Building Moduli Spaces from Monodromies

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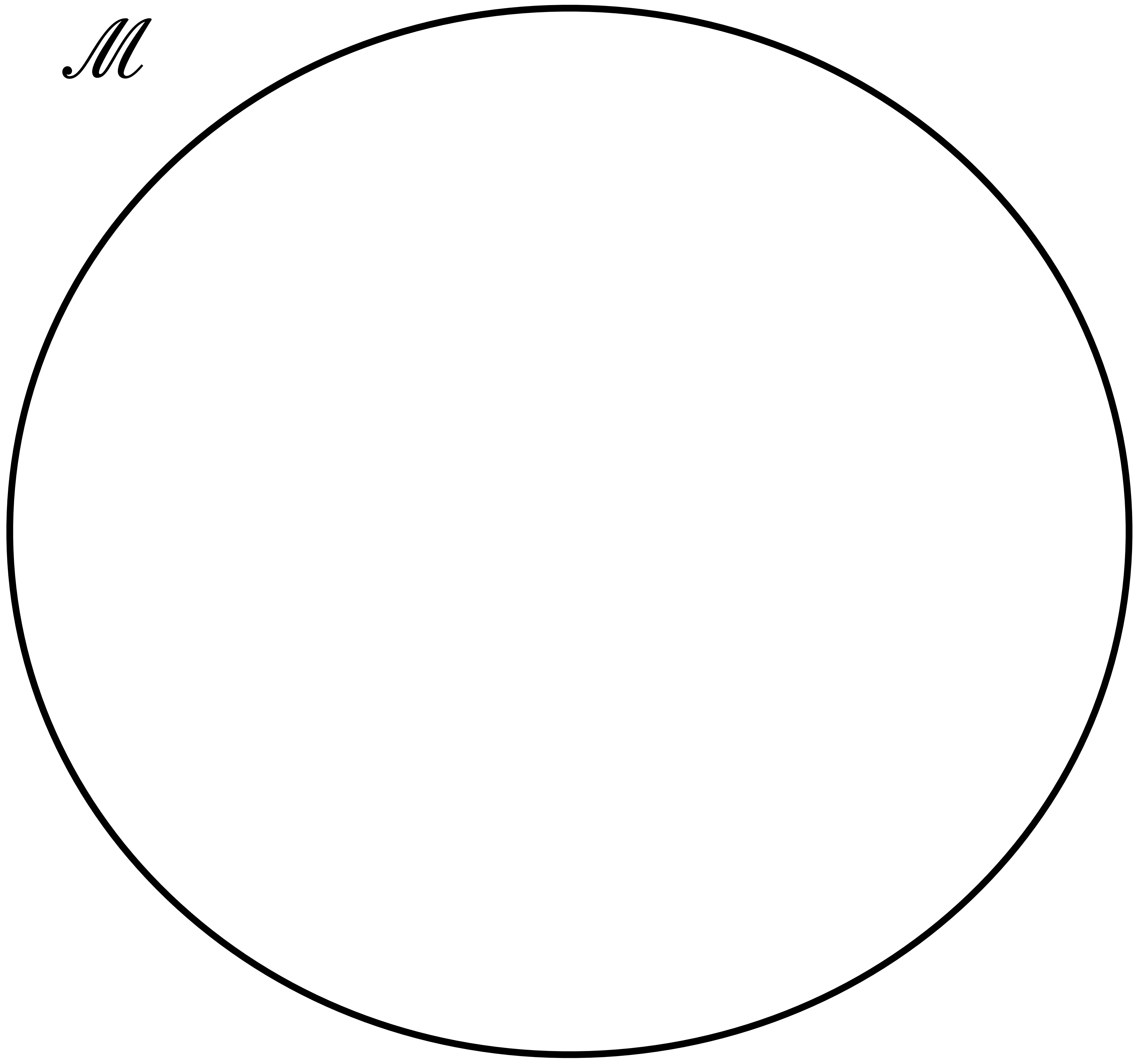
Motivation



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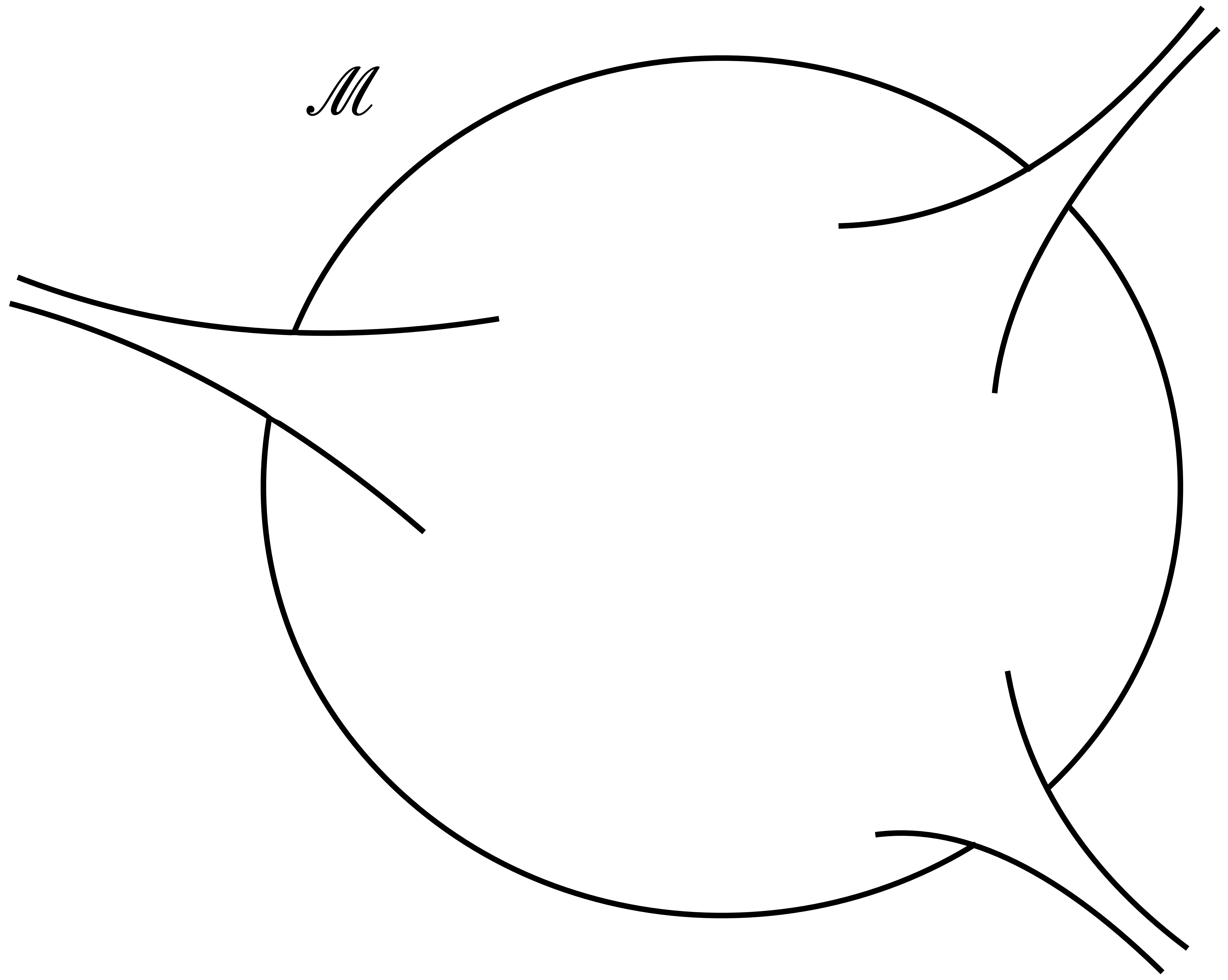

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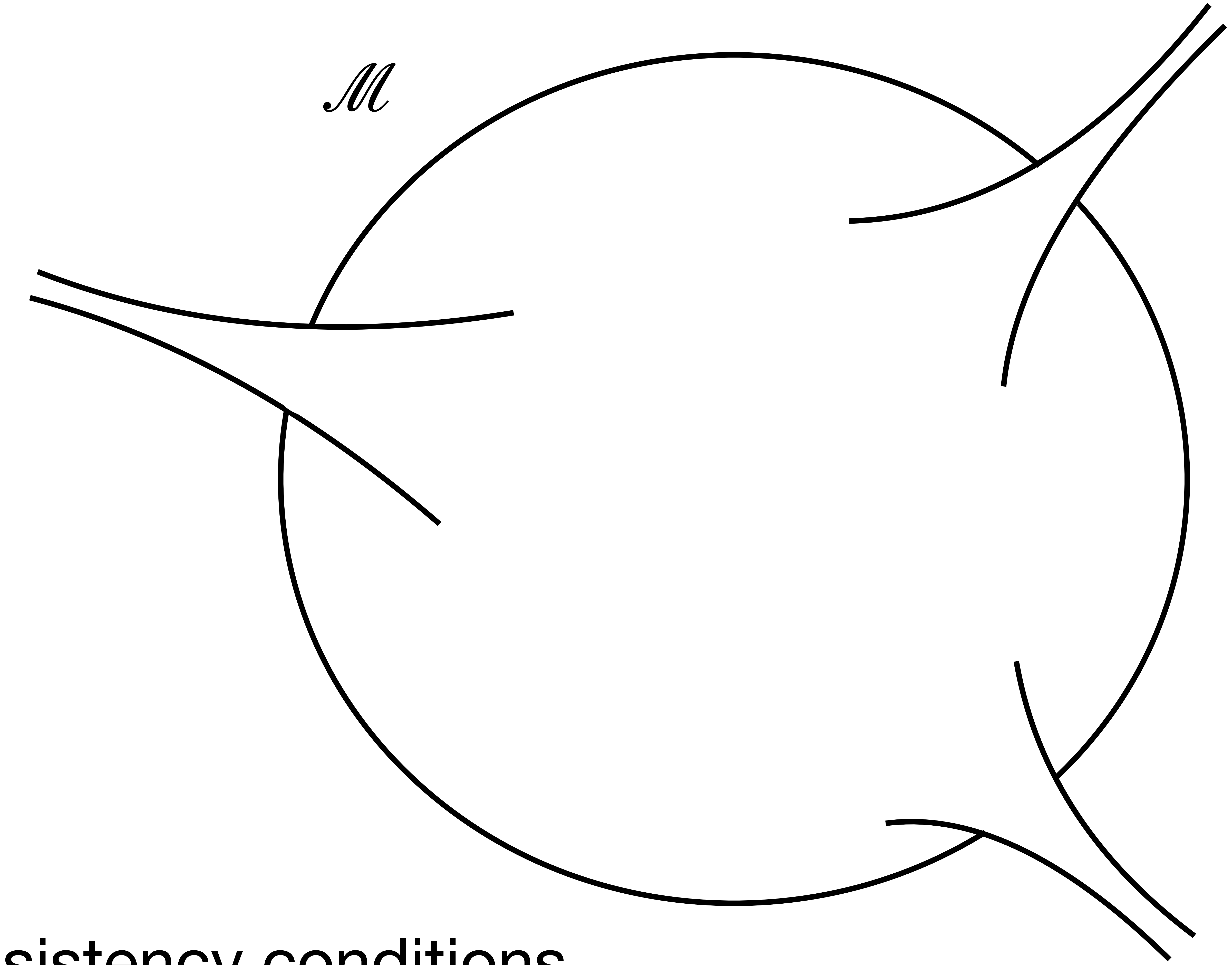

Central question:

What are the **global** consistency conditions for putting together asymptotic phases?

