Quantum field simulation of dynamics in curved spacetime and cosmological particle production

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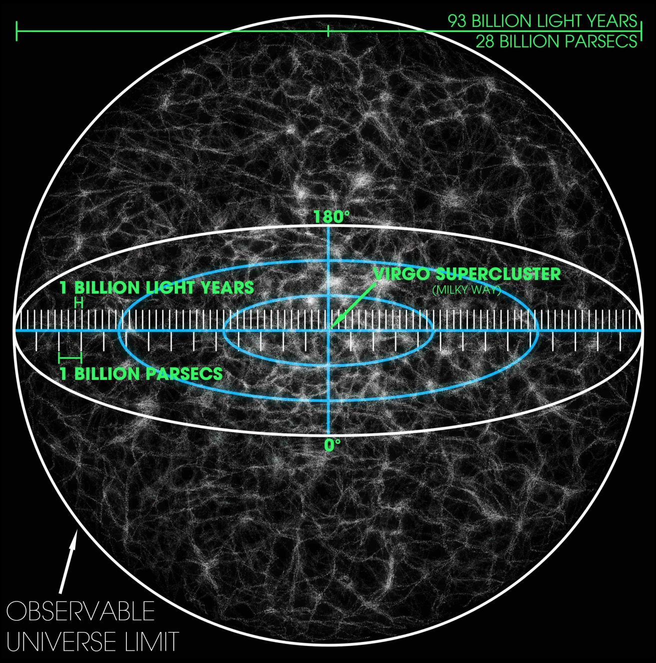
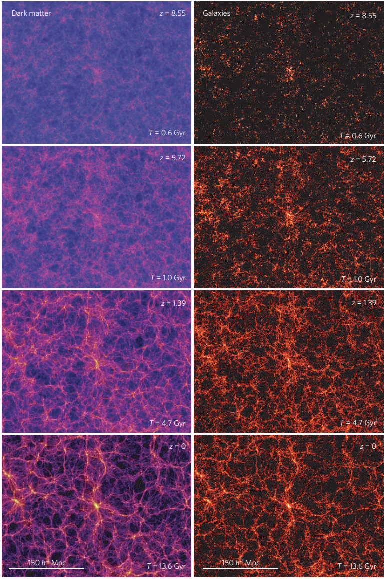
FRIEDRICH-SCHILLER-UNIVERSITÄT JENA

Collaboration & Publications

- Celia Viermann, Marius Sparn, Nikolas Liebster, Maurus Hans, Elinor Kath, Álvaro Parra-López, Mireia Tolosa-Simeón, Natalia Sánchez-Kuntz, Tobias Haas, Helmut Strobel, Stefan Floerchinger, Markus K. Oberthaler,
  Quantum field simulator for dynamics in curved spacetime
  [Nature 611, 260 (2022)]
- Mireia Tolosa-Simeón, Álvaro Parra-López, Natalia Sánchez-Kuntz, Tobias Haas, Celia Viermann, Marius Sparn, Nikolas Liebster, Maurus Hans, Elinor Kath, Helmut Strobel, Markus K. Oberthaler, Stefan Floerchinger,
  Curved and expanding spacetime geometries in Bose-Einstein condensates
  [Phys. Rev. A 106, 033313 (2022)]
- Natalia Sánchez-Kuntz, Álvaro Parra-López, Mireia Tolosa-Simeón, Tobias Haas, Stefan Floerchinger,
  Scalar quantum fields in cosmologies with 2+1 spacetime dimensions
  [Phys. Rev. D 105, 105020 (2022)]
- Mireia Tolosa-Simeón, Michael M. Scherer, S. Floerchinger,
  Analog of cosmological particle production in Dirac materials,
  [Phys. Rev. B (2024)]
- Christian F. Schmidt, Álvaro Parra-López, Mireia Tolosa-Simeón, Marius Sparn, Elinor Kath, Nikolas Liebster, Jelte Duchene, Helmut Stobel, Markus K. Oberthaler, Stefan Floerchinger,
  Cosmological particle production in a quantum field simulator as a quantum mechanical scattering problem,
  [arXiv:2406.08094]


93 BILLION LIGHT YEARS
28 BILLION PARSECS
180°
VIRGO SUPERCLUSTER
(MILKY WAY)
1 BILLION LIGHT YEARS
1 BILLION PARSECS
0°
OBSERVABLE
UNIVERSE LIMIT

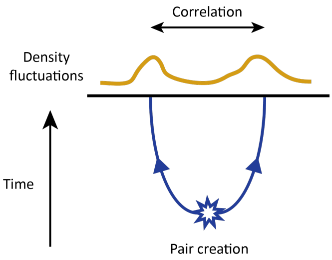
Evolution of cosmic large-scale structure



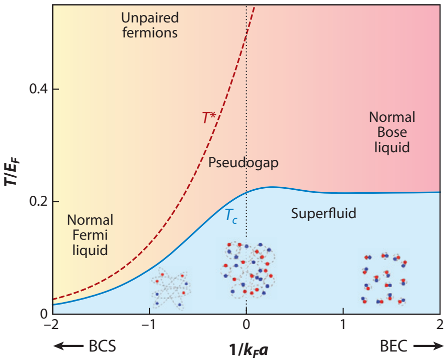
[Springel, Frenk & White, Nature 440, 1137 (2006)]

Quantum origin of fluctuations

- Universe was almost homogeneous at early times
- small fluctuations magnified by gravitational attraction
- primordial quantum fluctuations from inflation
  [Mukhanov & Chibisov (1981), Hawking (1982), Starobinsky (1982), Guth & Pi (1982), Bardeen, Steinhardt & Turner (1983), Fischler, Ratra & Susskind (1985)]



Ultracold quantum gases



- can be very well controlled experimentally
- develop and test quantum field theory
- finite density, finite temperature
- out-of-equilibrium
- quantum information

