A simple Hamiltonian for quantum simulation of QCD

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Main message

- Kogut-Susskind formulation for quantum is great.

However:

- You shouldn’t use Kogut-Susskind formulation for quantum simulation of SU(N) theory (N>2).

- Orbifold lattice is better.

Kogut-Susskind is hard.

$$\left[\hat{E}^{\alpha}_{\mu,\vec{n}}, \hat{U}_{\nu,\vec{n}'}\right] = a^{-2} g \delta_{\mu\nu} \delta_{\vec{n}\vec{n}'} \tau_{\alpha} \hat{U}_{\nu,\vec{n}'}$$

$$\left[\hat{E}^{\alpha}_{\mu,\vec{n}}, \hat{E}^{\beta}_{\nu,\vec{n}'}\right] = -i f^{\alpha\beta\gamma} a^{-2} g \delta_{\mu\nu} \delta_{\vec{n}\vec{n}'} \hat{E}^{\gamma}_{\nu,\vec{n}'}$$

$$\hat{U} \left|U\right\rangle = U \left|U\right\rangle$$

$$U \in \mathrm{SU}(N)$$

complicated

$$\hat{U}\,|U\rangle = U\,|U\rangle$$

$$U \in \mathrm{SU}(N)$$

Lattice regularisation is nightmare

$$\hat{x}\,|x\rangle = x\,|x\rangle$$

$$x \in \mathbb{R}$$

Lattice regularisation is trivial

$$\hat{U}\,|U\rangle = U\,|U\rangle$$

$$U \in \mathrm{SU}(N)$$

Lattice regularisation is nightmare

Electric field ~ momentum $\hat{E}$

$$\langle U|R, ij\rangle = \rho^{(R)}_{ij}(U)$$

'plane wave' = irreducible representation

Fourier transform is nightmare

$$\hat{x}\,|x\rangle = x\,|x\rangle$$

$$x \in \mathbb{R}$$

Lattice regularisation is trivial

momentum $\hat{p}$

$$\langle x|p\rangle = e^{ipx}$$

$$[\hat{x}, \hat{p}] = i$$

Fourier transform is trivial

$\boxed{\hat{x} \text{ and } \hat{p} \text{ are simpler}}$

- Fock basis truncation

$$\hat{a} = \frac{\hat{x}+i\hat{p}}{\sqrt{2}} \qquad \hat{a}\left|0\right\rangle = 0$$

$$\left|n\right\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\left|0\right\rangle \qquad 0 \leq n < \Lambda$$

- Coordinate basis truncation

$$\hat{x}\left|x\right\rangle = x\left|x\right\rangle \qquad -R \leq x \leq R$$

$$x_n = -R + n\delta_x, \qquad \delta_x = \frac{2R}{\Lambda - 1}$$

$$n = 0, 1, \cdots, \Lambda - 1$$

$$\boxed{\hat{x} \text{ and } \hat{p} \text{ are simpler}}$$

→ Scalar QFT is simpler

$$\hat{x}, \hat{p} \leftrightarrow \hat{\phi}_{\vec{n}}, \hat{\pi}_{\vec{n}}$$

(e.g., Jordan, Lee, Preskill, 2011)

$$[\hat{\phi}_{\vec{n}}, \hat{\pi}_{\vec{n}'}] = i\delta_{\vec{n}\vec{n}'}$$

→ Matrix model is simpler

$$\hat{x}, \hat{p} \leftrightarrow \hat{X}_{M,ij}, \hat{P}_{M,ij}$$

(e.g., Gharibyan, MH, Honda, Liu, 2021
Maldacena, 2023)

Orbifold lattice construction

(Kaplan, Katz, Unsal, 2003)

Original motivation = supersymmetric lattice

$$\text{SUSY Matrix model} \xrightarrow[\text{projection}]{\text{orbifold}} \text{SUSY Lattice}$$