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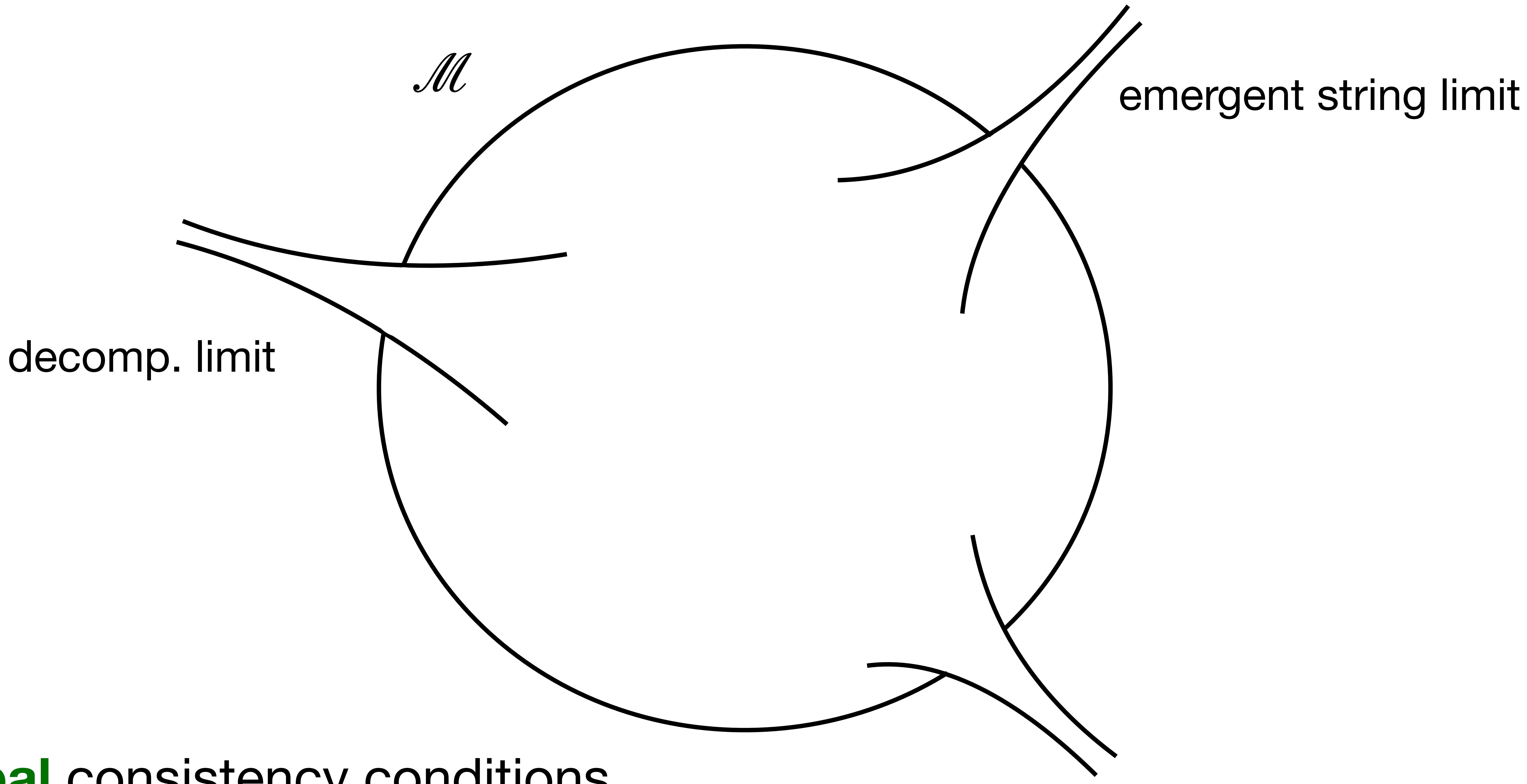

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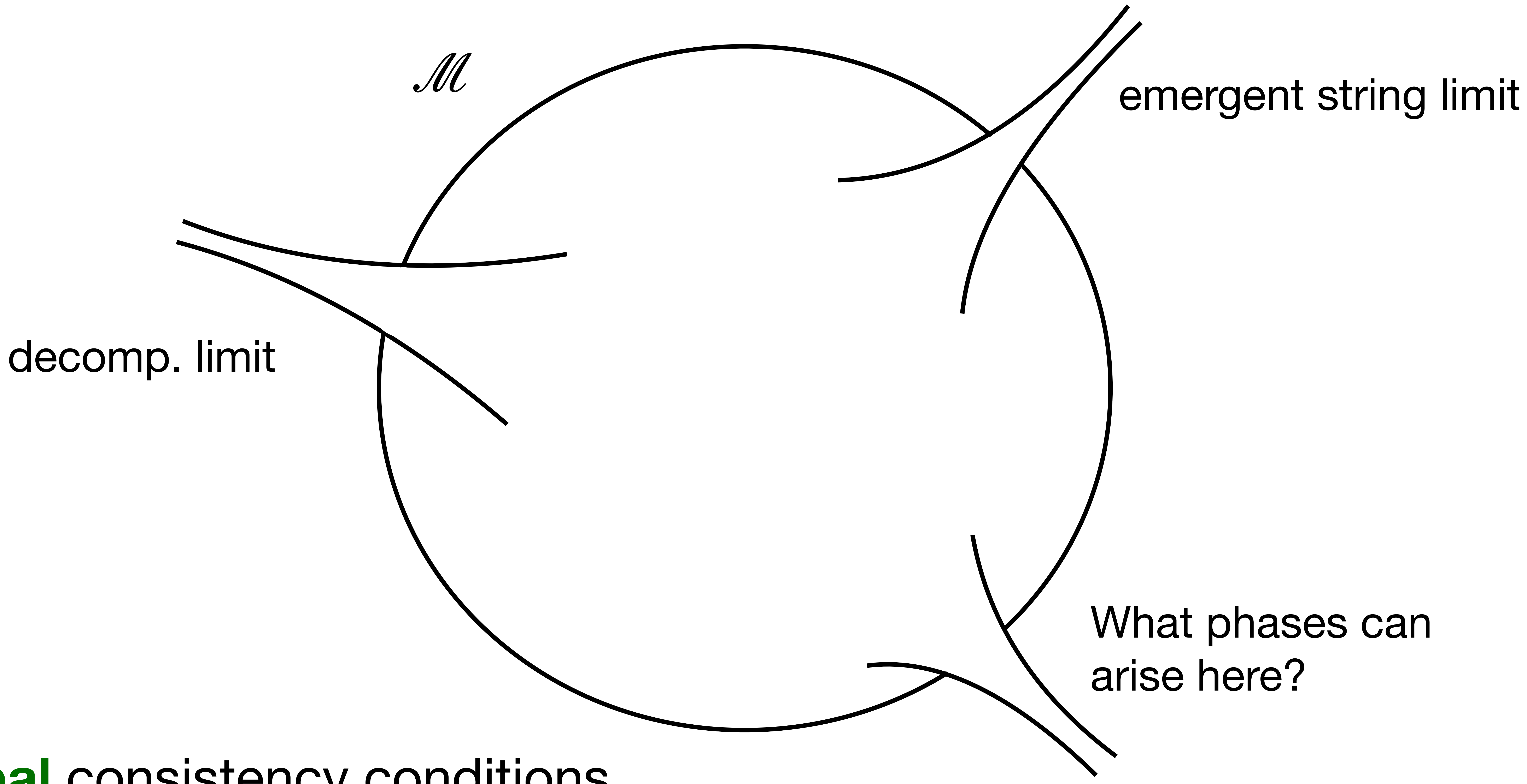

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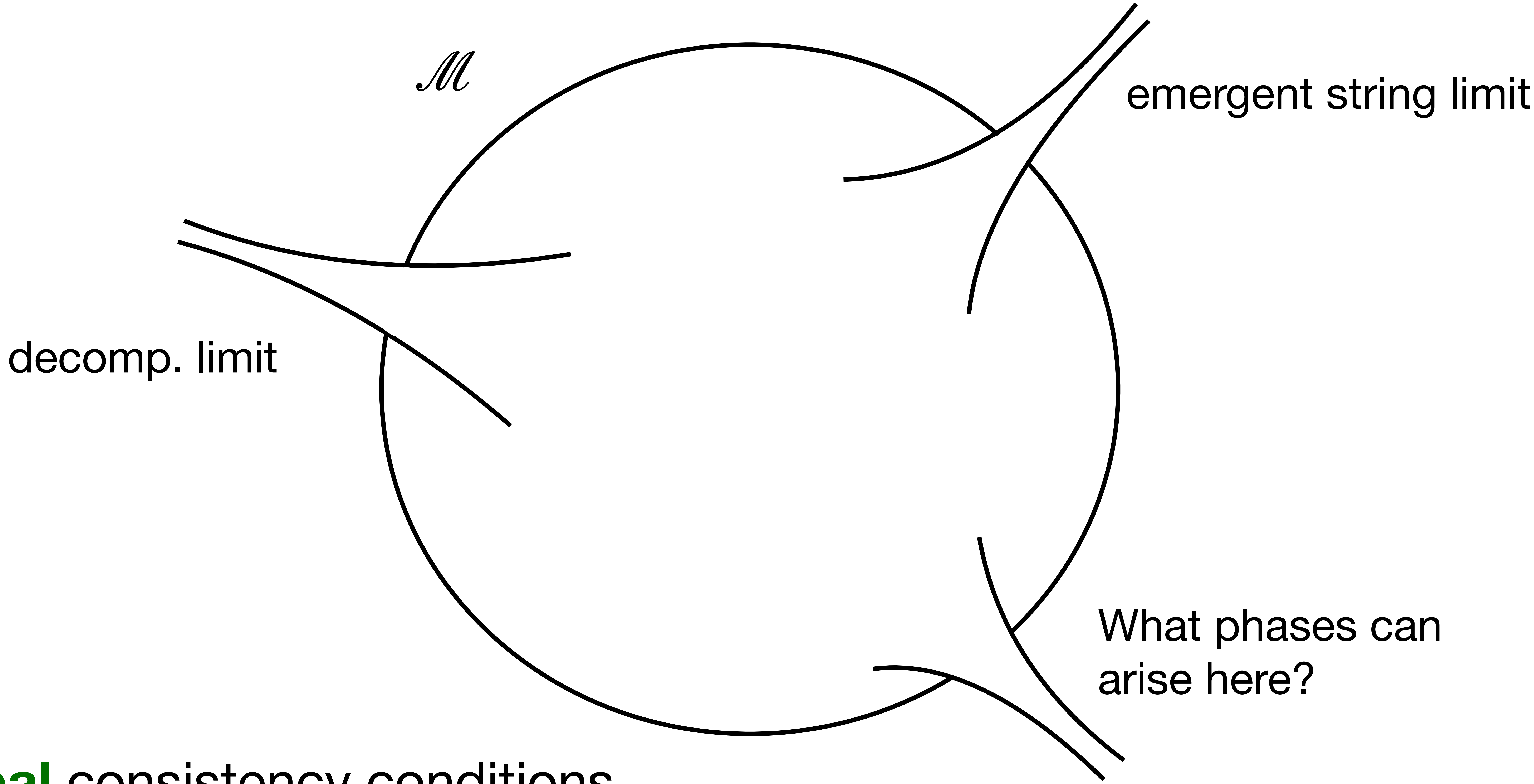