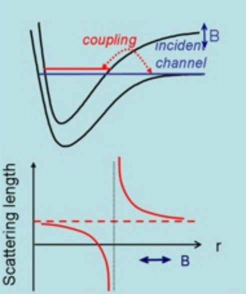
Non-relativistic quantum fields

- Bose-Einstein condensate in two dimensions
  [Gross (1961), Pitaevskii (1961)]

$$\Gamma[\Phi] = \int \mathrm{d}t\, \mathrm{d}^2x \Big\{ \hbar\Phi^*(t,\mathbf{x}) \left[i\frac{\partial}{\partial t} - V(t,\mathbf{x}) \right] \Phi(t,\mathbf{x}) \\ - \frac{\hbar^2}{2m} \boldsymbol{\nabla}\Phi^*(t,\mathbf{x}) \boldsymbol{\nabla}\Phi(t,\mathbf{x}) - \frac{\lambda(t)}{2} \Phi^*(t,\mathbf{x})^2 \Phi(t,\mathbf{x})^2 \Big\}$$

- low energy theory for bosonic atoms
- optical trap potential $V(t,\mathbf{x})$
- coupling strength $\lambda(t)$

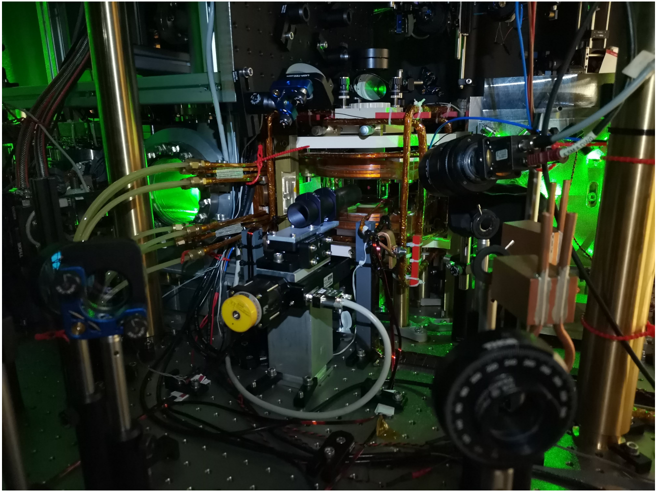
Feshbach resonance



- allow to control scattering length or effective s-wave interaction strength through magnetic field B
- can be made time-dependent by varying magnetic field

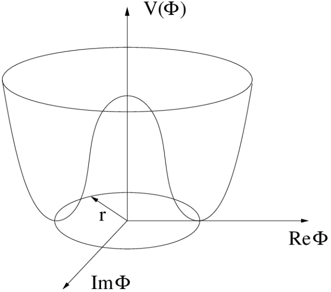
$$\frac{\lambda(t)}{2}\Phi^*(t,\mathbf{x})^2\Phi(t,\mathbf{x})^2$$

Experimental realization



[Markus K. Oberthaler group, Uni Heidelberg]

Superfluid and small excitations



- Complex non-relativistic field can be decomposed

$$\Phi = e^{iS_0}\left(\sqrt{n_0} + \frac{1}{\sqrt{2}}\left[\phi_1 + i\phi_2\right]\right)$$

- real fields ϕ_1 and ϕ_2 describe excitations on top of the superfluid
- low energy field $\phi_2(t, \mathbf{x})$
- stationary superfluid density $n_0(\mathbf{x})$ and vanishing superfluid velocity

$$\mathbf{v} = \frac{\hbar}{m}\boldsymbol{\nabla} S_0 = 0$$

Sound waves / phonons

- small energy excitations are sound waves or phonons
- propagate with finite velocity, similar to light
- local speed of sound

$$c_S(t, \mathbf{x}) = \sqrt{\frac{\lambda(t)\, n_0(\mathbf{x})}{m}}$$

- sound waves propagate along

$$ds^2 = -dt^2 + \frac{1}{c_S(t, \mathbf{x})^2}(d\mathbf{x} - \mathbf{v}\, dt)^2 = 0$$

- acoustic metric for vanishing fluid velocity $\mathbf{v} = 0$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{c_S(t,\mathbf{x})^2} & 0 \\ 0 & 0 & \frac{1}{c_S(t,\mathbf{x})^2} \end{pmatrix}$$

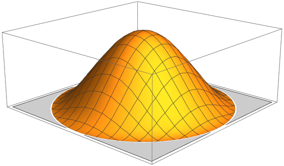
Relativistic scalar field

- Low energy theory for phonons (with $\phi = \phi_2/\sqrt{2m}$)

$$\Gamma[\phi] = \int \mathrm{d}t\, \mathrm{d}^2x \sqrt{g} \left\{ -\frac{1}{2} g^{\mu\nu}\, \partial_\mu \phi\, \partial_\nu \phi \right\}$$

- metric determinant $\sqrt{g} = \sqrt{-\det(g_{\mu\nu})}$
- acoustic metric depends on space and time like the space-time metric in general relativity
- phonons behave like a real, massless, relativistic scalar field in a curved spacetime !
- quantum simulator for QFT in curved space

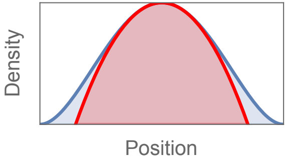
Density profiles



- assume specifically for $r = |\mathbf{x}| < R$

$$n_0(r) = \bar{n}_0 \times \left[1 - \frac{r^2}{R^2}\right]^2$$

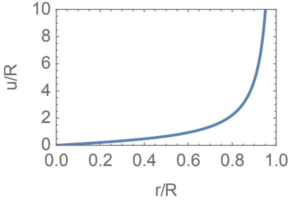
- experimental realization with optical trap and digital micromirror device
- approximate realization in harmonic trap



Acoustic spacetime geometry

- variable transform to $0 \leq u < \infty$

$$u(r) = \frac{r}{1 - \frac{r^2}{R^2}}$$



- leads to Friedmann-Lemaitre-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t) \left(\frac{du^2}{1 - \kappa u^2} + u^2 d\varphi^2 \right)$$

