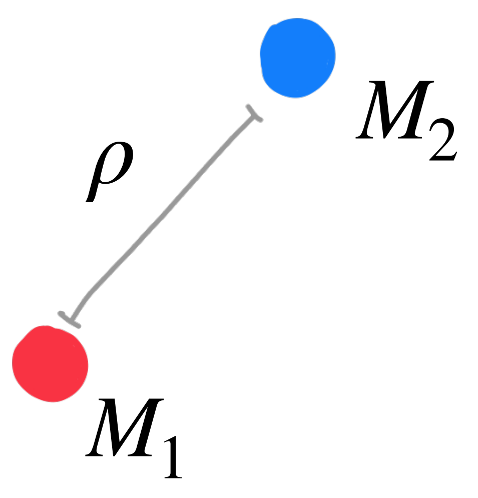

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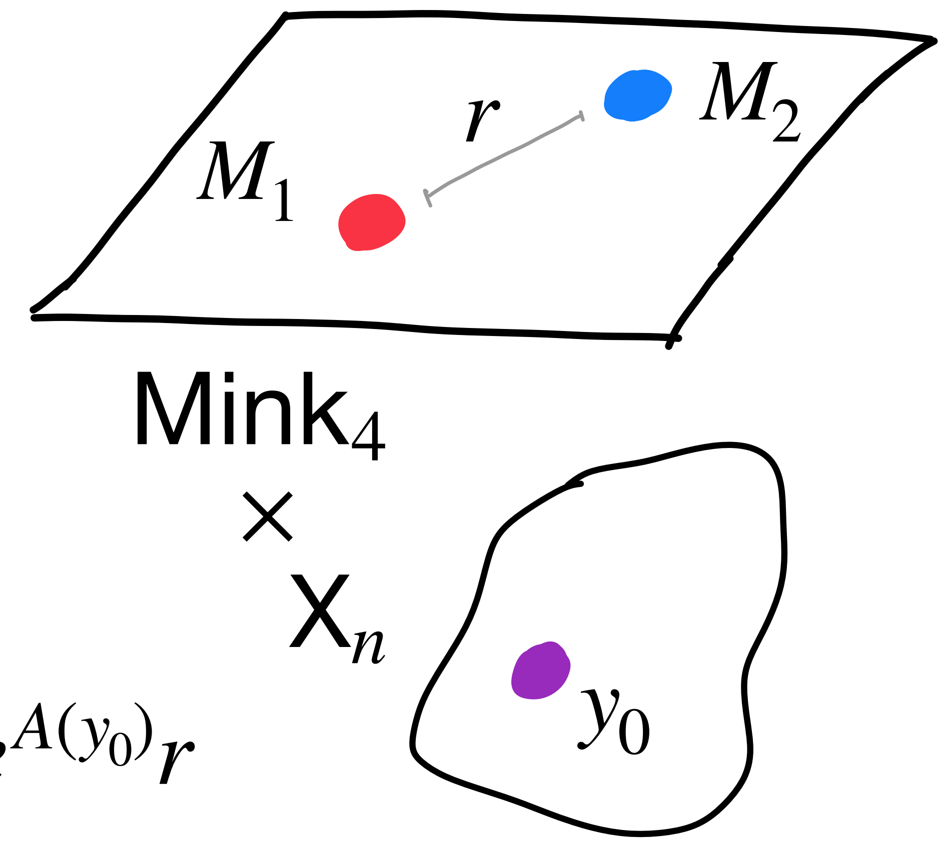

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$U_{4d}(\rho) \to U_D(\rho)$

for $\rho \to 0$

$$e^{f(y_0)} \lim_{r\to 0} r^n \sum_{k=0}^{\infty} \psi_k^2(y_0) e^{-m_k r} = \frac{m_{Pl,4}^2}{m_{Pl,D}^{D-2}} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$$

- We can also obtain a mathematically rigorous proof of this formula by directly comparing the local behavior of Green's functions

Proof of the Weyl law: integrated argument

$$e^{f(y_0)} \lim_{r\to 0} r^n \sum_{k=0}^{\infty} \psi_k^2(y_0) e^{-m_k r} = \frac{m_{Pl,4}^2}{m_{Pl,D}^{D-2}} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$$

$$\xrightarrow{\int_X \sqrt{g_n}}$$

$$\lim_{r\to 0} r^n \sum_{k=0}^{\infty} e^{-m_k r} = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \text{Vol}$$