scalar field

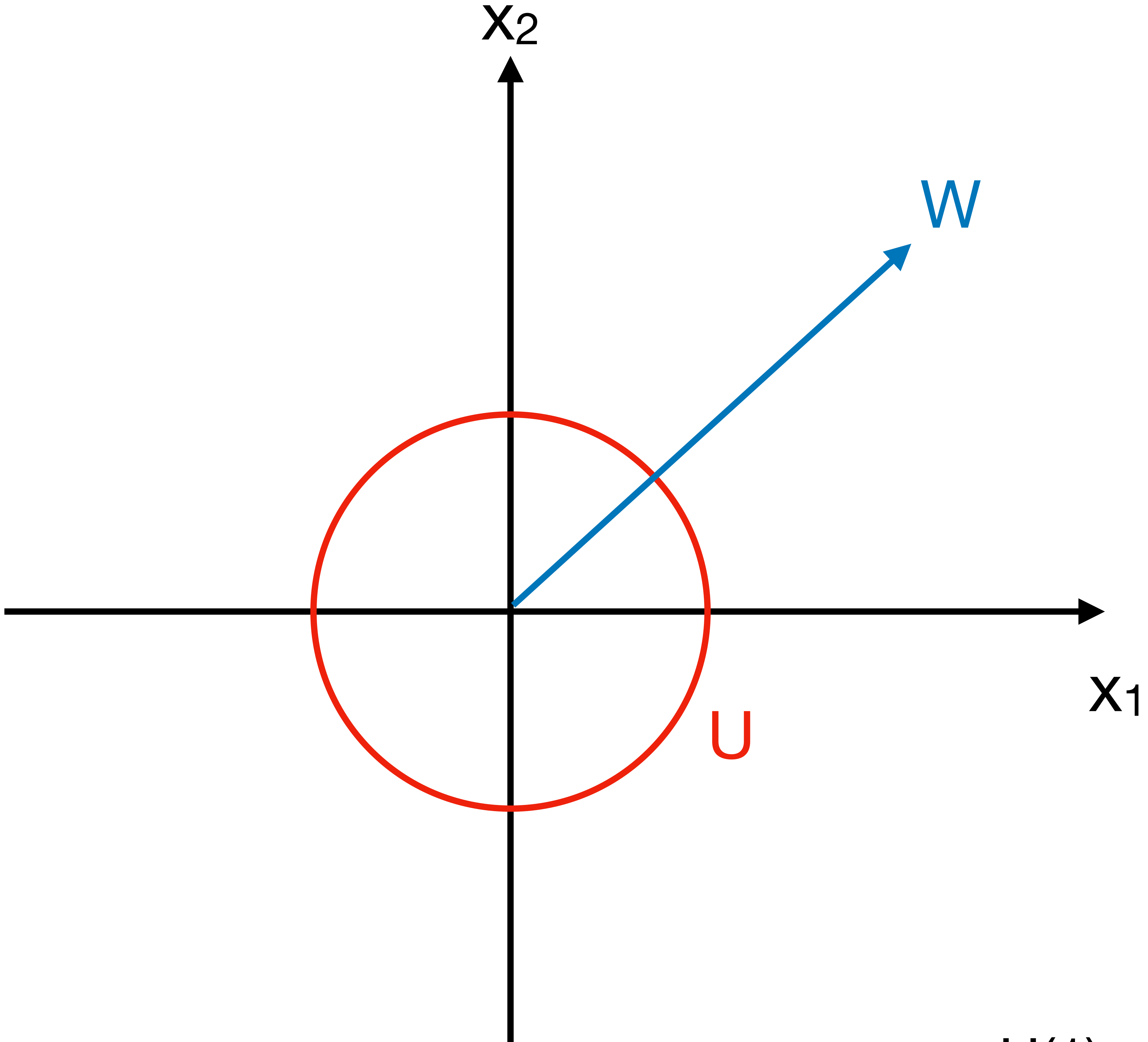
$$X_1 + iX_2 = Z = c \cdot W \cdot U = c \cdot e^{as} \cdot e^{iaA}$$

gauge field

positive definite Hermitian

unitary

$$L = \int d^3x \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(D_\mu s_I)^2 + \frac{g_{4\mathrm{d}}^2}{4}[s_I, s_J]^2\right)$$



U(1) case

Forget about SUSY

Orbifold lattice for Yang-Mills

(Gharibyan, MH, Honda, Liu, 2020; Bergner, MH, Rinaldi, Schafer, 2024)