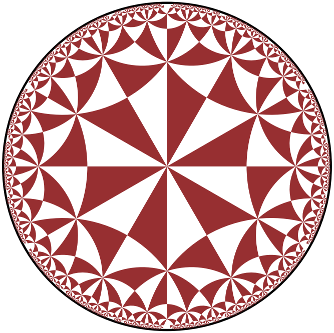
- negative spatial curvature

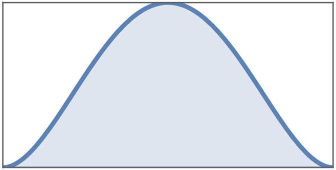
$$\kappa = -4/R^2$$

- scale factor

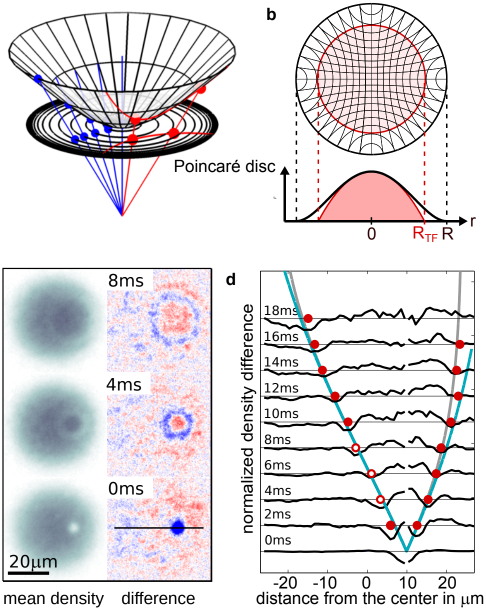
$$a(t) = \sqrt{\frac{m}{\bar{n}_0} \frac{1}{\lambda(t)}}$$

Hyperbolic geometry

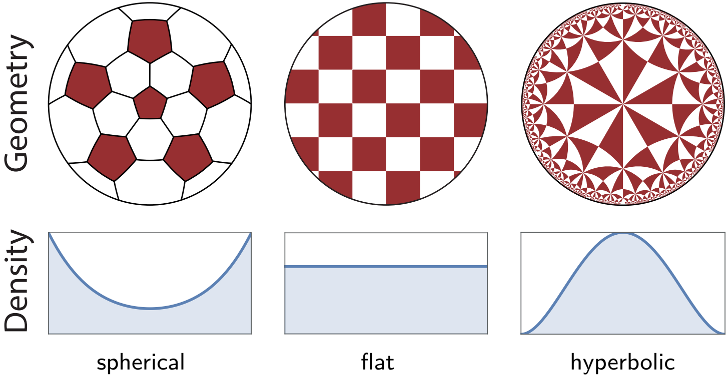


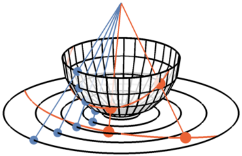


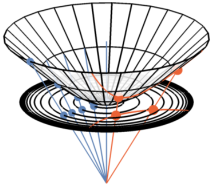
Experimental realization in a Bose-Einstein condensate



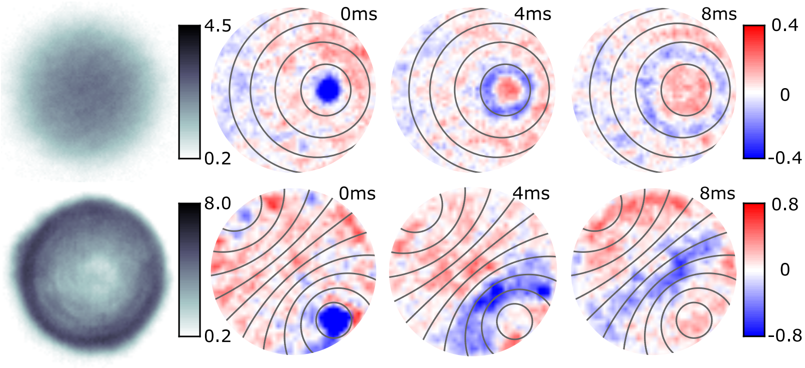
Geometries with constant spatial curvature



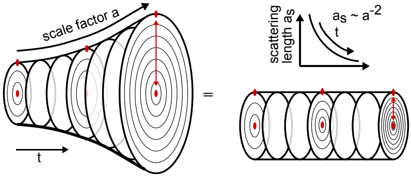


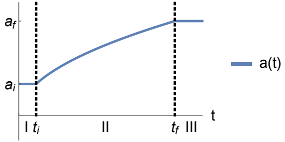


Propagating sound waves



Expansion and particle production





- time-dependent scattering length induces time-dependent metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{du^2}{1-\kappa u^2} + u^2 d\varphi^2\right)$$

- particle concept works well in regions I and III but not in region II
- vacuum state in region I leads to state with particles in region III
- expanding space leads to particle production
- analytic calculations possible for power law scale factors

$$a(t) = \text{const} \times t^{\gamma}$$

Laplace operator

- Laplace-Beltrami operator with spatial curvature

$$\Delta = \begin{cases} |\kappa| \left[\frac{1}{\sin\theta} \partial_\theta \left(\sin\theta \, \partial_\theta \right) + \frac{1}{\sin^2\theta} \partial_\varphi^2 \right] & \text{for} \quad \kappa > 0 \\ \partial_u^2 + \frac{1}{u} \partial_u + \frac{1}{u^2} \partial_\varphi^2 & \text{for} \quad \kappa = 0 \\ |\kappa| \left[\frac{1}{\sinh\sigma} \partial_\sigma \left(\sinh\sigma \, \partial_\sigma \right) + \frac{1}{\sinh^2\sigma} \partial_\varphi^2 \right] & \text{for} \quad \kappa < 0 \end{cases}$$

- eigenfunctions

$$\mathcal{H}_{km}(u, \varphi) = \begin{cases} Y_{lm}(\theta, \varphi) & \text{for} \quad \kappa > 0 & \text{with} \quad l \in \mathbb{N}_0, m \in \{-l, \dots, l\} \\ X_{km}(u, \varphi) & \text{for} \quad \kappa = 0 & \text{with} \quad k \in \mathbb{R}_0^+, m \in \mathbb{Z} \\ W_{lm}(\sigma, \varphi) & \text{for} \quad \kappa < 0 & \text{with} \quad l \in \mathbb{R}_0^+, m \in \mathbb{Z} \end{cases}$$