- The effect of the warping completely disappeared using $\int_X \sqrt{g} e^f \psi_k \psi_l = \text{Vol}_f \delta_{kl}$ $\quad m_{Pl,4}^2 = m_{Pl,D}^{D-2} \text{Vol}_f$
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- To proceed, we make the ansatz that at large k, $m_k^2 \sim \alpha^2 k^{2/\nu}$, and we determine α and ν from
    - For $r \ll 1$ the sum can be well approximated by an integral

$$\varepsilon = (\alpha r)^\nu$$

$$\varepsilon \sum_{k=0}^{\infty} f(k\varepsilon) \sim \int_0^\infty dp f(p)$$

$$\longrightarrow \sum_k e^{-m_k r} \underset{r\to 0}{\sim} \sum_k e^{-\alpha k^{1/\nu} r} \sim \frac{1}{(\alpha r)^\nu} \int_0^\infty dp\, e^{-p^{1/\nu}} = \frac{\Gamma(\nu+1)}{(\alpha r)^\nu} \longrightarrow \begin{cases} \nu = n \\ \alpha^2 = 4\pi\Gamma(1+n/2)^{2/n} \mathrm{Vol}^{-2/n} \end{cases}$$

- More rigorously, the result follows from Karamata theorem

[Karamata '30]

Proof of the Weyl law: integrated argument

$$e^{f(y_0)} \lim_{r\to 0} r^n \sum_{k=0}^{\infty} \psi_k^2(y_0) e^{-m_k r} = \frac{m_{Pl,4}^2}{m_{Pl,D}^{D-2}} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \quad \xrightarrow{\int_X \sqrt{g_n}} \quad \lim_{r\to 0} r^n \sum_{k=0}^{\infty} e^{-m_k r} = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \text{Vol}$$

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- More rigorously, the result follows from Karamata theorem [Karamata '30]
- However, the exchange of limits is not delicate to justify in general, in particular in presence of singularities
- Using the RCD theory we can rigorously prove the Weyl law for solutions with D-brane singularities for which the spectrum is discrete (D6, D7, D8)

Quantum ergodicity

$$e^{f(y_0)}\lim_{r\to 0} r^n \sum_{k=0}^{\infty} \psi_k^2(y_0) e^{-m_k r} = \frac{m_{Pl,4}^2}{m_{Pl,D}^{D-2}} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$$

- A finite number of modes does not contribute to the lhs, it's the infinite tail at large k that gives a non-zero contribution to the limit. We need to estimate $\psi_k(y_0)^2$ for $k \gg 1$.

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- A finite number of modes does not contribute to the lhs, it's the infinite tail at large k that gives a non-zero contribution to the limit. We need to estimate $\psi_k(y_0)^2$ for $k \gg 1$.

- Few results are known for eigenfunctions. But for $f = 0$, if the space is ergodic it is also Quantum Ergodic

- Ergodic:
  - Probability of finding a classical particle in a region of phase space proportional to the volume of that region

- Quantum ergodic:
  - Probability of finding a quantum particle in a region B proportional to the volume of that region

[Schinrelman, '74; de Verdière, '85; Zelditch, '87]

$$\forall f \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\gamma(t), \dot{\gamma}(t)) dt = \int_{S^\star X} f\omega \quad \Longrightarrow \quad \lim_{\substack{k \to \infty \\ k \notin e}} \frac{\int_B \sqrt{g}\psi_k^2}{\int_X \sqrt{g}\psi_k^2} = \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(X)}$$

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- In such spaces, $\psi_k^2(y_0)$ for large k oscillates around a constant →

$$\lim_{r\to 0} r^n \sum_{k=0}^{\infty} e^{-m_k r} = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \text{Vol}$$ ✓

  - 'Most' spaces are ergodic, but not all, e.g. when there are symmetries

- Is there a similar notion for the eigenfunctions of the warped Laplacian $(f \neq 0)$?

Weighted quantum ergodicity?

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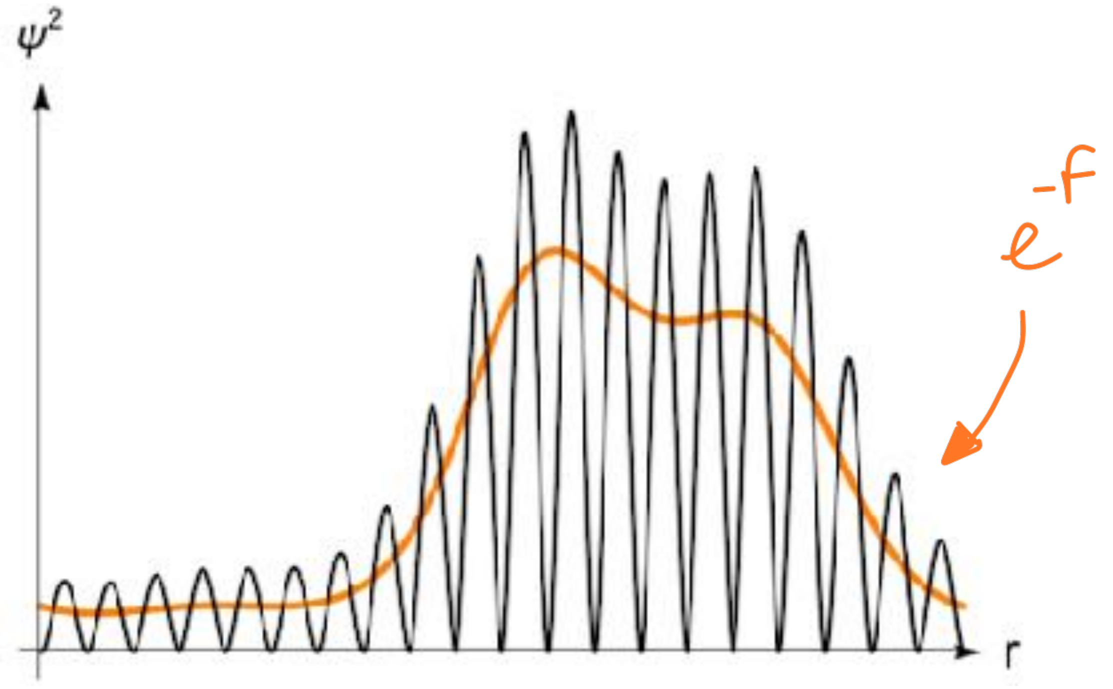
- Weighted Quantum Ergodicity