Bosonic part

$$\hat{H} = \sum_{\vec{n}} \mathrm{Tr}\left(\sum_{j=1}^{3} \hat{P}_{j,\vec{n}}\hat{\bar{P}}_{j,\vec{n}} + \frac{g_{4\mathrm{d}}^2}{2a^3}\left|\sum_{j=1}^{3}\left(\hat{Z}_{j,\vec{n}}\hat{\bar{Z}}_{j,\vec{n}} - \hat{\bar{Z}}_{j,\vec{n}-\hat{j}}\hat{Z}_{j,\vec{n}-\hat{j}}\right)\right|^2 + \frac{2g_{4\mathrm{d}}^2}{a^3}\sum_{j<k}\left|\hat{Z}_{j,\vec{n}}\hat{Z}_{k,\vec{n}+\hat{j}} - \hat{Z}_{k,\vec{n}}\hat{Z}_{j,\vec{n}+\hat{k}}\right|^2\right) + \Delta\hat{H}\,.$$

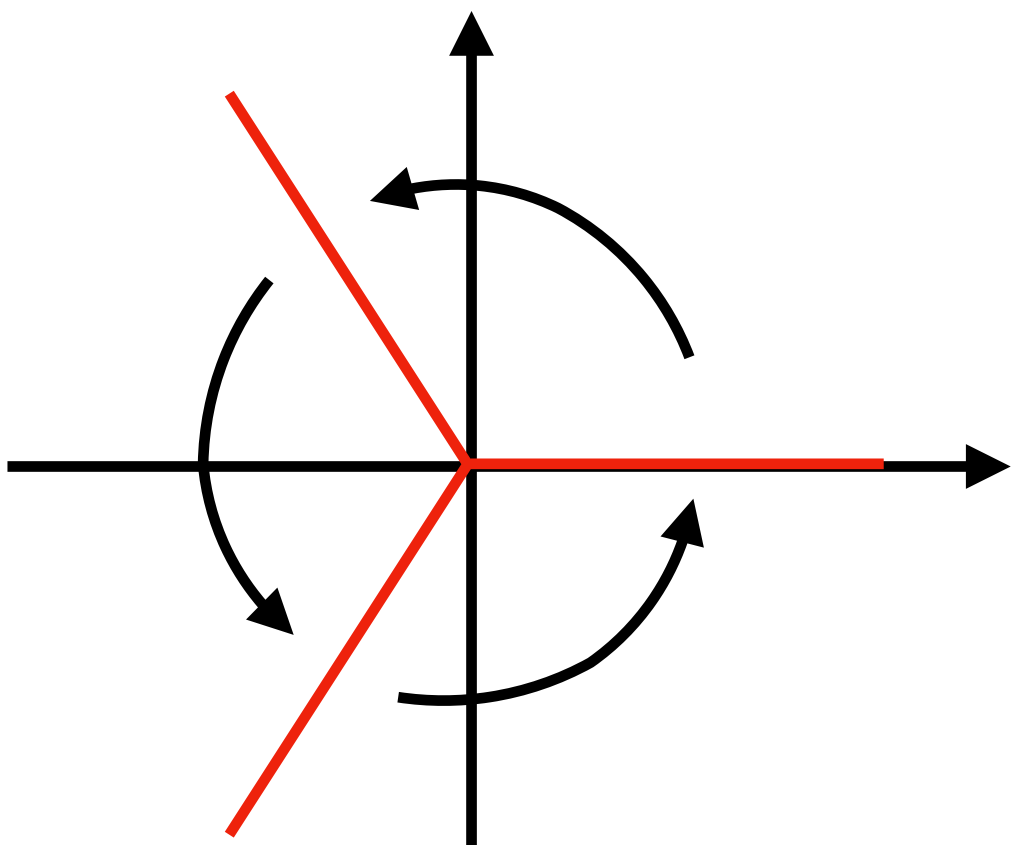
$$\Delta\hat{H} \equiv \frac{m^2 g_{4\mathrm{d}}^2}{2a}\sum_{\vec{n}}\sum_{j=1}^{3}\mathrm{Tr}\left|\hat{Z}_{j,\vec{n}}\hat{\bar{Z}}_{j,\vec{n}} - \frac{a}{2g_{4\mathrm{d}}^2}\right|^2 + \frac{\mu^2 g_{4\mathrm{d}}^2}{2a}\sum_{\vec{n}}\sum_{j=1}^{3}\left|\frac{1}{N}\mathrm{Tr}(\hat{Z}_{j,\vec{n}}\hat{\bar{Z}}_{j,\vec{n}}) - \frac{a}{2g_{4\mathrm{d}}^2}\right|^2$$

Scalar mass term
→YM at low energy

$$[\hat{Z}_{j,\vec{n},pq}, \hat{\bar{P}}_{k\vec{n}',rs}] = i\delta_{jk}\delta_{\vec{n}\vec{n}'}\delta_{ps}\delta_{qr}$$

history

Why is it called "orbifold"?