- eigenvalues with $k = |\kappa| l$

$$h(k) = \begin{cases} -k(k + \sqrt{|\kappa|}) & \text{for} \quad \kappa > 0 \\ -k^2 & \text{for} \quad \kappa = 0 \\ -\left(k^2 + \frac{1}{4}|\kappa|\right) & \text{for} \quad \kappa < 0 \end{cases}$$

Eigenfunctions

- positive spatial curvature $\kappa > 0$: spherical harmonics

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{(l-m)!}{(l+m)!}}\, e^{im\varphi}\, P_{lm}(\cos\theta),$$

- vanishing spatial curvature $\kappa = 0$: Bessel functions

$$X_{km}(u, \varphi) = e^{im\varphi}\, J_m(ku),$$

- negative spatial curvature $\kappa < 0$: sperical harmonics with complex angular momentum

$$W_{lm}(\sigma, \varphi) = (-i)^m \frac{\Gamma(il + 1/2)}{\Gamma(il + m + 1/2)}\, e^{im\varphi}\, P^m_{il-1/2}(\cosh\sigma),$$

Mode functions and Bogoliubov transforms

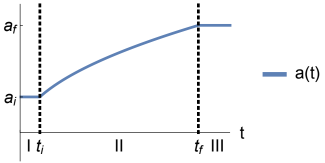
- field gets expanded in modes

$$\phi(t,u,\varphi) = \int_{k,m} \left[\hat{a}_{km}\mathcal{H}_{km}(u,\varphi)v_k(t) + \hat{a}^\dagger_{km}\mathcal{H}^*_{km}(u,\varphi)v^*_k(t)\right]$$

- temporal mode functions satisfy

$$\ddot{v}_k(t) + 2\frac{\dot{a}(t)}{a(t)}\dot{v}_k(t) + \frac{k^2+|\kappa|/4}{a^2(t)}v_k(t) = 0$$

- vacuum state only unique for $\dot{a}(t)=0$ where $v_k(t) \sim e^{-i\omega_k t}$
- Bogoliubov transforms between different choices of $\hat{a}_{km}$ and vacuum states



Bogoliubov transforms

- in region I one has positive frequency modes v_k and corresponding operators. Define vacuum

$$\hat{a}_{km}|\Omega\rangle = 0$$

- similar in region III positive frequency modes u_k with

$$\hat{b}_{km}|\Psi\rangle = 0$$

- Bogoliubov transform mediates between them

$$u_k = \alpha_k v_k + \beta_k v_k^*, \qquad v_k = \alpha_k^* u_k - \beta_k u_k^*$$

- operators are related by

$$\hat{b}_{km} = \alpha_k^* \hat{a}_{km} - \beta_k^* (-1)^m \hat{a}_{k,-m}^\dagger$$

- condition $|\alpha_k|^2 - |\beta_k|^2 = 1$
- constant term in spectrum $N_k = |\beta_k|^2$
- oscilating term $\Delta N_k = \mathrm{Re}[\alpha_k \beta_k e^{2i\omega_k t}]$

Cosmology in $d = 2+1$ spacetime dimensions

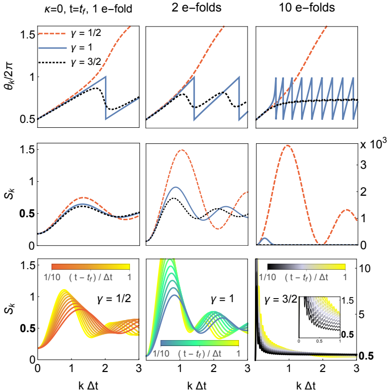
- analytic solutions for many choices of

$$a(t) = \text{const} \times t^{\gamma}$$

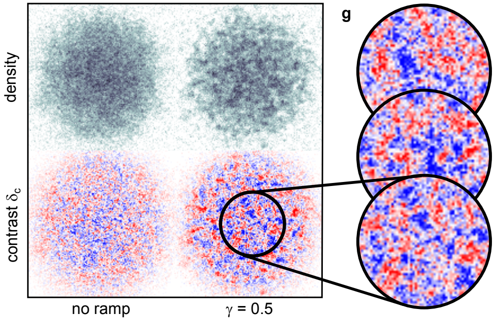
- correlation function in momentum space proportional to

$$S_k(t) = \frac{1}{2} + N_k + A_k \cos\left(\theta_k + 2\omega_k t\right)$$

- depends on number of e-folds, exponent γ and time after expansion ceases



Observation of particle production

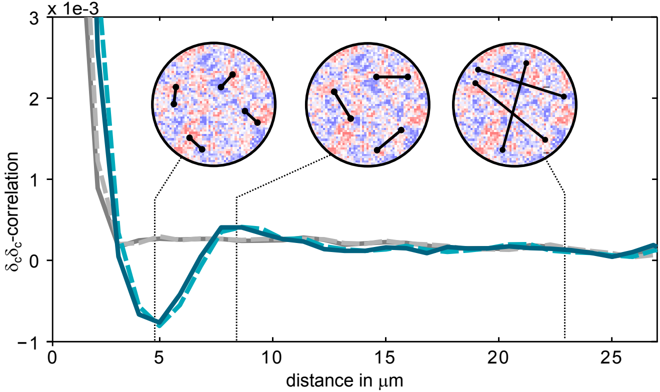


- rescaled density contrast

$$\delta_c(t,\mathbf{x}) = \sqrt{\frac{n_0(\mathbf{x})}{\bar{n}_0^3}}\,[n(t,\mathbf{x}) - n_0(\mathbf{x})]$$
$$\sim \partial_t \phi(t,\mathbf{x})$$

- allows to access correlation functions of relativistic scalar field by observation of density fluctuations