- An explicit 1-dimensional check, $X = S^1$, $f = \sin(x) + \cos^3(x)$



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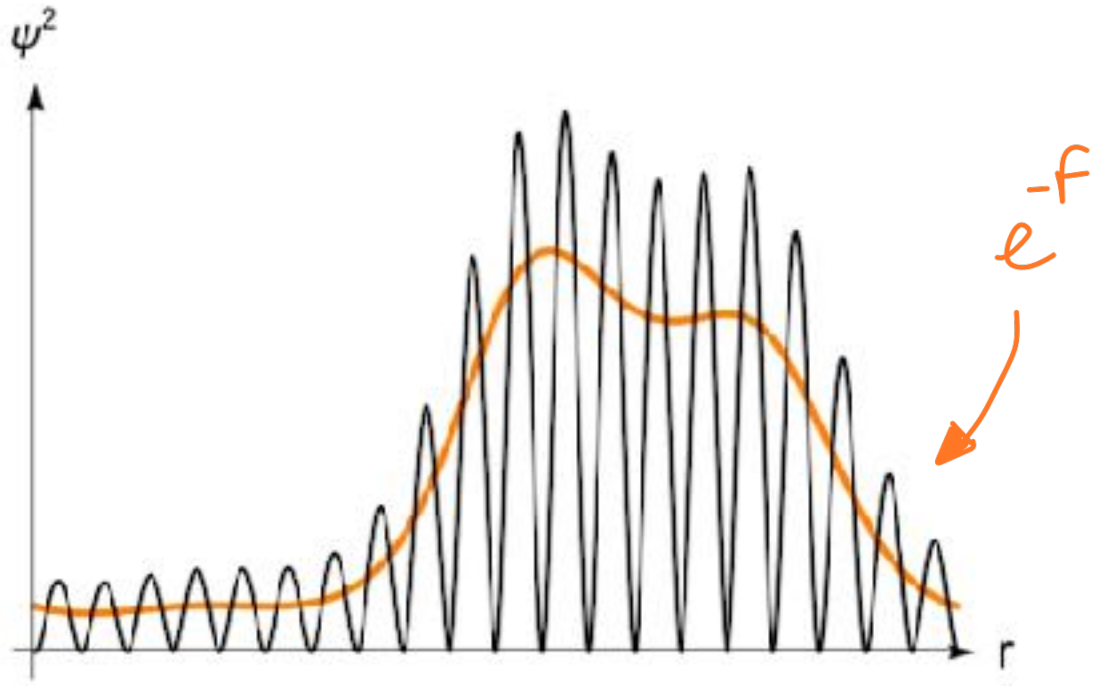

- Integrating over a B both sides and using $m_{Pl,4}^2 = m_{Pl,D}^{D-2}\text{Vol}_f$ gives again