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What are the **global** consistency conditions for putting together asymptotic phases?

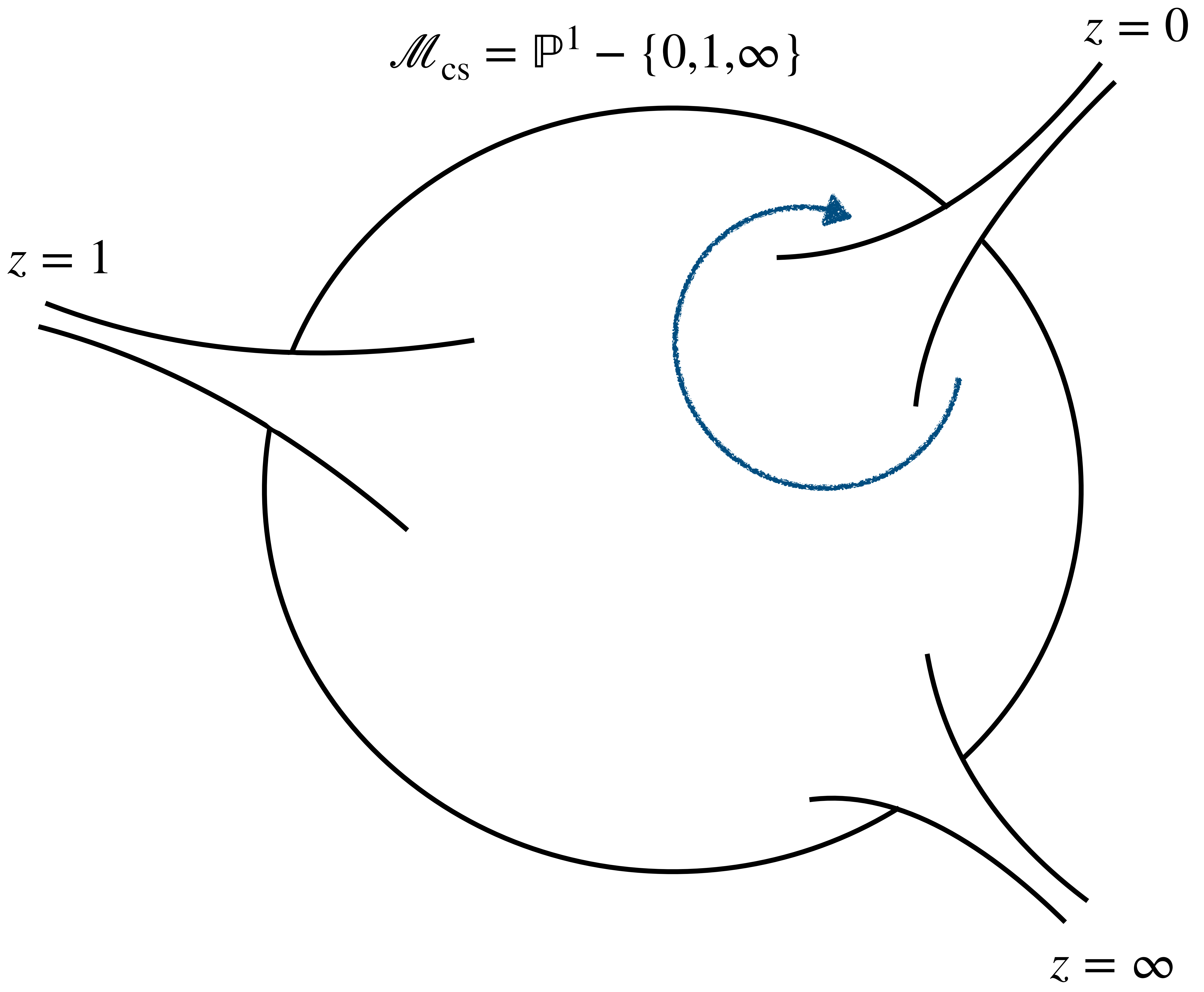
see also: [Etheredge, Heidenreich, Rudelius, Ruiz, Valenzuela; to appear]

Monodromies

Circling a boundary point induces a monodromy:

$$\mathbf{\Pi}(z) \mapsto \mathbf{\Pi}(e^{2\pi i}z) = M \cdot \mathbf{\Pi}(z)$$

$(M \in SL(2,\mathbb{Z}), Sp(4,\mathbb{Z}), SO(3,2;\mathbb{Z}))$



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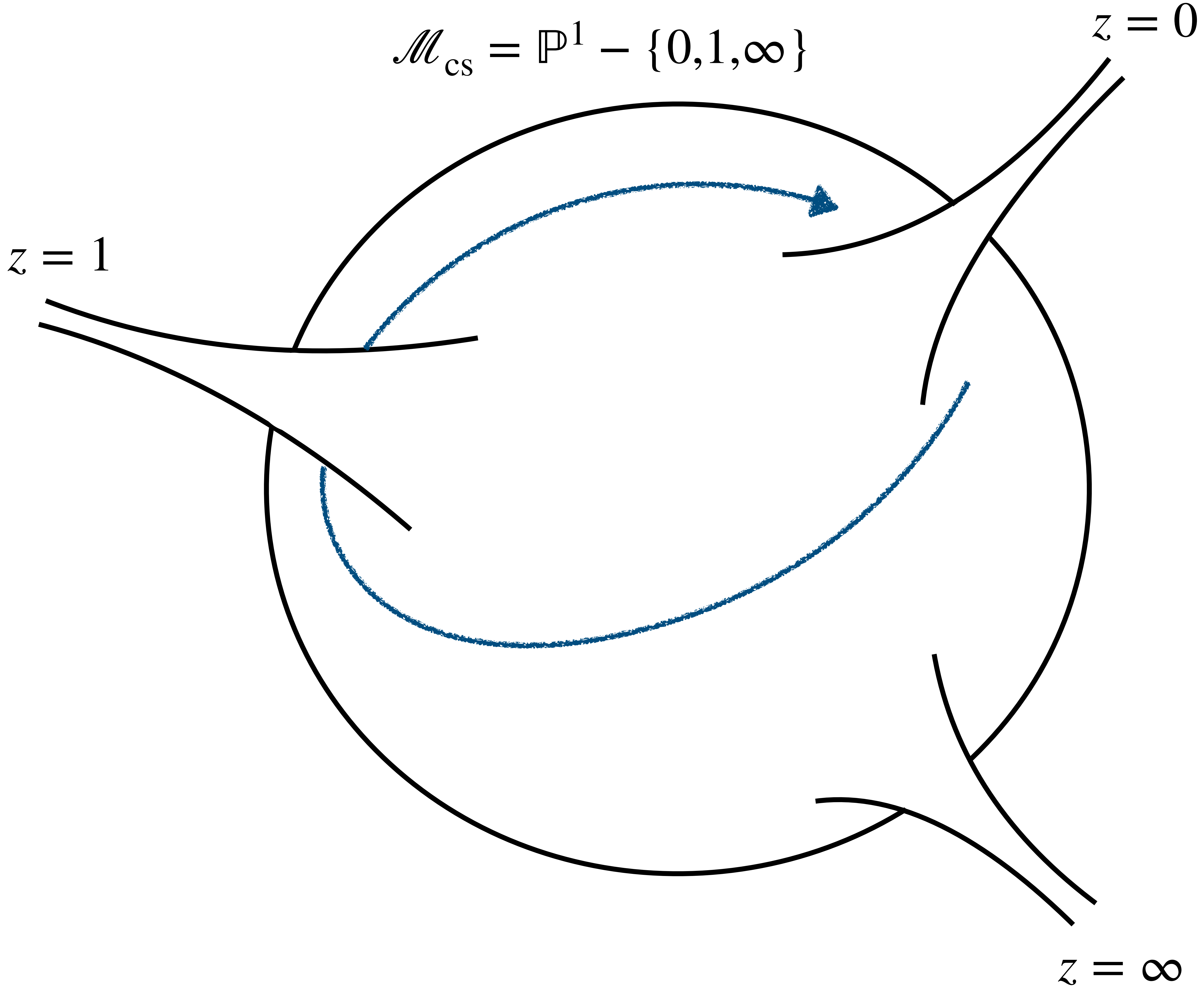