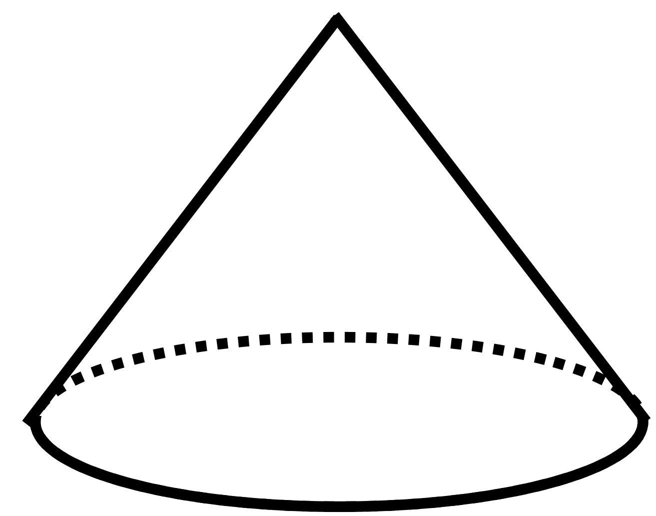


$\mathbb{Z}_k$ transformation

$$z \to e^{i\theta} z$$
$$k\theta \equiv 0 \bmod 2\pi$$

orbifold projection

$$z \sim e^{i\theta} z$$



history

Why is it called "orbifold"?

$$\mathrm{AdS}_5 \times \mathrm{S}^5 \quad \Longrightarrow \quad \mathrm{AdS}_5 \times (\mathrm{S}^5/\mathbb{Z}_k)$$

orbifold projection

$$z \sim e^{i\theta} z$$

history

Why is it called "orbifold"?

$$\mathrm{AdS}_5 \times \mathrm{S}^5 \longrightarrow \mathrm{AdS}_5 \times (\mathrm{S}^5/\mathbb{Z}_k)$$

orbifold projection

$$z \sim e^{i\theta} z$$

$$\boxed{\begin{aligned} Z_1 &= X_1 + iX_2 \\ Z_2 &= X_3 + iX_4 \\ Z_3 &= X_5 + iX_6 \end{aligned}}$$

Maximally supersymmetric Yang-Mills theory → Quiver gauge theory

orbifold projection

history

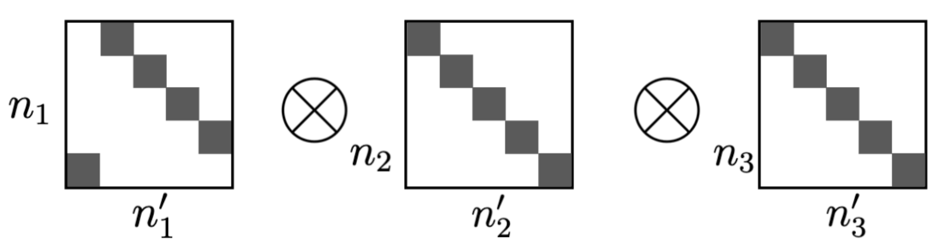
Why is it called "orbifold"?

Matrix Model → Orbifold Lattice

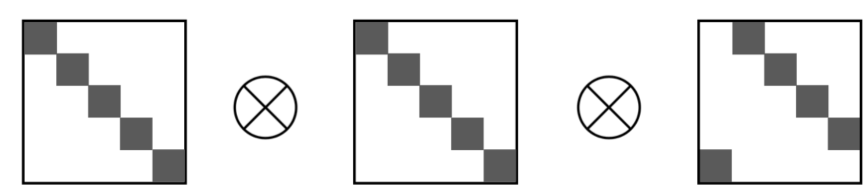
(Special kind of quiver gauge theory)

$$\mathrm{Tr}\left(\frac{1}{2}\sum_{I}(D_tX_I)^2+\frac{g_{1\mathrm{d}}^2}{4}\sum_{I,J}[X_I,X_J]^2\right)$$