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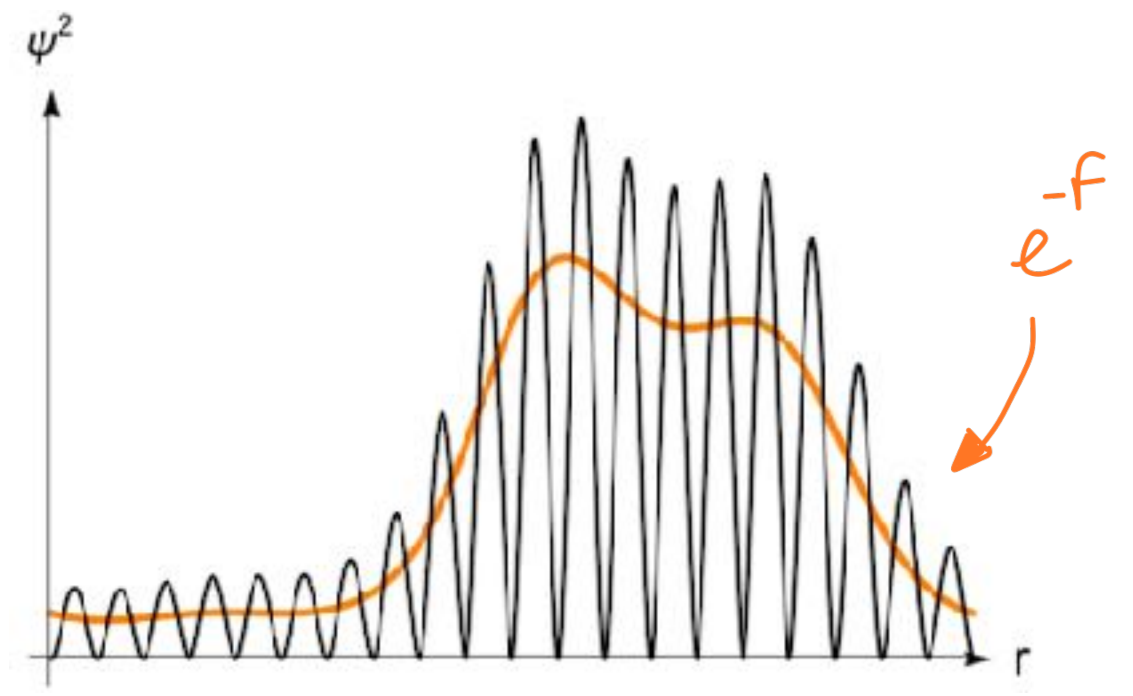

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- This is now a conjecture: it is not known to follow from a classical counterpart
    - If f is smooth it can be argued for it starting by mapping to a Schrödinger problem and using QE
    - But for physical sources f is not smooth

Recap and future directions

- The spin 2 Kaluza-Klein spectrum can be studied in general, without specifying the background
  - By controlling the *effective* curvature is possible to prove universal bounds on masses of spin 2 KK fields by using theorems in Optimal Transport Theory
    - Can we prove stronger bounds on $m_{KK}^2/|\Lambda|$ that exclude, or prove, separation of scales in certain regimes and for certain sources?

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- Can we apply similar techniques to other spins?
  - What are the corresponding operators? What controls their spectrum?

Thank You!

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$$\text{Ric} \geqslant (n-1)K^2 \implies$$

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- Classical theorems of this kind are not directly applicable to KK

  1. In general $f \neq 0$, so theorems on standard Laplacian are not directly useful
  2. The equations of motion do not constrain the Ricci tensor enough

Bounding the curvature

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Ric_f^N: Effective Ricci curvature tensor, in $N = 2 - d < 0$ effective dimensions

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Raychaudhuri: ξ tangent to geodesics

$$\theta \equiv \nabla_m \xi^m \quad \text{(expansion)}$$
$$-\nabla_\xi \theta \geqslant R_{mn} \xi^m \xi^n + \frac{1}{n}\theta^2$$

$f \neq 0$

$$\theta_f \equiv e^{-f} \nabla_m (e^f \xi^m) \quad \text{(weighted expansion)}$$
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- This notion of curvature is studied in Bakry-Emery geometry, and Optimal Transport theory
  - It also admits a rigorous definition for non-smooth metric spaces, in terms of concavity of entropy, allowing to rigorously write the supergravity equations for singular spaces. 'Synthetic curvature'

[GBDL, De Ponti, Mondino, Tomasiello, 2212.02511 & WIP]

The Reduced Energy Condition

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$$\tilde{T}_{mn} \equiv m_D^{2-D}\left(T_{mn} - \frac{T^{(d)}}{d} g_{mn}\right)$$

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For fluxes, scalar fields, scalar potentials, D-dim cosmological constants and localized sources with positive tension

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$$\text{Ric}_{mn}^{2-d,f} \geqslant -|\Lambda| \longrightarrow N < 0:$$

Useful to obtain stronger physical results on the spectrum of Δ_f, but mathematical techniques less developed. Very recent mathematical progress [e.g.: Ohta, Takatsu '10, '11, Ohta '13, Milman, '14, Woolgar, Wylie '17]

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$$\Downarrow$$

[GBDL, Tomasiello '21]

$$\text{Ric}^{\infty,f}_{mn} \geqslant -\left(|\Lambda| + \frac{\sigma^2}{D-2}\right)$$

→ $N = \infty$: Weaker physical consequences: it requires to control ∇f independently. Mathematical theory more mature, more results availiable