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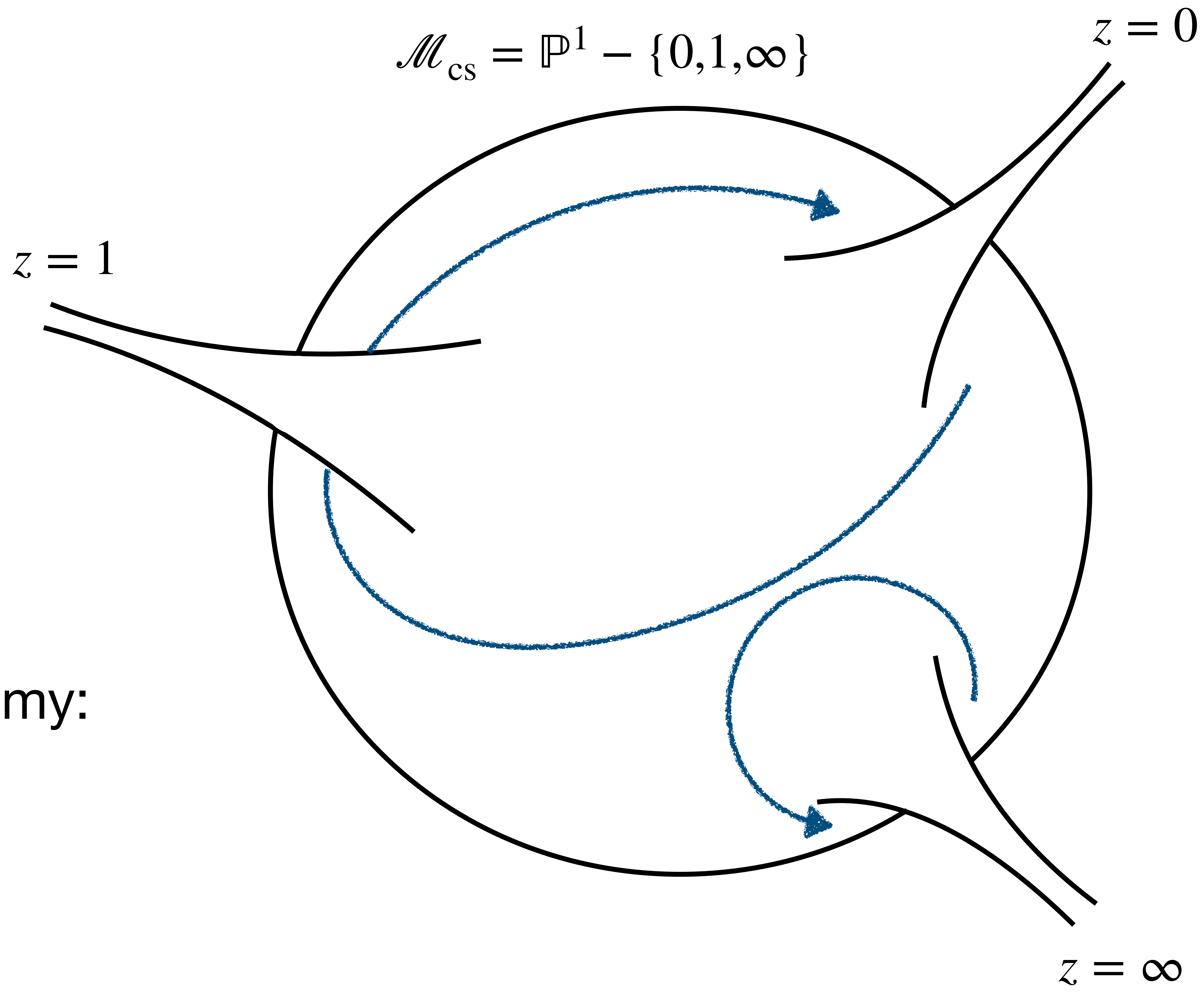
Equivalent loops have same monodromy:

$$M_0 M_1 = (M_\infty)^{-1}$$



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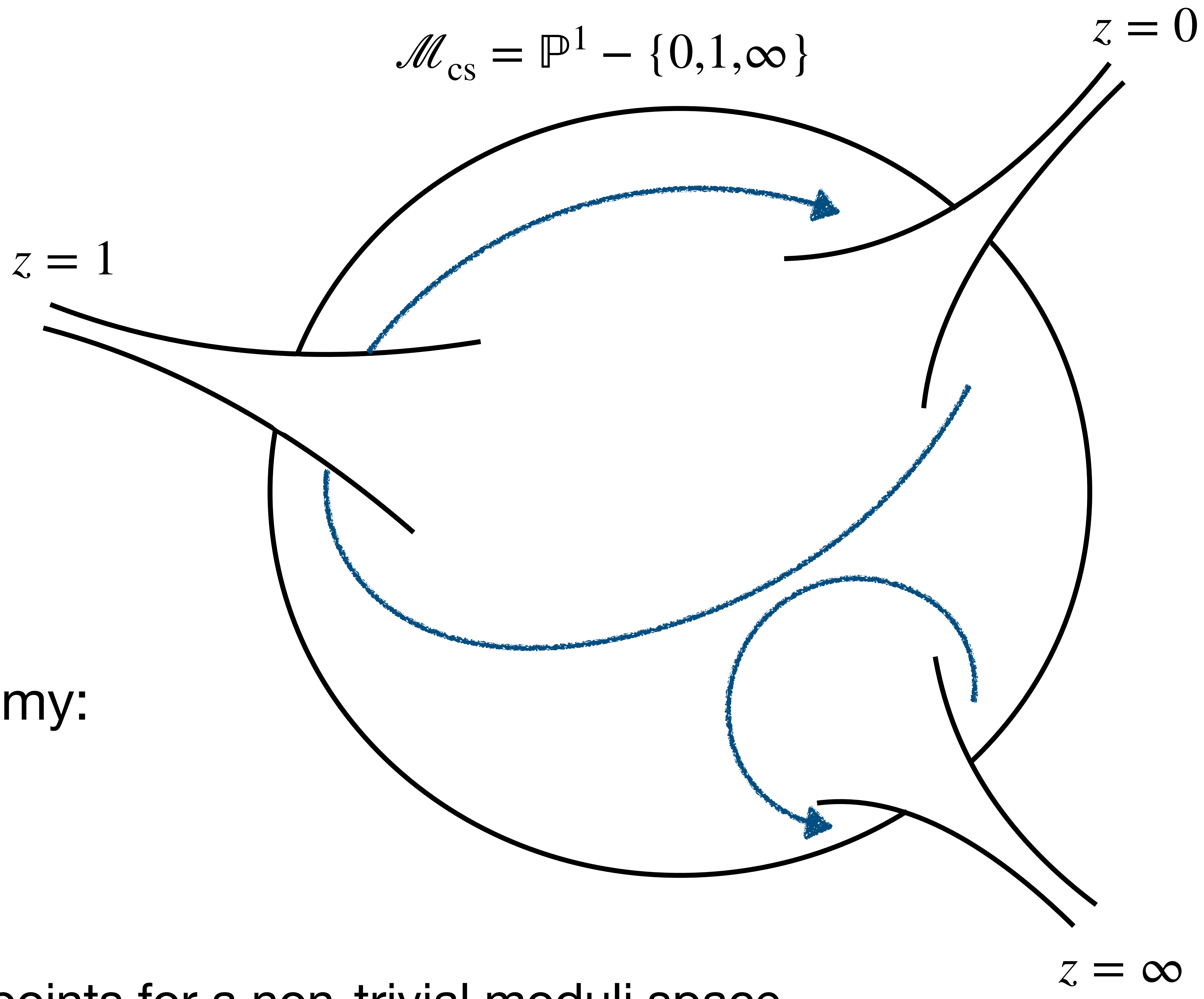

Side-remark: need at least **three** singular points for a non-trivial moduli space

(monodromy group must be infinite order and completely reducible [Griffiths, ’70])

F-theory on Calabi-Yau fourfolds

Kähler potential and flux superpotential:

$$e^{-K_{\text{cs}}} = \int_{Y_4} \bar{\Omega}(\bar{z}) \wedge \Omega(z) = \bar{\mathbf{\Pi}}^T(\bar{z})\Sigma\mathbf{\Pi}(z)$$

$$\Omega(z) \in H^{4,0}$$

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Dependence on **complex structure moduli** encoded in period vector:

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This talk:
Hodge numbers $h^{3,1} = h^{2,2} = 1$

Large complex structure periods

Periods in LCS regime:

[Gerhardus, Jonkers '16;
Cota, Klemm, Schimannek '18;
Marchesano, Prieto, Wiesner '21]

$$\Pi_{\mathrm{LCS}} = \begin{pmatrix} 1 \\ -t \\ -\frac{1}{2}t^2 - \frac{1}{2}t + \frac{c_2}{24\kappa} \\ \frac{\kappa}{6}t^3 + \frac{\kappa}{4}t^2 + \frac{\kappa}{8}t + \frac{ic_3\zeta(3)}{8\pi^3} - \frac{c_2}{48} \\ \frac{\kappa}{24}t^4 + \frac{c_2}{48}t^2 + \frac{ic_3 t\zeta(3)}{8\pi^3} - \frac{c_4}{3456} - \frac{5}{12} \end{pmatrix}$$

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(covering coordinate: $z = e^{2\pi i t}$)

Monodromy under $t \mapsto t+1$:

$$M_{\mathrm{LCS}}(\kappa, c_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ \frac{1}{24}\left(c_2 + 13\kappa\right) & -\frac{\kappa}{2} & -\kappa & 1 & 0 \\ \frac{1}{24}\left(c_2 + \kappa\right) & -\frac{1}{24}\left(c_2 + \kappa\right) & 0 & 1 & 1 \end{pmatrix}$$

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Encode **topological data** of mirror Calabi-Yau

Finiteness of monodromy groups

- (Non-effective) Finiteness theorem by [Deligne '81]

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→ apply to Calabi-Yau fourfold moduli spaces

Quasi-unipotence of monodromies

Driving principle behind classification: quasi-unipotence

$$(M^l - \mathbb{I})^d \neq 0\,, \qquad (M^l - \mathbb{I})^{d+1} = 0\,,$$

geometric proof by [Landman, '73]
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- Nilpotence degree $d = 0,1,\ldots,4$ (complex dimension of Calabi-Yau manifold)
- Finite order $l = 1,2,3,4,5,6,8,10,12$ (possible orders for a $GL(5,\mathbb{Q})$ matrix)

Argument for quasi-unipotence [Schmid, '73]

Jordan decomposition $M = M_u M_s$ (M_s semi-simple, $M_u - 1$ nilpotent)