Density contrast correlation function

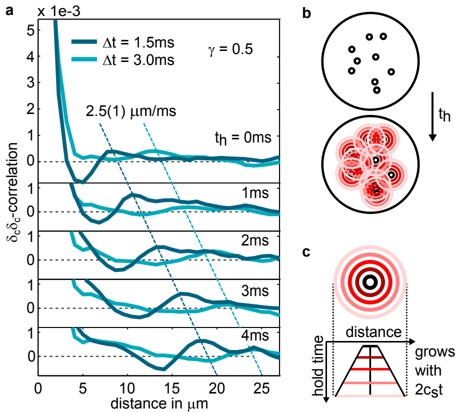


- correlation function

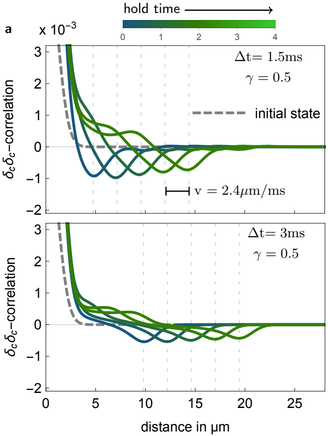
$$\langle \delta_c(\mathbf{x})\delta_c(\mathbf{y}) \rangle$$

- before and after expansion

Time dependent correlation functions after expansion



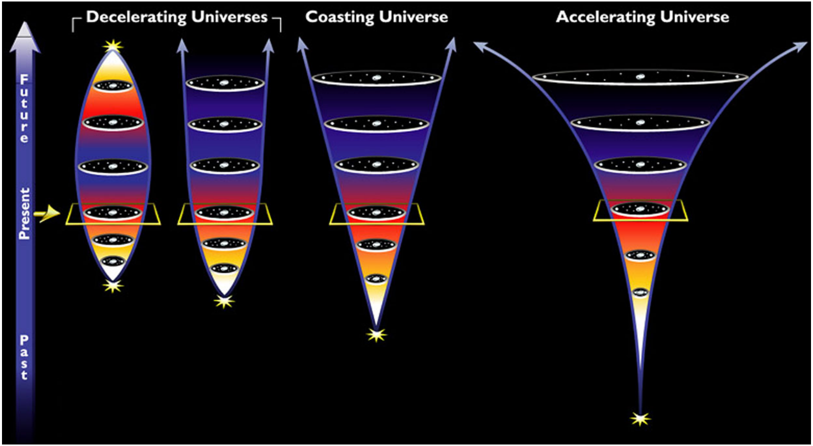
Experiment



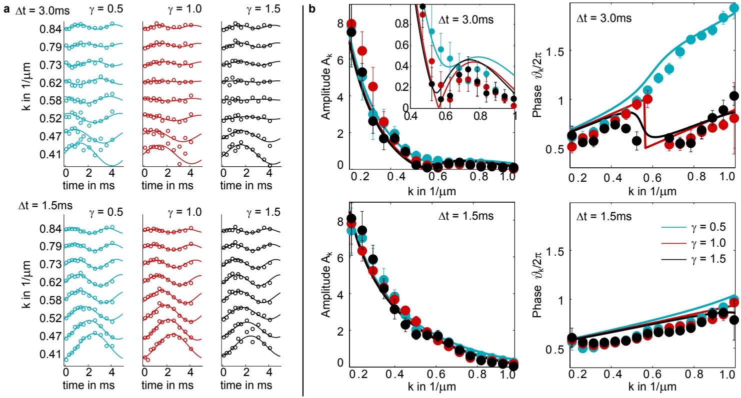
Theory

- analgous to baryon accoustic or Sakharov oscillations in cosmology
- optical resolution important for detailed shape

Expansion history



Oscillations in Fourier space



- Fourier spectrum of excitations

$$S_k(t) = \frac{1}{2} + N_k + A_k \cos(2\omega_k(t - t_{\mathrm{f}}) + \vartheta_k)$$

- decelerated, coasting and accelerated expansion
- good agreement with analytic theory (solid lines)

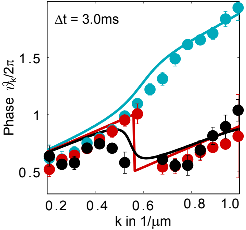
Quantum recurrences

- uniform expansion with $a(t) = Qt$ is special
- shows quantum recurrences of the incoming vacuum state at special values of wavenumber k

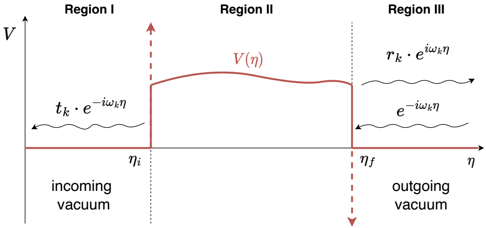
$$k_n = \frac{a_\mathsf{f} - a_\mathsf{i}}{\Delta t} \left[\left(\frac{n\pi}{\ln\left(a_\mathsf{f}/a_\mathsf{i}\right)} \right)^2 + \frac{1}{4} \right]^{\frac{1}{2}},$$

with integer $n = 1, 2, 3, \ldots$

- at these points one has trivial Bogoliubov coefficient $\beta_k = 0$
- can be seen experimentally as a discontinuity in the phase !



The scattering analogy



- evolution equation

$$\ddot{v}_k(t) + 2\frac{\dot{a}(t)}{a(t)}\dot{v}_k(t) + \frac{k^2 + |\kappa|/4}{a^2(t)}v_k(t) = 0$$

- can be rewritten with rescaled mode function and conformal time

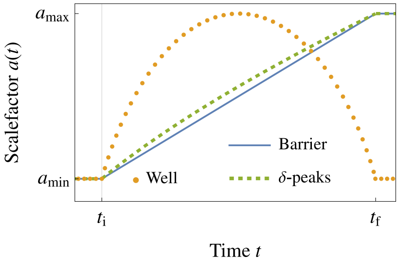
$$\psi_k(\eta) = \sqrt{a(t)}v_k(t), \qquad dt = a(t)d\eta$$

- results in stationary Schrödinger equation

$$\frac{d^2}{d\eta^2}\psi_k(\eta) + [E - V(\eta)]\,\psi_k(\eta) = 0$$

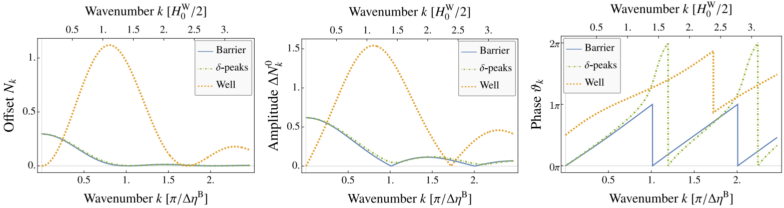
with $V(\eta) = \dot{a}^2/4 + \ddot{a}a/2$ and $E = -h(k) = k^2$