$$\sigma \equiv |\sup \nabla f|$$

Some eigenvalue bounds for $N < 0$

$\mathrm{Ric}^{2-d,f}_{mn} \geqslant -|\Lambda|$

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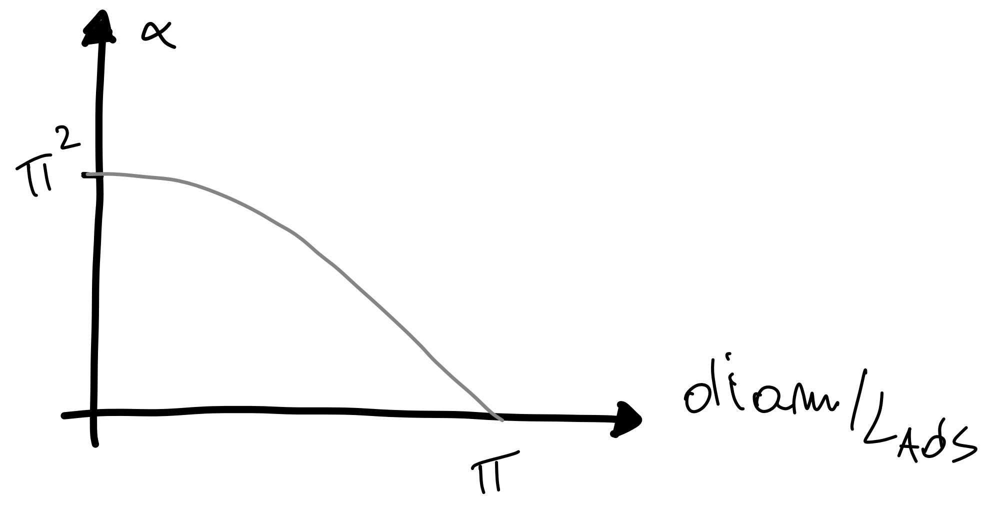

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Intuitive, but now rigorous even with D-brane singularities and warping

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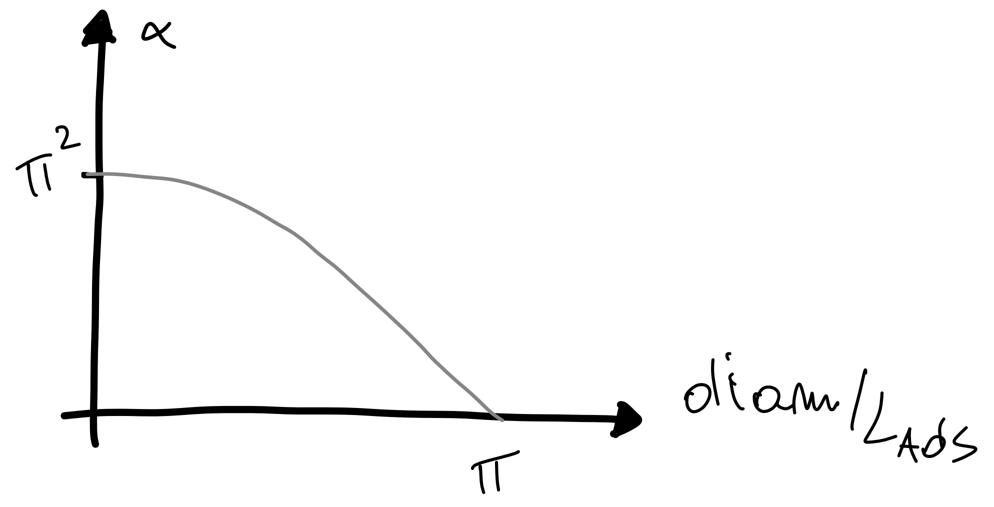

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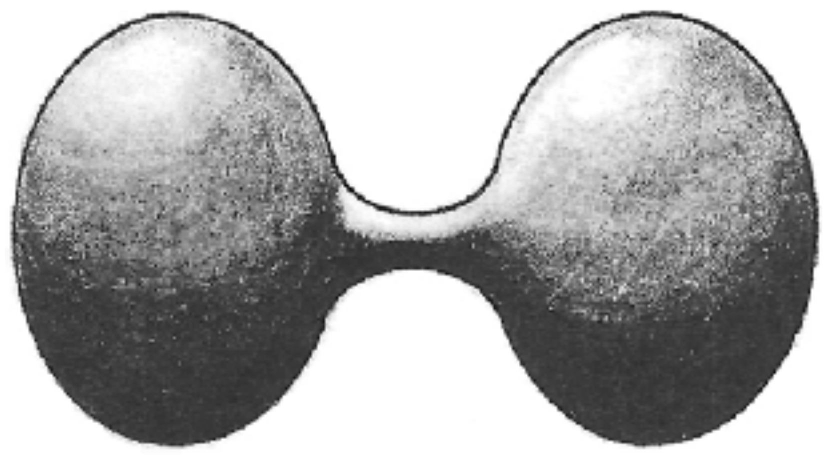
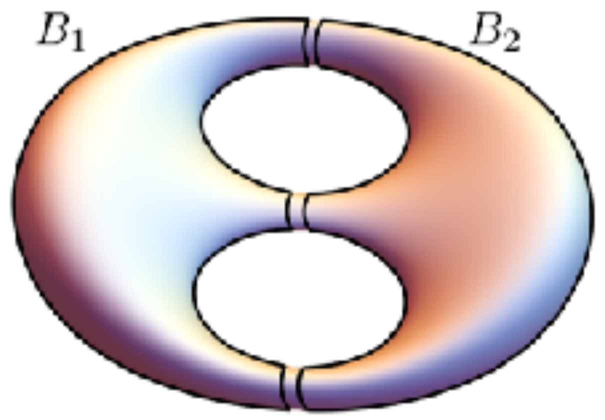
Define a generalization of Cheeger constants: [Generalization of Cheeger '69, Buser '82]

$$h_k \equiv \inf_{B_0,\ldots,B_k} \max_{0\leqslant i\leqslant k} \frac{\text{Per}_f(B_i)}{\text{Vol}_f(B_i)}$$