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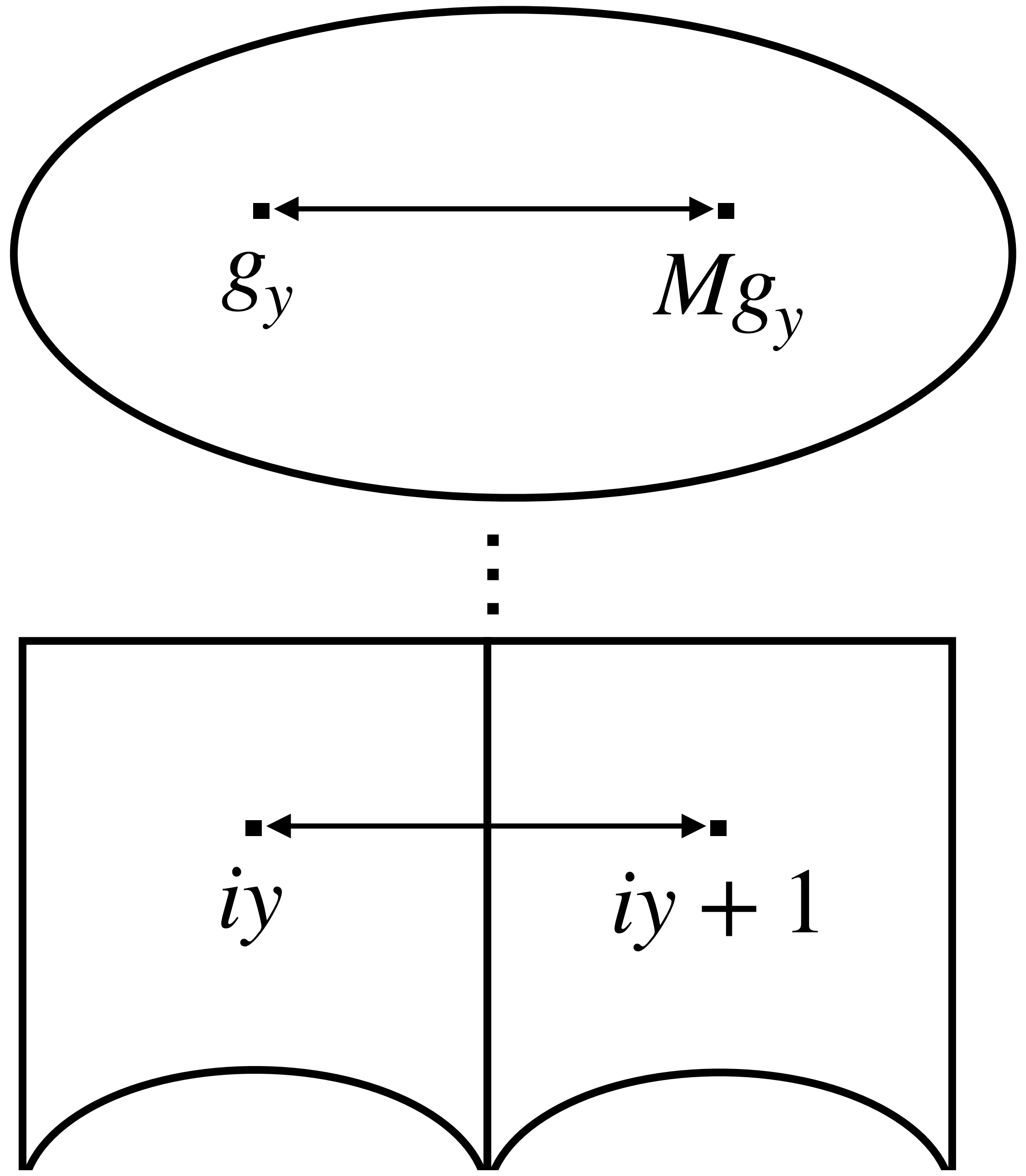
(analogue of $SL(2)/SO(2)$)



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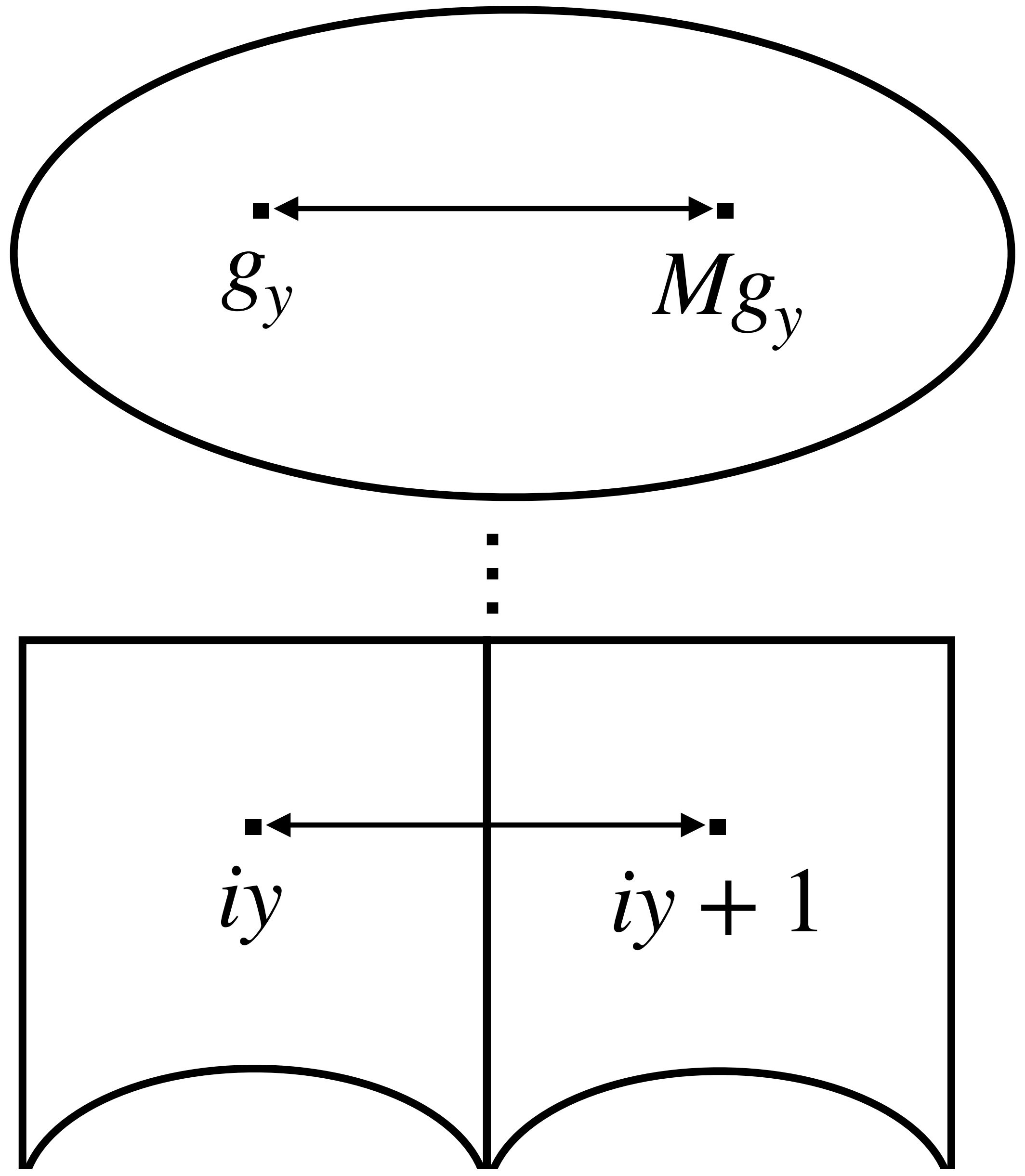

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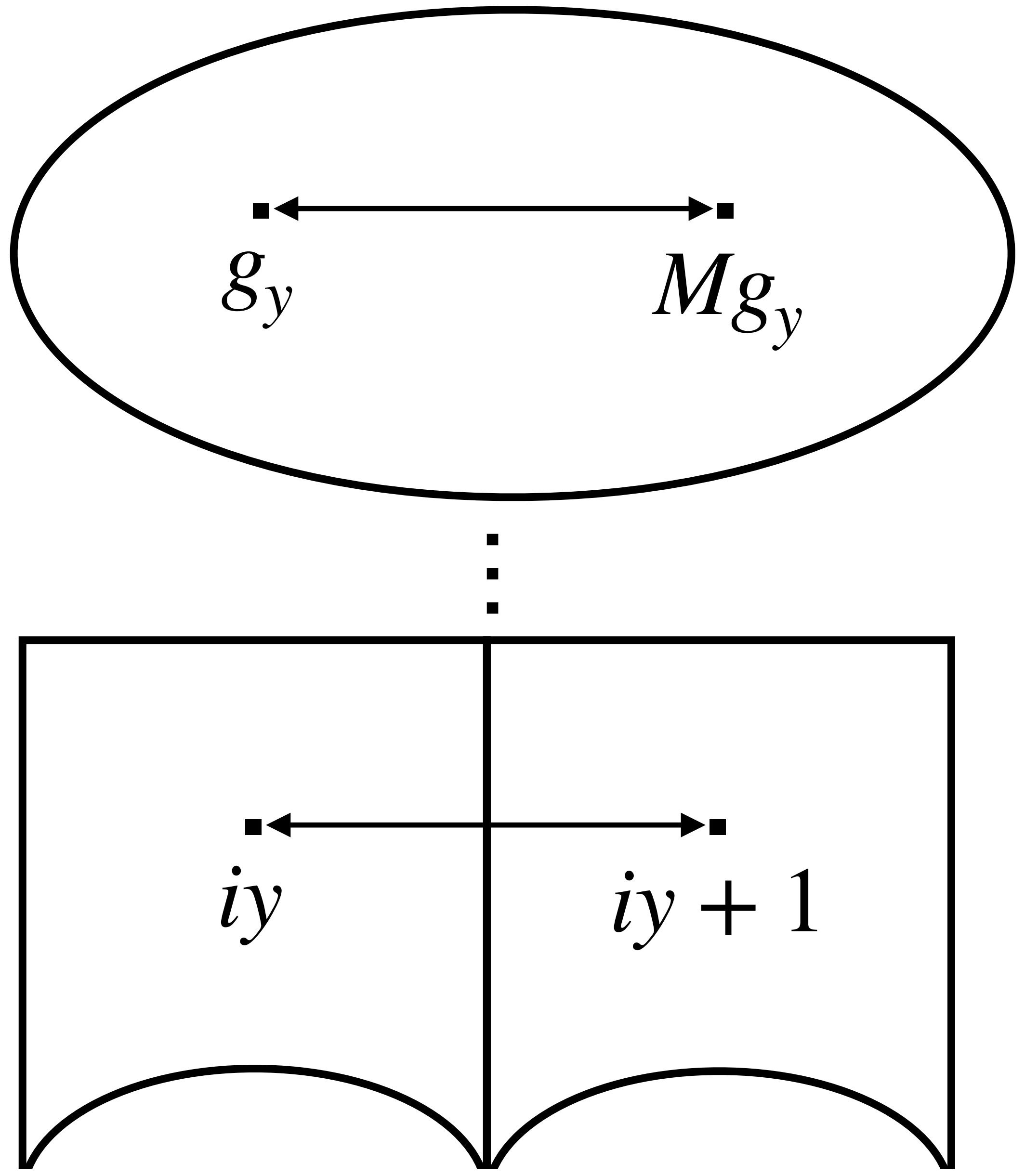