Some example potentials



- potential $V(\eta) = \dot{a}^2/4 + \ddot{a}a/2$ has Dirac peaks when $\dot{a}$ has discontinuity
- coasting universe $a \sim t$ leads to square barrier
- "radiation dominated" universe $a \sim t^{2/3}$ has only Dirac peaks
- particular anti-bounce leads to square well

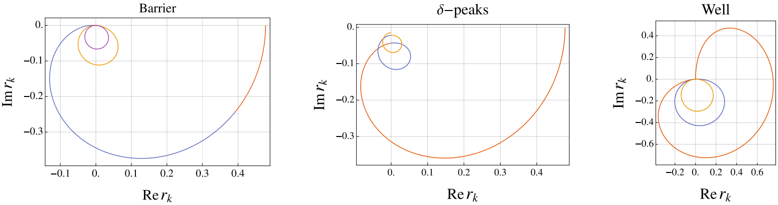
Resulting particle spectra



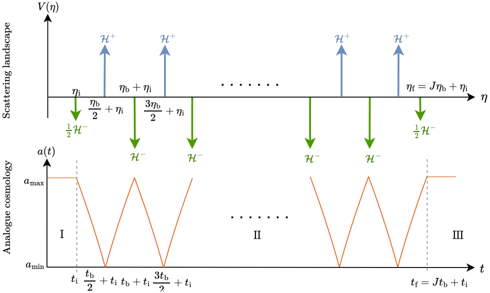
- resulting particle spectra

$$S_k(t) = \frac{1}{2} + N_k + \Delta N_k^0 \cos(2\omega_k(t - t_{\mathrm{f}}) + \vartheta_k)$$

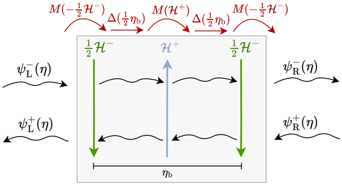
- reflection amplitude has zero crossings that explain phase jumps

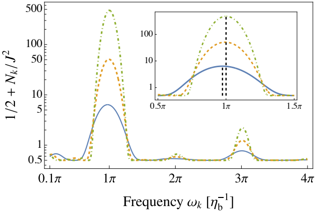


Periodic universes



- combination of expanding and contracting phases where $a \sim t^{2/3}$
- potential landscape with attractive and repulsive Dirac peaks
- can be solved with transfer matrix method



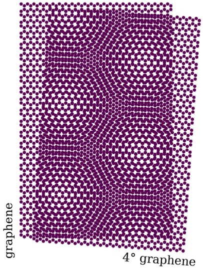


Relativistic fermions in materials

- low energy theory of Dirac materials

$$\Gamma[\Psi] = \int dt d^2x \left\{ -\bar{\Psi} \left[\gamma^0 \partial_t + v_F(t) \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + \Delta(t)\Gamma \right] \Psi \right\}$$

- time dependent Fermi velocity $v_F(t)$
  - change in twist angle for bilayer graphene
  - change in pressure
  - light pulses
- time-dependent gap or mass parameter $\Delta(t)\Gamma$ can be
  - breaking spatial inversion $\Gamma = \mathbb{1}$
  - Kekulé modulation of hoping $\Gamma = \gamma^3 \cos(\alpha) + \gamma^5 \sin(\alpha)$
  - Haldane mass breaking time parity $\Gamma = \gamma^{35}$
- can be manipulated with fast electronics



Fermions in curved spacetime

- action for Dirac fermions in general spacetime

$$\Gamma[\Psi] = \int dt d^2x \sqrt{g} \left\{ -\bar{\Psi} \left[\gamma^{\alpha} e_{\alpha}{}^{\mu} \partial_{\mu} (\partial_{\mu} + \Omega_{\mu}) + m\Gamma \right] \Psi \right\}$$

- tetrad field $e_{\alpha}{}^{\mu}$ inverse to $e^{\alpha}{}_{\mu}$ so that

$$g_{\mu\nu}(x) = e^{\alpha}{}_{\mu}(x) e^{\beta}{}_{\nu}(x) \eta_{\alpha\beta}$$

- spin connection $\Omega_{\mu} = \omega_{\mu\alpha\beta}[\gamma^{\alpha}, \gamma^{\beta}]/8$ with

$$\omega_{\mu\alpha\beta} = -\eta_{\alpha\gamma} \left[\partial_{\mu} e^{\gamma}{}_{\nu} - \Gamma^{\rho}_{\mu\nu} e^{\gamma}{}_{\rho} \right] e_{\beta}{}^{\nu}$$

  and Levi-Civita connection $\Gamma^{\rho}_{\mu\nu}$
- local Lorentz transformations
- general coordinate transformations

Weyl scaling transformation

- transform Dirac fields (with conformal weight $\Delta_\Psi = (d-1)/2 = 1$)

$$\Psi(x) \to e^{-\zeta(x)}\Psi(x), \qquad \bar{\Psi}(x) \to e^{-\zeta(x)}\bar{\Psi}(x)$$

- transform tetrad field

$$e^{\alpha}{}_{\mu}(x) \to e^{\zeta}(x) e^{\alpha}{}_{\mu}(x)$$

  and accordingly metric like

$$g_{\mu\nu}(x) \to e^{2\zeta(x)} g_{\mu\nu}(x)$$

- spin connection transforms like

$$\omega_{\mu\alpha\beta} \to \omega_{\mu\alpha\beta} + \left[e_{\alpha\mu} e_{\beta}{}^{\nu} - e_{\beta\mu} e_{\alpha}{}^{\nu}\right] \partial_\nu \zeta$$