Example small h_1:

$(f = 0)$





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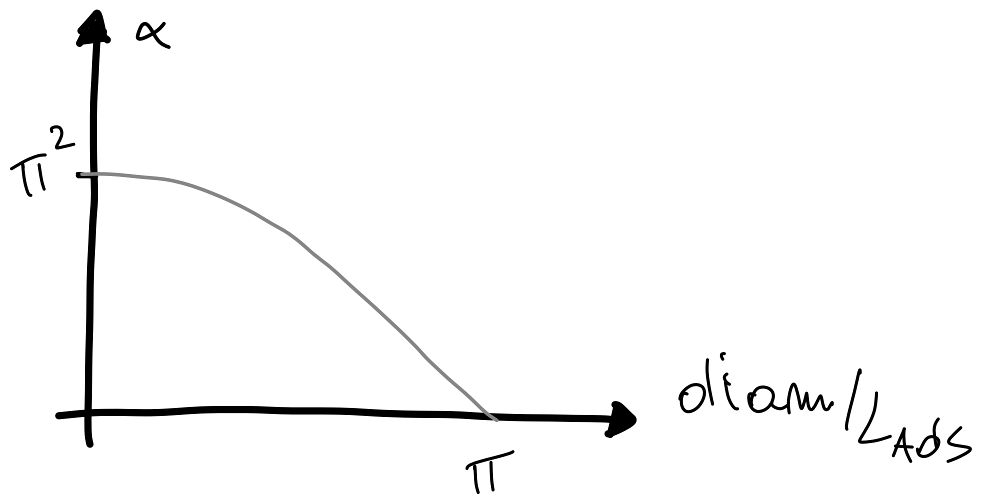

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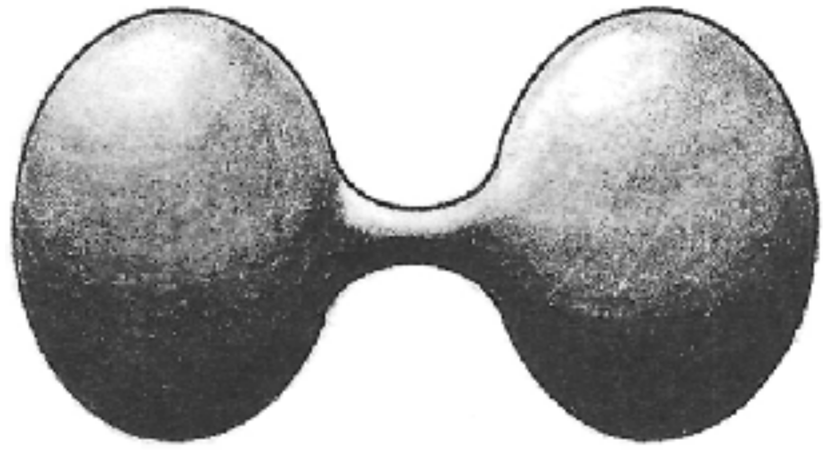

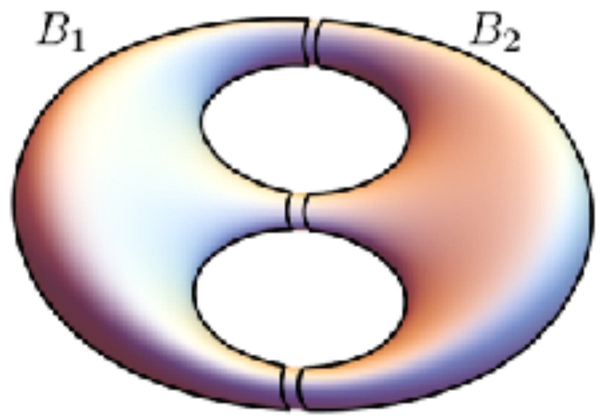

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Th. 3: $$m_k^2 \geqslant Ck^{-6}h_k^2$$

If h_k is not small, m_k cannot be small!