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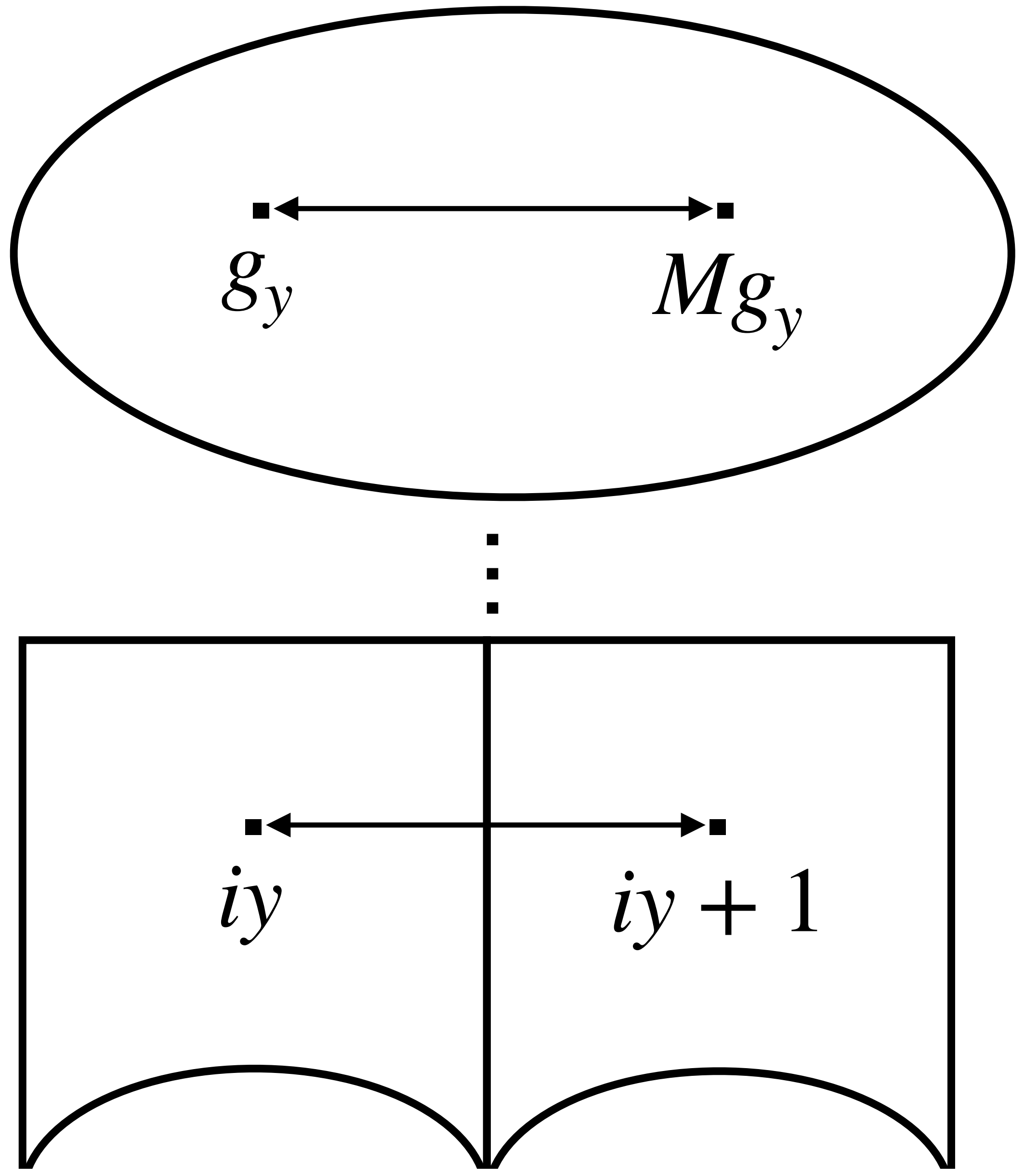

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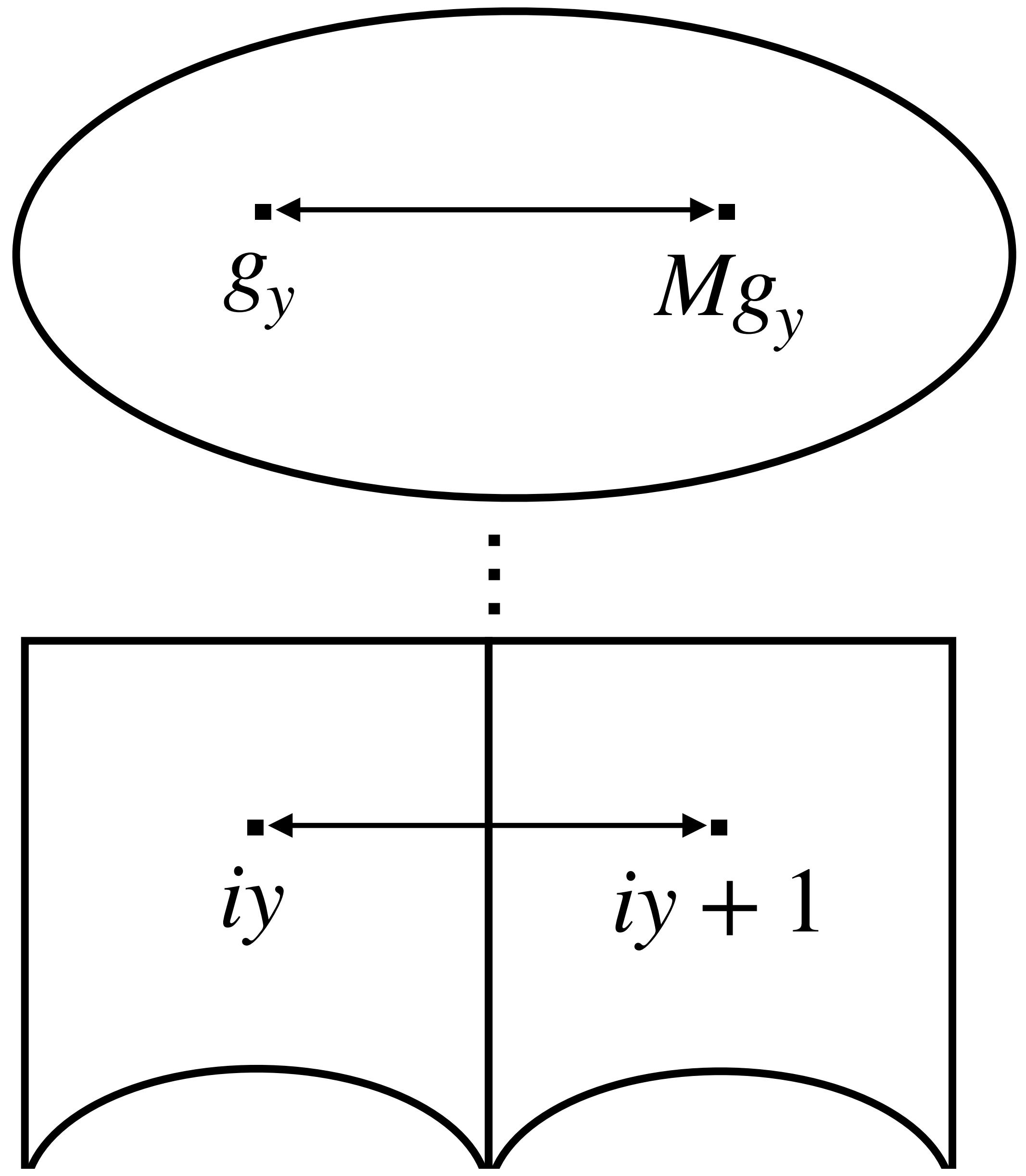

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Warm-up: T2 monodromies

- Monodromies in $SL(2,\mathbb{Z})$:

$$M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 1 & -\kappa \\ 0 & 1 \end{pmatrix} \qquad M_\infty = (M_0 M_1)^{-1} = \begin{pmatrix} 1-\kappa & \kappa \\ -1 & 1 \end{pmatrix}$$

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An example, $d = 0, l = 3$: $\quad M_\infty^3 - 1 = (\kappa - 3) \begin{pmatrix} 2\kappa - \kappa^2 & \kappa^2 - \kappa \\ 1 - \kappa & \kappa \end{pmatrix} = 0\,,$

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$$M_\infty^6 - 1 = (\kappa - 1)(\kappa - 3) \begin{pmatrix} \kappa^4 - 7\kappa^3 + 14\kappa^2 - 7\kappa & -\kappa^4 + 6\kappa^3 - 9\kappa^2 + 2\kappa \\ \kappa^3 - 6\kappa^2 + 9\kappa - 2 & -\kappa^3 + 5\kappa^2 - 5\kappa \end{pmatrix} = 0\,,$$

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$\Longrightarrow$ solutions $\kappa = 3,2,1,4$

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Periods are solutions to the hypergeometric differential operator

$$L = \theta^2 - \mu z(\theta + a_1)(\theta + a_2) \qquad \theta = z\frac{d}{dz}$$

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Periods are given by **hypergeometric functions**:

$$\varpi_0 = {}_2F_1\left(a_1, a_2; 1; \mu z\right)\ , \qquad \varpi_1 = \frac{i}{\sqrt{\kappa}} \cdot {}_2F_1\left(a_1, a_2; 1; 1 - \mu z\right)$$

Reverse-engineer geometries

[Hosono, Klemm, Theisen, Yau '93]

Expand fundamental period in large complex structure regime:

(example: $\kappa = 1$)

$$\varpi_0 = \sum_{n=0}^{\infty} \frac{(6n)!}{n!(2n)!(3n)!} z^n = 1 + 60z + 13860z^2 + 4084080z^3 + \mathcal{O}(z^4)$$

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(example: $\kappa = 1$)

degree of hypersurface

$$\varpi_0 = \sum_{n=0}^{\infty} \frac{(6n)!}{n!(2n)!(3n)!} z^n = 1 + 60z + 13860z^2 + 4084080z^3 + \mathcal{O}(z^4)$$