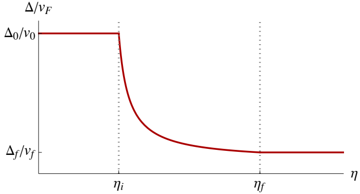
- gap term is **not** invariant

$$m\Gamma = e^{\zeta(x)} m\Gamma$$

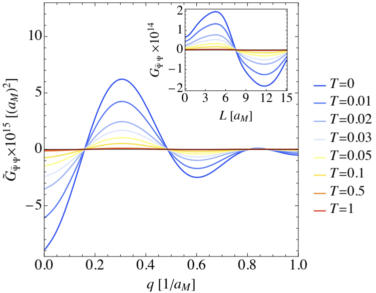
- allows to transform a time-dependent mass term into a constant mass term
- only ratio $\Delta(t)/v_F(t)$ matters for particle production

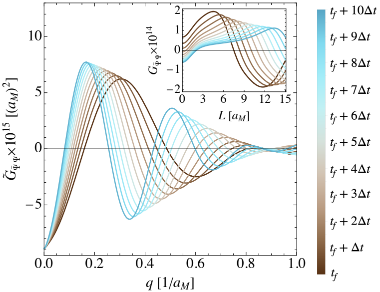
Fermionic particle production

- time dependence of ratio Δ/v_F



- leads to particle production





Conclusion

- Bose-Einstein condensates can act as quantum simulators for quantum fields in curved spacetime
- symmetric spaces with constant curvature can be realized with specific density profiles
- experimental realization achieved in two spatial dimensions
- time-dependent coupling allows to simulate expansion or contraction
- particle production
- Sacherov oscillations after expansion allow detailed investigations
- scattering analogy picture allows to gain insights into many possible "cosmologies"
- fermion production in expanding geometry could be realized with Dirac materials
- extensions to three dimensions, other geometries, different field content, and more, to come
- Geometric fields (metric, tetrad, spin connection, Weyl gauge fields, ...) allow to study very interesting regime of non-equilibrium physics

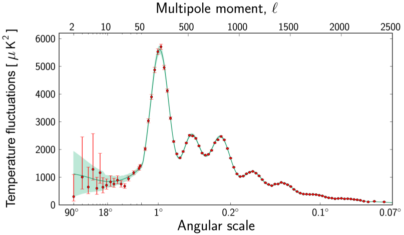
Backup

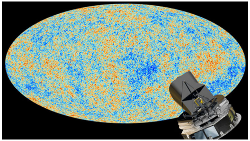
Symmetries and Wigners classification

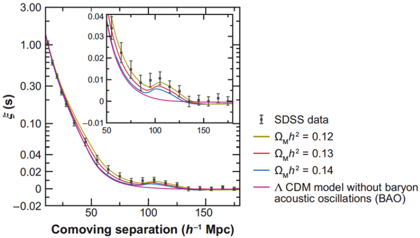
Particles as representations of space-time symmetries [Eugene P. Wigner (1939)]

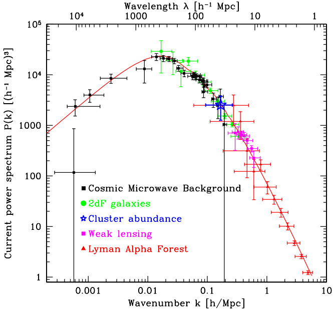
- translations in space and time $\leftrightarrow$ momentum, energy, mass
- rotations and Lorentz boosts $\leftrightarrow$ spin / helicity
- what happens when translational symmetries get broken?

Baryon acoustic oscillations

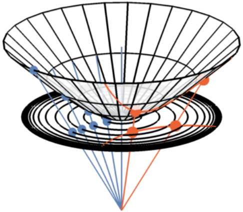








Hyperbolic geometry in Minkowski space



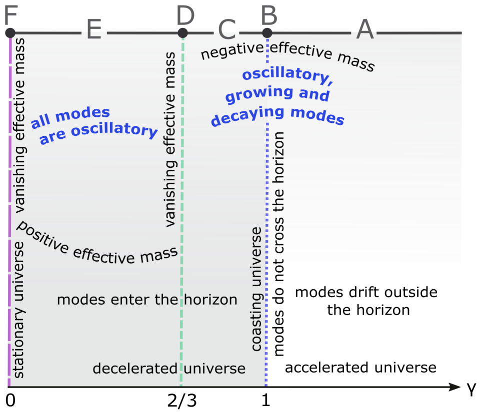
- start with Minkowski space $ds^2 = dX^2 + dY^2 - dZ^2$
- consider hyperboloid ("mass shell") $X^2 + Y^2 - Z^2 = -R^2/4$
- stereographic projection to Poincaré disc

Horizon crossing

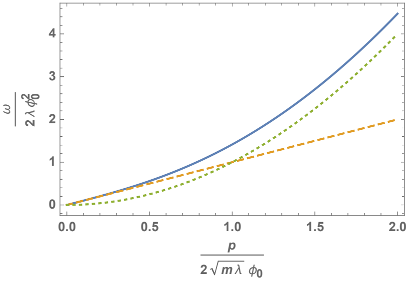
- power law expansion

$$a(t) = \text{const} \times t^{\gamma}$$

- can be decelerating, coasting or accelerating



Bogoliubov dispersion relation



- Quadratic part of action for excitations

$$S_2 = \int dt\ d^3x \left\{ -\frac{1}{2}(\phi_1, \phi_2) \begin{pmatrix} -\frac{\nabla^2}{2m} + 2\lambda n_0 & \partial_t \\ -\partial_t & -\frac{\nabla^2}{2m} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\}$$

- Dispersion relation

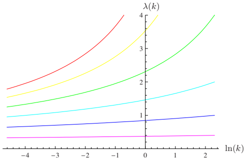
$$\omega = \sqrt{\left(\frac{\mathbf{p}^2}{2m} + 2\lambda\phi_0^2\right)\frac{\mathbf{p}^2}{2m}}$$

becomes linear for

$$\mathbf{p}^2 \ll 4\lambda m n_0 = \frac{2}{\xi^2}$$

Renormalization in two dimensions

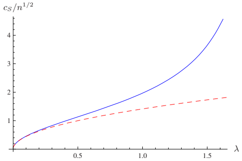
[S. Floerchinger, C. Wetterich, *Superfluid Bose gas in two dimensions*, PRA 79, 013601 (2009)]