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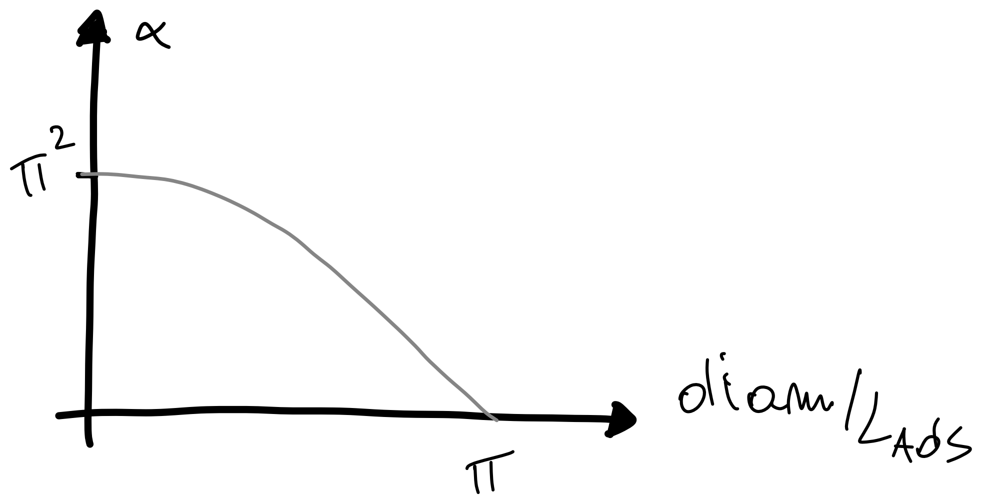

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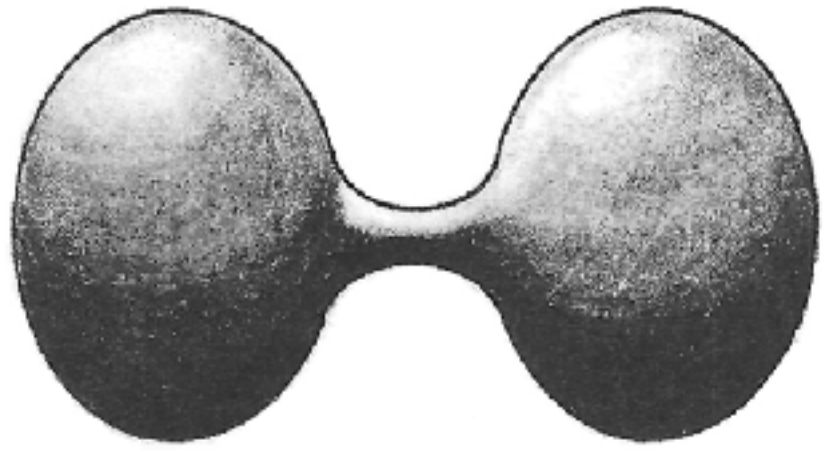

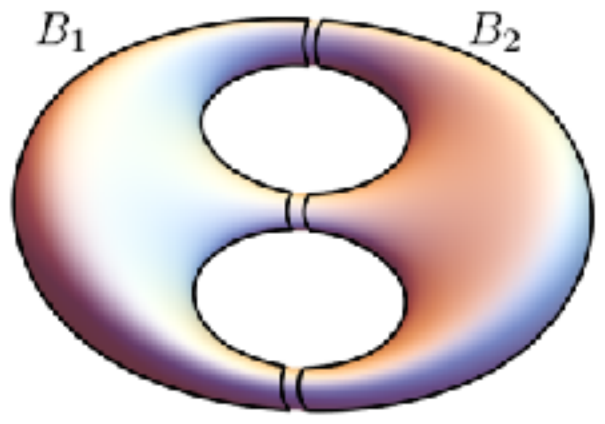

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$$m_k^2 \geqslant Ck^{-6}h_k^2$$

$\Longrightarrow$

Rigorous even in presence of O-planes

Can be used to check sep. of scale in explicit proposed examples. e.g. in

[**D**eWolfe, **G**iryavets, **K**achru, **T**aylor, '05
Acharya, Benini, Valandro '06,
Junghans '20, Marchesano, Palti, Quirant, Tomasiello '20]

$$h_1^2 \sim N^{-1/2}\,, |\Lambda| \sim N^{-3/2}$$

If $h_k$is not small, m_k cannot be small!

Some eigenvalue bounds for $N = \infty$

$$\mathrm{Ric}^{\infty,f}_{mn} \geqslant -\left(|\Lambda| + \frac{\sigma^2}{D-2} \right)$$

$$\sigma \equiv |\sup \nabla f|$$

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Th. 4:

$$m_k^2 \leqslant n\left(|\Lambda| + \frac{D-1}{D-2}\sigma^2\right) + \gamma(n)\frac{k^2}{\text{diam}^2}$$

Th. 5:

$$m_1^2 \geqslant \frac{\pi^2}{\text{diam}^2}\exp\left(-c(n)\text{diam}\sqrt{|\Lambda| + \frac{\sigma^2}{D-2}}\right)$$

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- Valid also in presence of D-brane singularities:

[GBDL, De Ponti, Mondino, Tomasiello, '21, using De Ponti, Mondino '19]

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$$m_1^2 \leqslant \max\left\{\frac{21}{10}h_1\sqrt{\Lambda + \frac{\sigma^2}{D-2}}, \frac{22}{5}h_1^2\right\}$$