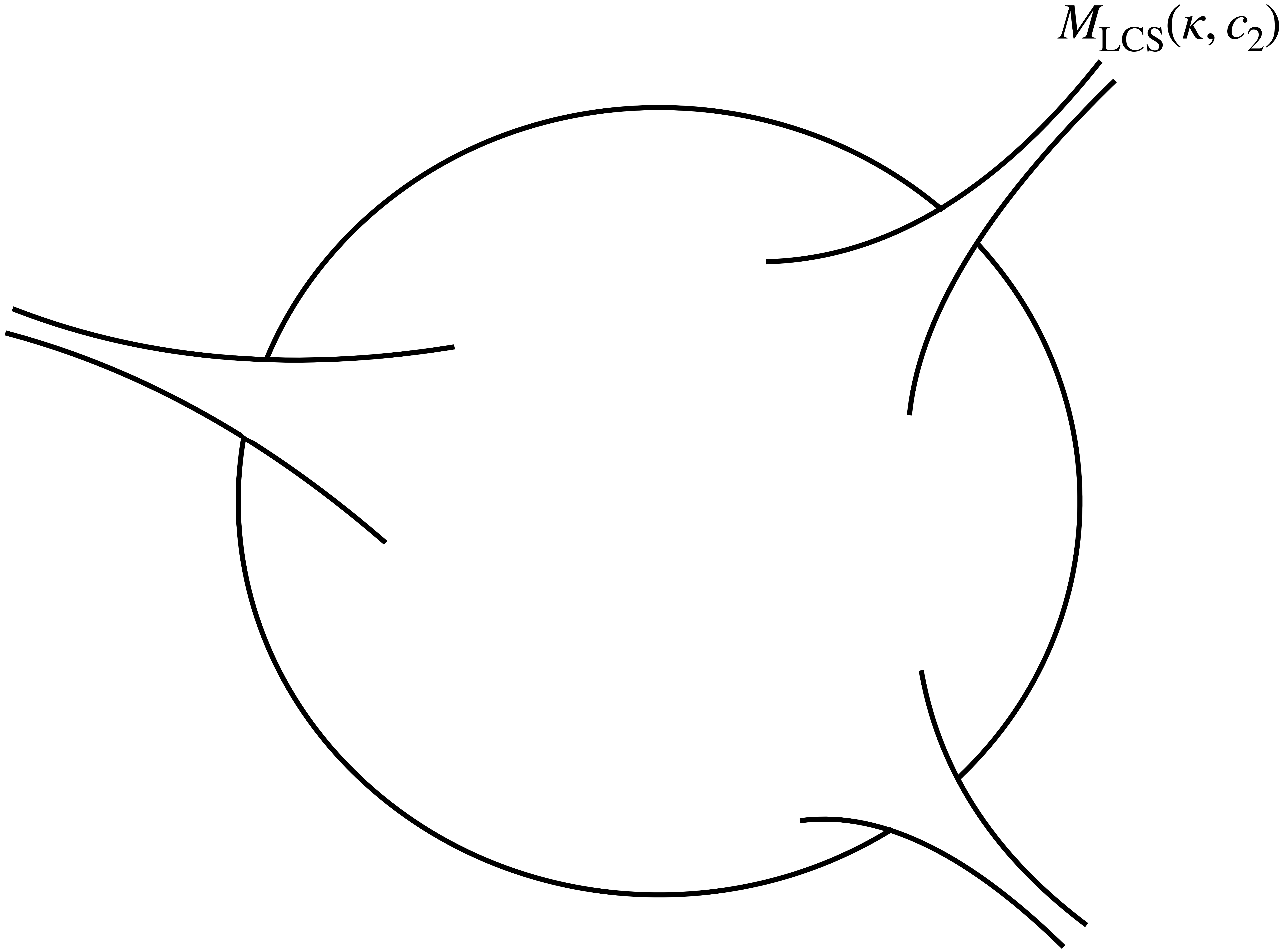
weights of projective space

$\Longrightarrow$ complete intersection Calabi-Yau $X_6(1,2,3)$: sextic in $\mathbb{P}^2[1,2,3]$

Warm-up: T2 landscape

(a_1, a_2)	$(\frac{1}{6}, \frac{5}{6})$	$(\frac{1}{4}, \frac{3}{4})$	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{2}, \frac{1}{2})$
κ	1	2	3	4
μ	432	64	27	16
(d, l)	$(0, 6)$	$(0, 4)$	$(0, 3)$	$(1, 2)$
Modular group	$\Gamma_1(1)$	$\Gamma_1(2)$	$\Gamma_1(3)$	$\Gamma_1(4)$
Elliptic curve	$X_6(1, 2, 3)$	$X_4(1^2, 2)$	$X_3(1^3)$	$X_{2,2}(1^4)$

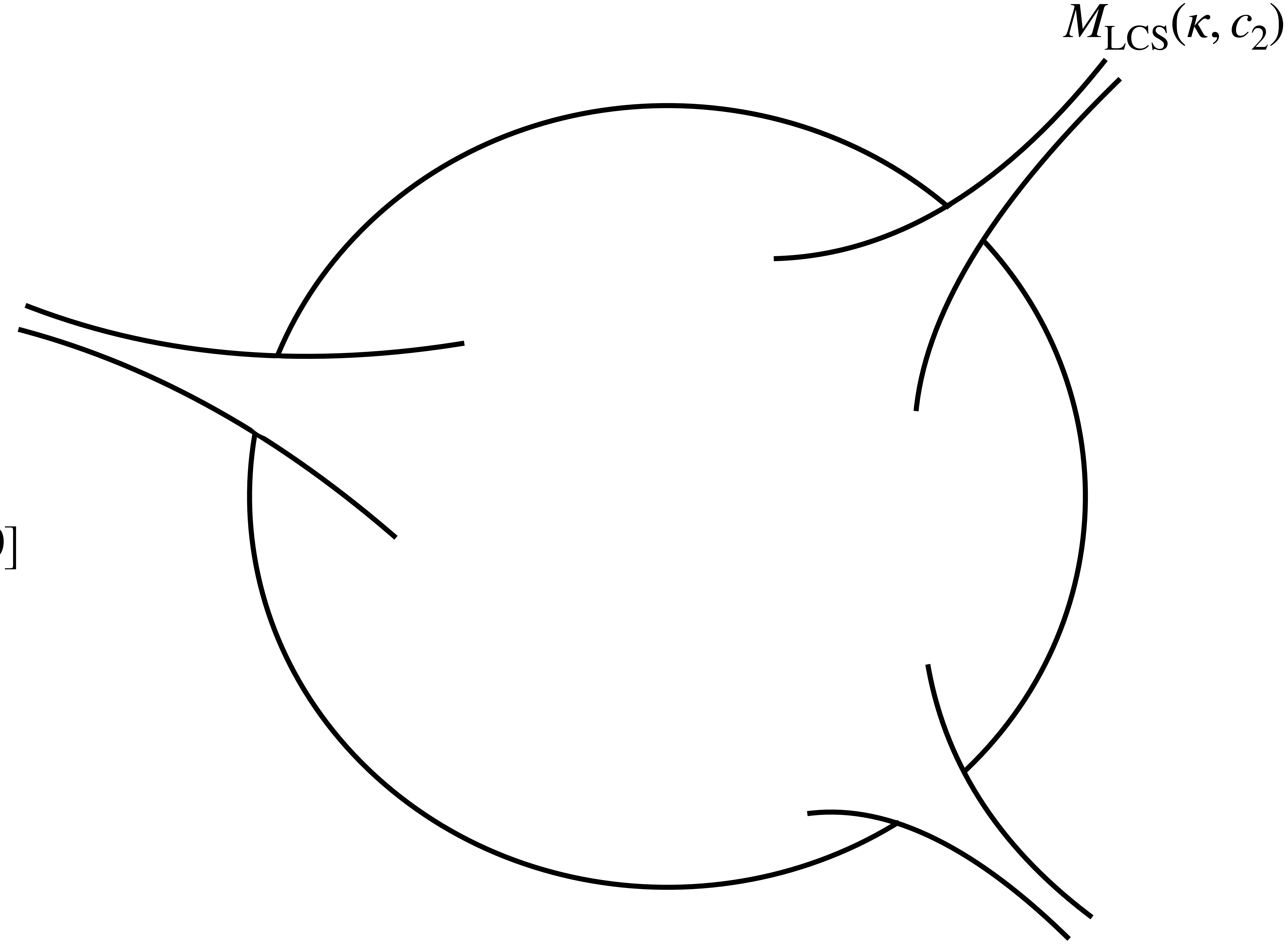
Back to Calabi-Yau fourfolds



Back to Calabi-Yau fourfolds

$$M_C = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[Grimm, Ha, Klemm, Klevers ’09]



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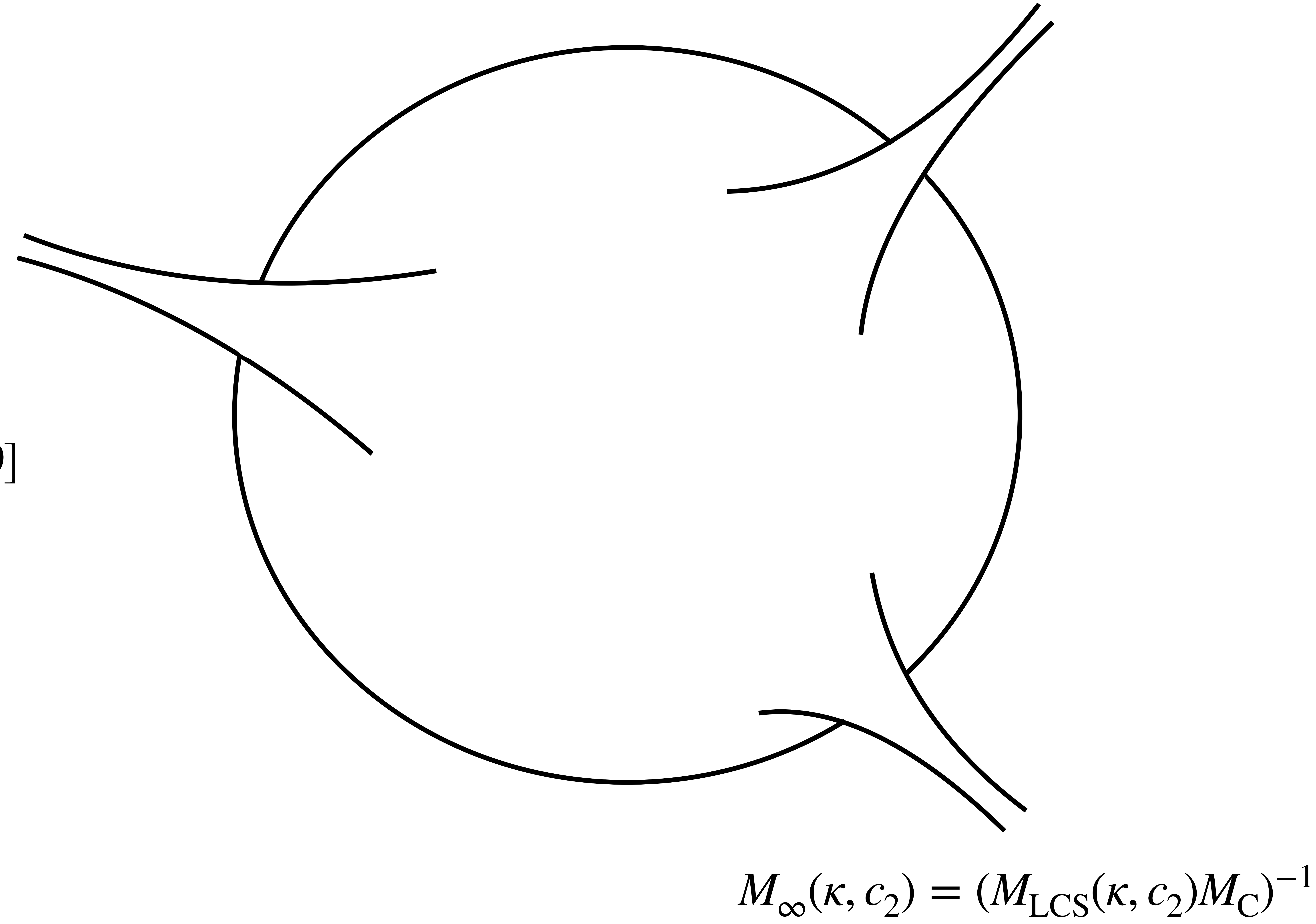