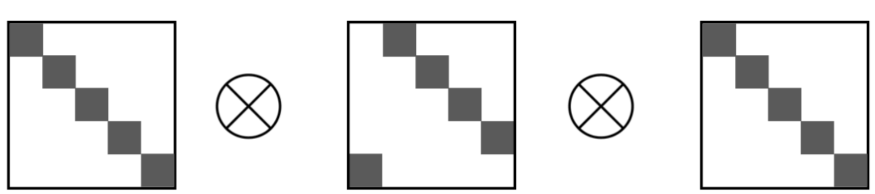
$$x_{\vec{n},pq}\equiv x_{n_1,n_2,n_3,p;n_1+1,n_2,n_3,q}$$



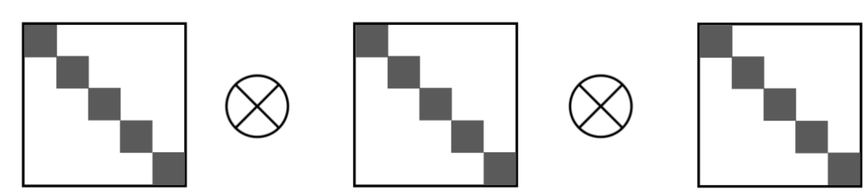
$$z_{\vec{n},pq}\equiv z_{n_1,n_2,n_3,p;n_1,n_2,n_3+1,q}$$



$$y_{\vec{n},pq}\equiv y_{n_1,n_2,n_3,p;n_1,n_2+1,n_3,q}$$



$$A^t_{\vec{n},pq}\equiv A^t_{n_1,n_2,n_3,p;n_1,n_2,n_3,q}$$



Forget about SUSY

Orbifold lattice for QCD

Fermionic part

(Bergner, MH, Rinaldi, Schafer, 2024)

$$\hat{H}_{\text{naive}} = a^3 \sum_{\vec{n}} \left\{ \frac{i}{2a} \sum_{j=1}^{3} \left(\hat{\bar{\psi}}_{\vec{n}} \gamma^j \hat{U}_{j,\vec{n}} \hat{\psi}_{\vec{n}+\hat{j}} - \hat{\bar{\psi}}_{\vec{n}+\hat{j}} \gamma^j \hat{U}^\dagger_{j,\vec{n}} \hat{\psi}_{\vec{n}} \right) + m \hat{\bar{\psi}}_{\vec{n}} \hat{\psi}_{\vec{n}} \right\}$$

$$\Downarrow$$

$$\hat{H}_{\text{naive}} = a^3 \sum_{\vec{n}} \left\{ \frac{i}{2a} \sqrt{\frac{2g_{\text{4d}}^2}{a}} \sum_{j=1}^{3} \left(\hat{\bar{\psi}}_{\vec{n}} \gamma^j \hat{Z}_{j,\vec{n}} \hat{\psi}_{\vec{n}+\hat{j}} - \hat{\bar{\psi}}_{\vec{n}+\hat{j}} \gamma^j \hat{\bar{Z}}_{j,\vec{n}} \hat{\psi}_{\vec{n}} \right) + m \hat{\bar{\psi}}_{\vec{n}} \hat{\psi}_{\vec{n}} \right\}$$

Forget about SUSY

Orbifold lattice for QCD

(Bergner, MH, Rinaldi, Schafer, 2024)

Fermionic part

$$\hat{H}_{\mathrm{Wilson}} = a^3 \cdot \frac{i}{2a} \sum_{j=1}^{3} \sum_{\vec{n}} \left(\hat{\bar{\psi}}_{\vec{n}} - \hat{\bar{\psi}}_{\vec{n}+\hat{j}} \hat{U}^{\dagger}_{j,\vec{n}} \right) \left(\hat{\psi}_{\vec{n}} - \hat{U}_{j,\vec{n}} \hat{\psi}_{\vec{n}+\hat{j}} \right)$$

$$\hat{H}_{\mathrm{Wilson}} = a^3 \cdot \frac{i}{2a} \sum_{j=1}^{3} \sum_{\vec{n}} \left(\hat{\bar{\psi}}_{\vec{n}} - \sqrt{\frac{2g_{4\mathrm{d}}^2}{a}} \hat{\bar{\psi}}_{\vec{n}+\hat{j}} \hat{\bar{Z}}_{j,\vec{n}} \right) \left(\hat{\psi}_{\vec{n}} - \sqrt{\frac{2g_{4\mathrm{d}}^2}{a}} \hat{Z}_{j,\vec{n}} \hat{\psi}_{\vec{n}+\hat{j}} \right)$$

Forget about SUSY

Orbifold lattice for QCD

(Bergner, MH, Rinaldi, Schafer, 2024)

U(N) → SU(N)

$$Z = c \cdot W \cdot U$$

Complex

U(N)

- No difference in the IR if U(1) is not asymptotically free at UV
- To remove U(1) explicitly, we can add $\Delta \hat{H} \propto |\mathrm{Im}\det \hat{Z}|^2$

Easy for SU(2) and SU(3).
Harder for larger N.