Th. 7:

$$m_k^2 < k^2 \max\left\{\frac{14112}{25}\left(\Lambda + \frac{\sigma^2}{D-2}\right), \frac{2816}{5}m_1^2\right\}$$

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- In tension with the spin 2 swampland conjecture in the limit $h_1 \to 0$, h_2, σ fixed

[Klawer, Lust, Palti '18]

[Bachas '19]

$R_n = 0 + \text{Casimir} \to \Lambda_4 < 0$

[GBDL, De Ponti, Mondino, Tomasiello, '22]

- With a compact internal space, Casimir energy density can be automatically generated
- If the space has small circles, with antiperiodic BCs for fermions, Casimir energies are of the form

[Arkani-Hamed, Dubovsky, Nicolis, Villadoro '07]

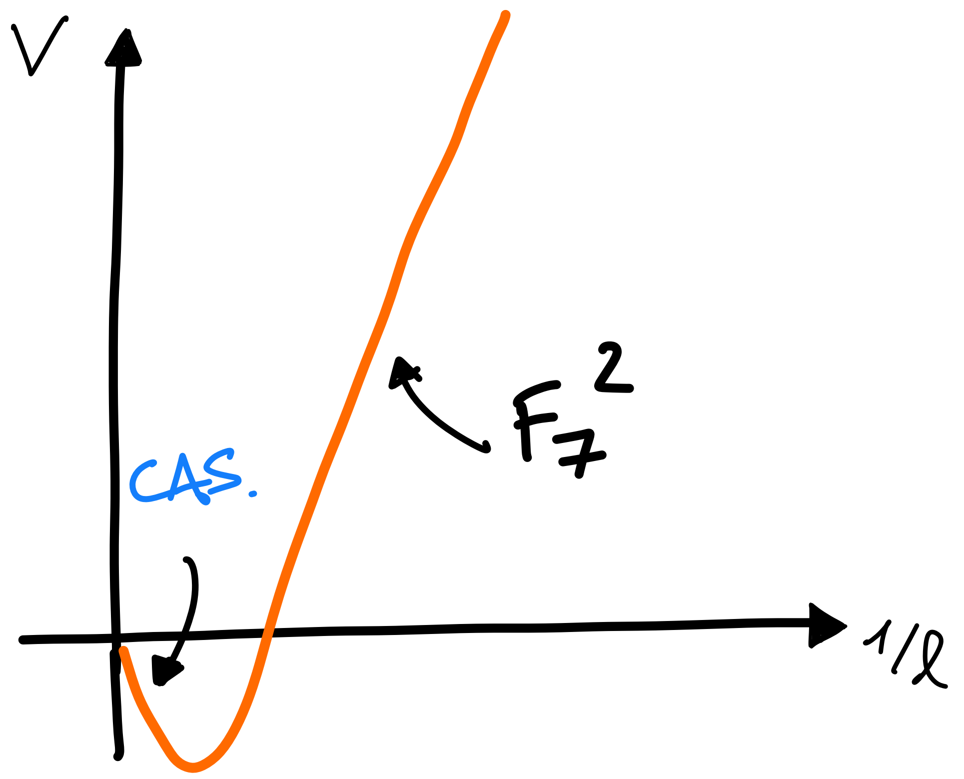
[cf. Maldacena, Milekhin, Popov '18]

$$T_{ij} \sim R_c(y)^{-D} g_{ij} \qquad T_{ab} \sim -\frac{D-k}{k} R_c(y)^{-D} g_{ab}$$

other directions

circle directions

small circle size



- Then solve the semi-classical equations:

$$-\frac{2}{\sqrt{-g_D}} \frac{S_D^{\text{(class.)}}}{\delta g_{MN}^D} = \langle T_{MN}^{\text{(Cas.)}} \rangle$$

- Explicitly in M-theory on $\text{AdS}_4 \times T^7$:

$$ds_{11}^2 = L_4^2 ds_{AdS_4}^2 + R_c^2 ds_{T^7}^2$$

$$T_{\mu\nu}^{Cas} = |\rho_c| \ell_{11}^9 R_c^{-11} g_{\mu\nu} \qquad T_{ij}^{Cas} = -\frac{4}{7} |\rho_c| \ell_{11}^9 R_c^{-11} g_{ij}$$

$$F_7 = f_7 vol_{T^7} \qquad \frac{1}{\ell_{11}^6} \int F_7 = N_7$$

$$\Longrightarrow \frac{L_4^2}{R_c^2} = \frac{2401}{4608} \frac{N_7^6}{\rho_c^4} \gg 1$$

$$\frac{R_c^{11}}{\ell_{11}^{11}} \sim N_7^{22/3} \gg 1$$

QG effects under control

parametric separation of scales!

- Non-susy and unstable for M2 bubble nucleation
- Compatible with AdS distance conjecture, $m_{KK}^2 \sim |\Lambda|^{1/d}$

[Lust, Palti, Vafa, '19
Gonzalo, Ibáñez, Valenzuela , '21]

- [Also non stable dS possible in this way but not under parametric control]