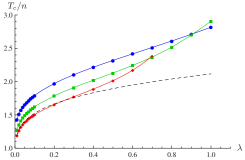


- scale-dependent coupling in two dimensions

$$k\frac{\partial}{\partial k}\lambda = \frac{\lambda^2}{4\pi}$$

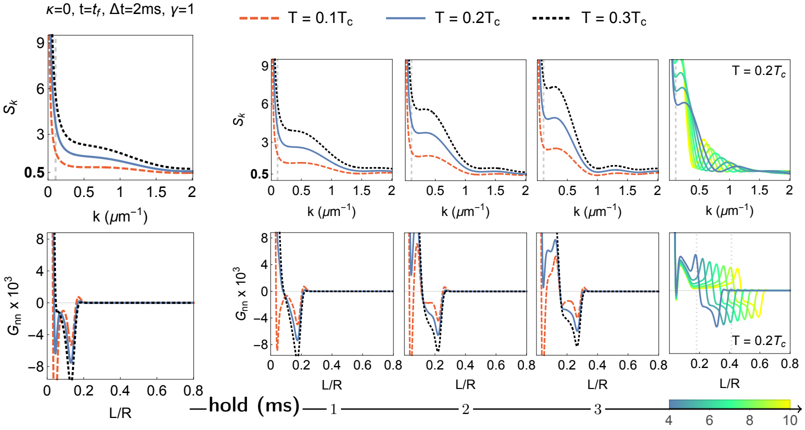
- sound velocity and critical temperature



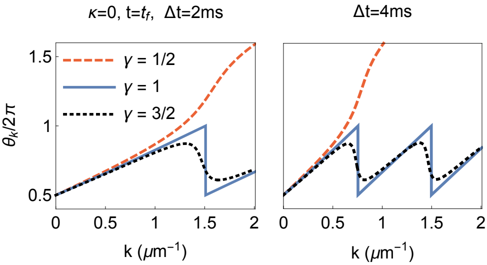


Temperature dependence

- initial state not necessarily vacuum
- allow finite temperature T, leads to enhanced fluctuations

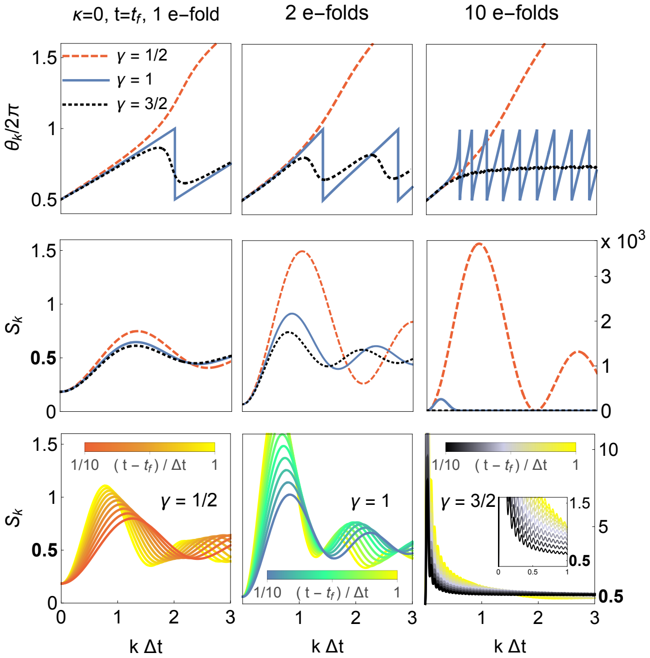


Phases



κ=0, t=t_f, Δt=2ms
Δt=4ms
γ = 1/2
γ = 1
γ = 3/2
$\theta_k/2\pi$
1.5
1
0.5
0
0.5
1
1.5
2
k (μm^{-1})
k (μm^{-1})

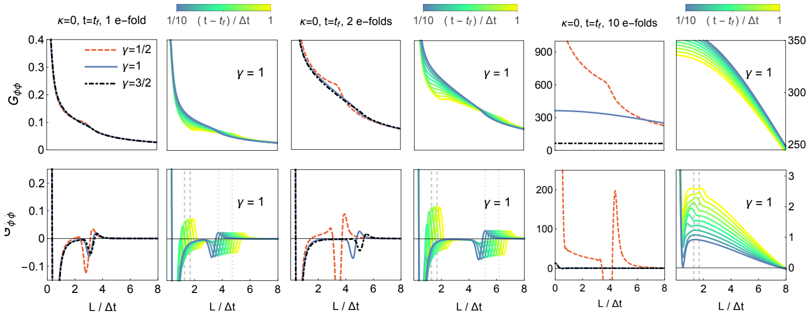
More e-folds



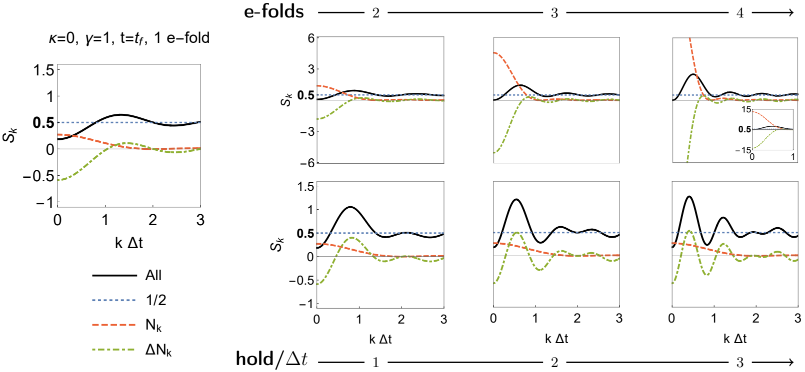
κ=0, t=t_f, 1 e−fold
2 e−folds
10 e−folds
γ = 1/2
γ = 1
γ = 3/2
θ_k/2π
S_k
x 10^3
1/10 (t − t_f) / Δt 1
γ = 1/2
γ = 1
γ = 3/2
k Δt

Correlation functions

- correlation functions in position space with Gaussian window function for UV regularization



Power spectra



Horizons and inflation