However, SU(∞) = U(∞)

How easy?

$$\hat{H} = \sum_{\vec{n}} \mathrm{Tr}\left(\sum_{j=1}^{3} \hat{P}_{j,\vec{n}} \hat{\bar{P}}_{j,\vec{n}} + \frac{g_{4\mathrm{d}}^2}{2a^3} \left| \sum_{j=1}^{3} \left(\hat{Z}_{j,\vec{n}} \hat{\bar{Z}}_{j,\vec{n}} - \hat{\bar{Z}}_{j,\vec{n}-\hat{j}} \hat{Z}_{j,\vec{n}-\hat{j}} \right) \right|^2 \right.$$

$$\left. + \frac{2g_{4\mathrm{d}}^2}{a^3} \sum_{j<k} \left| \hat{Z}_{j,\vec{n}} \hat{Z}_{k,\vec{n}+\hat{j}} - \hat{Z}_{k,\vec{n}} \hat{Z}_{j,\vec{n}+\hat{k}} \right|^2 \right) + \Delta \hat{H} \, .$$

$$\Delta \hat{H} \equiv \frac{m^2 g_{4\mathrm{d}}^2}{2a} \sum_{\vec{n}} \sum_{j=1}^{3} \mathrm{Tr} \left| \hat{Z}_{j,\vec{n}} \hat{\bar{Z}}_{j,\vec{n}} - \frac{a}{2g_{4\mathrm{d}}^2} \right|^2 \quad .$$

$$+ \frac{\mu^2 g_{4\mathrm{d}}^2}{2a} \sum_{\vec{n}} \sum_{j=1}^{3} \left| \frac{1}{N} \mathrm{Tr}(\hat{Z}_{j,\vec{n}} \hat{\bar{Z}}_{j,\vec{n}}) - \frac{a}{2g_{4\mathrm{d}}^2} \right|^2$$

4-boson interaction $\hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_4$

$$x_n = -R + n\delta_x, \qquad \delta_x = \frac{2R}{\Lambda - 1}$$

$$n = 0, 1, \cdots, \Lambda - 1$$

$$\hat{x} = \sum_{n=0}^{\Lambda-1} x_n \left|n\right\rangle \left\langle n\right| = -R \cdot \mathbf{1} + \delta_x \cdot \hat{n}$$

$$\left|n\right\rangle = \left|b_1\right\rangle \left|b_2\right\rangle \cdots \left|b_K\right\rangle \ , \qquad b_i = 0 \text{ or } 1\,, \qquad n = b_1 + 2b_2 \cdots + 2^{K-1} b_K$$

$$\hat{n} = \frac{\hat{\sigma}_{z,1} + 1}{2} + 2 \cdot \frac{\hat{\sigma}_{z,2} + 1}{2} + \cdots + 2^{K-1} \cdot \frac{\hat{\sigma}_{z,K} + 1}{2}$$

$$\boxed{\hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_4 \sim \sum \hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \hat{\sigma}_z}$$

$$\hat{p}^2 = \frac{1}{\delta_X^2}\sum_{n=0}^{\Lambda-1}\{2\left|n\right\rangle\left\langle n\right| - \left|n+1\right\rangle\left\langle n\right| - \left|n\right\rangle\left\langle n+1\right|\}$$

identity

just add or subtract 1

- Just add or subtract 1.
- Or, diagonalize it via Quantum Fourier Transform.

Fourier transform is easy,
unlike Kogut-Susskind

$$\Delta\hat{H} \propto |\mathrm{Im}\det\hat{Z}|^2 \quad \leftarrow \quad \hat{\sigma}_z^{\otimes 6}$$ for SU(3)

Future Directions

- Study of QCD via orbifold lattice
  - How large scalar mass? RG flow? Better lattice fermions?
  - Euclidean lattice simulation is enough
- Resource estimate for quantum simulation
- Quantum algorithm
- Quantum simulation of matrix model (before QCD)
  - Quantum Gravity in the Lab
  - SYK and spin models are technically similar to orbifold lattice