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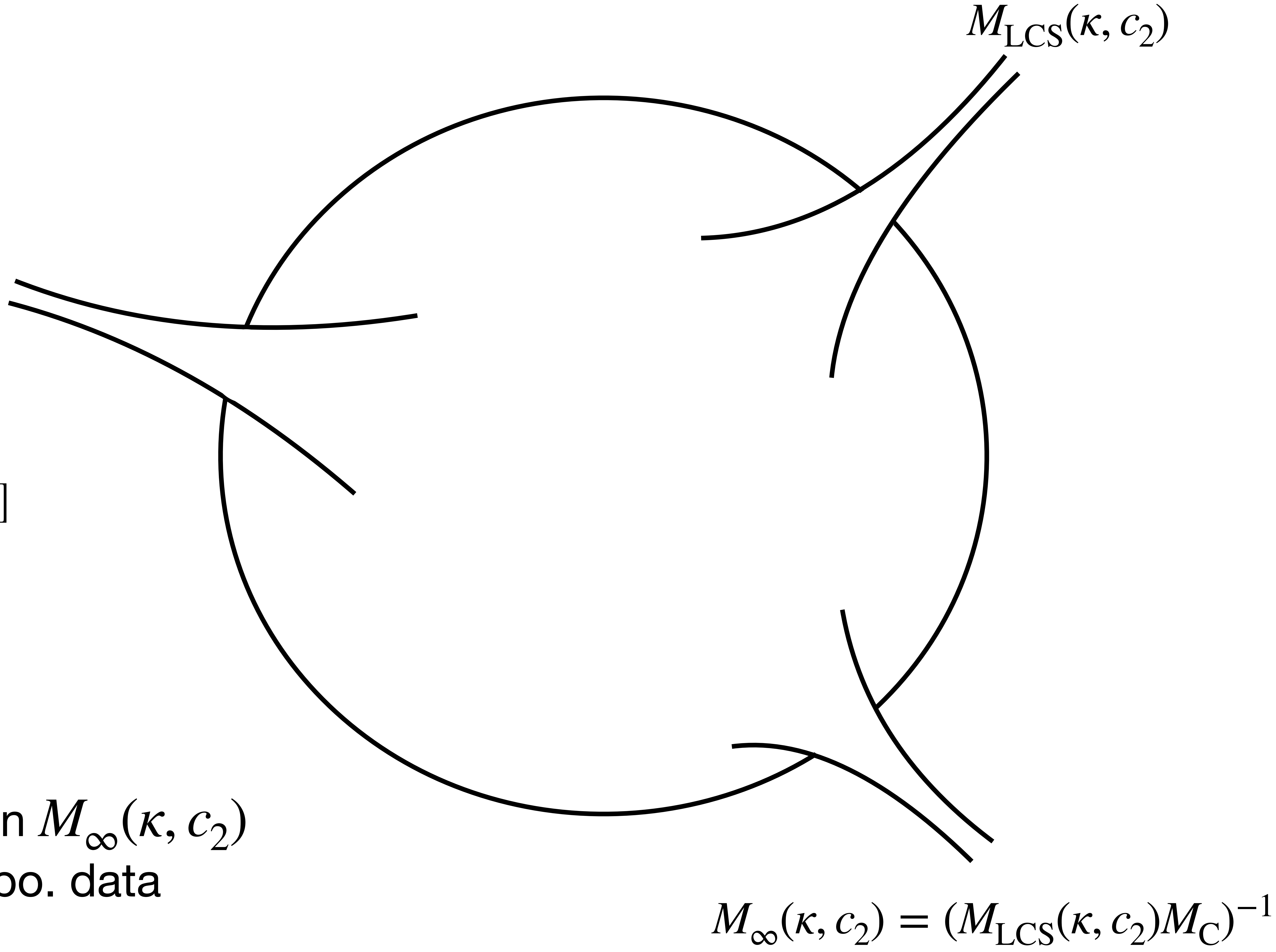

$\Longrightarrow$ impose quasi-unipotence on $M_\infty(\kappa, c_2)$ and solve for topo. data

Example

Impose a finite order monodromy of order $l = 6$:

$$(M_\infty(\kappa, c_2))^6 - \mathbb{1} = 0$$

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$\Longrightarrow$ data of the sextic in $\mathbb{P}^5$, (without doing a geometrical computation)

Landscape of monodromy groups

[DvdH, '24]

(κ, a)	(6,4)	(4,4)	(2,3)	(10,5)	(2,4)	(4,3)	(12,5)
degree d	0						
order l	6	8	10		12		

(a) *Finite order monodromies.*

(κ, a)	(8,4)	(2,2)	(18,6)	(16,6)	(8,5)	(24,7)	(32,8)
degree d	1			2			4
order l	4	6		4	6		2

(b) *Infinite order monodromies.*

$a = (\kappa + c_2)/24$

Computing the periods

- Periods solve the hypergeometric equation:

$$L = \theta^5 - \mu z(\theta + a_1)(\theta + a_2)(\theta + a_3)(\theta + a_4)(\theta + a_5) \qquad \theta = z\frac{d}{dz}$$

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- Can determine the CICY from series expansion of this period
- Other 4 periods have similar expressions in hypergeometric functions

Calabi-Yau fourfold landscape

a_1, a_2, a_3, a_4, a_5	Type	Mirror	μ	(κ, a)	c_2	c_3	c_4
$\frac{1}{5}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}$	F	$X_{2,5}(1^7)$	$2^2 5^5$	$(10, 5)$	110	-420	2190
$\frac{1}{10}, \frac{3}{10}, \frac{1}{2}, \frac{7}{10}, \frac{9}{10}$	F	$X_{10}(1^5, 5)$	$2^{10} 5^5$	$(2, 3)$	70	-580	5910
$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	LCS	$X_{2^5}(1^{10})$	2^{10}	$(32, 8)$	160	-320	960
$\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}$	CY3	$X_{2,3,3}(1^8)$	$2^2 3^6$	$(18, 6)$	126	-324	1206
$\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}$	C	$X_{2,2,2,3}(1^9)$	$2^6 3^3$	$(24, 7)$	144	-336	1152
$\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}$	C	$X_{2,2,4}(1^8)$	2^{12}	$(16, 6)$	128	-384	1632
$\frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{7}{8}$	F	$X_{2,8}(1^6, 4)^*$	2^{18}	$(4, 4)$	92	-600	4908
$\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}$	F	$X_6(1^6)$	6^6	$(6, 4)$	90	-420	2610
$\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{11}{12}$	F	$X_{2,2,12}(1^6, 4, 6)^{**}$	$2^{14} 3^6$	$(2, 4)$	94	-972	11814
$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}$	CY3	$X_{4,4}(1^6, 2)$	2^{14}	$(8, 4)$	88	-304	1464
$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$	F	$X_{3,4}(1^7)$	$2^8 3^3$	$(12, 5)$	108	-336	1476
$\frac{1}{6}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}$	F	$X_{4,6}(1^5, 2, 3)^*$	$2^{12} 3^3$	$(4, 3)$	68	-320	2028
$\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{5}{6}$	CY3	$X_{6,6}(1^4, 2, 3^2)^*$	$2^{10} 3^3$	$(2, 2)$	46	-244	1734
$\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6}$	C	$X_{2,2,6}(1^7, 3)^*$	$2^{10} 3^6$	$(8, 5)$	112	-528	3264

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- 9 CY4 already known

[Cabo-Bizet, Klemm, Lopes '14]

- 5 CY4 are new

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CY3-point of $X_{6,6}(1^4,2,3^2)$

Period expansion around the CY3-point:

$$\Pi(\tau) = \begin{pmatrix} 1 \\ \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ 0 \\ \tau \\ (\frac{1}{2} + \frac{i\sqrt{3}}{2})\tau \end{pmatrix} + \frac{i}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ -1 \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix} + \mathcal{O}(e^{2\pi i \tau}) \qquad \tau = \log[z]/2\pi i$$

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- Complex structure coordinate parametrizes the **string coupling**

D7-brane superpotential

Fourfold periods are known to encode **open-string** physics

[Grimm-Ha-Klemm-Klevers '09; Alim-Hecht-Jockers-Mayr-Mertens-Soroush '09; Jockers-Mayr-Walcher '09; Clinghler-Donagi-Wijnholt '12]

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Remaining period: superpotential induced by **worldvolume flux** of D7-branes

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$$= \frac{q_{\mathrm{D7}}}{\pi^2}\sqrt{z}\left(1 - \frac{2187}{2}z + \frac{9298091736}{1225}z^2 - \frac{4236443047215}{49}z^3 + \mathcal{O}(z^4)\right) \qquad z = e^{2\pi i \tau}$$

Conclusions & outlook

- Monodromies give a powerful tool in charting the landscape

- New $\mathcal{N} = 1$ moduli spaces to be explored further

  (e.g. in searching for flux vacua, cf. [Plauschinn, Schlechter '23; Lüst '24]

- Singularities at infinity $\Longrightarrow$ novel phases of $\mathcal{N} = 1$ string compactifications