Topological Quantum Field Theory lecture notes

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Caution: These preliminary notes are likely to have errors and typos!
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Contents

1 Introduction

First of all, by topological quantum field theory (TQFT), we mean a QFT which has no dependence on a spacetime metric. This is in stark contrast to "ordinary" QFTs which normally have a dependence on the metric and distances and so on. For example we deal with correlation functions $\langle \theta_1(x_1), \ldots, \theta_n(x_n) \rangle$ which depend on coordinates. These have singularities when operators "approach each other" and this involves some notion of distance.

TQFTs instead are simple models of QFTs in which there are no singularities and everything is simpler and easier to mathematically define. In particular there is an axiomatic mathematical formulation of a TQFT.

But then a natural question arises: does it have applications for physics? Is it useful for mathematics? In physics, TFTs arise a sector of a larger QFT, typically in supersymmetric theories. They also arise in condensed matter systems, an important example being that of the integer quantum hall effect. Other applications include associating TQFTs to "ordinary" QFTs. From the math point of view, from TQFTs one can obtain topological invariants and knot invariants as path integrals or expectation values of extended operators.

The plan for the course is the following:

- Discussion of topology/geometry in quantum mechanical systems.
- Chern Simons theory as an example of a TQFT.
- Axiomatic definition of TQFT.
- Examples.
- Supersymmetry and supersymmetric twists.
- Further possible topics include:
  - TQFTs with boundary or defects,
  - Extended TQFTs.

2 Topology and Geometry in quantum mechanical systems

Keywords: canonical quantization, Aharonov Bohm, magnetic monopoles.

2.1 Classical mechanics: Hamiltonian formalism

If we consider a particle moving in $\mathbb{R}^3$ we have a phase space $\mathbb{R}^6$ indicating the position q and momentum q of the particle (q, p). If we were to consider n particles the phase space would be $\mathbb{R}^{6n}$. Let us indicate the phase space by Γ. The Hamiltonian is a function on phase space, i.e. a function $H : \Gamma \to \mathbb{R}$ which for a free particle (or n free particles) is given by:

$$H = \frac{p^2}{2m} \quad \text{or} \quad \sum_i \frac{p_i^2}{2m}$$

The equations of motion then are given by:

$$\dot{q}_j = \frac{\partial H}{\partial p_j}$$
$$\dot{p}_j = -\frac{\partial H}{\partial q_j}$$

The Hamiltonian is the first example of an observable, i.e. a function $\Gamma \to \mathbb{R}$. I can also regard the single q_i and p_i as observables, they're the functions which give me that component of the position or momenta at every point in Γ.

We also have an additional structure given by the Poisson bracket between observables:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{1}$$

Considering the observables q_i and p_i we get the following relations:

$$\{q_i, q_j\} = 0$$
$$\{p_i, p_j\} = 0$$
$$\{q_i, p_j\} = \delta_{ij}$$

The last two relations can be rendered more general, by considering a general observable $f(q,p)$:

$$\frac{d}{dt}f = \sum\left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right) = \sum\left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right) = \{f, H\}$$

2.2 Geometry of phase space

The simplest case is to simply consider $\mathbb{R}^{6n}$, however this comes with additional structure. Let us start with an interlude on linear algebra to recall the necessary tools.

2.2.1 Interlude: linear algebra

It's natural to think of maps between vector spaces which "preserve the structure" that vector spaces have, that is addition and multiplication by a scalar. In particular, consider a map $A : V \to W$ between vector spaces V and W over a field F. This map is linear if it satisfies the following:

$$A(\lambda \otimes v \oplus \mu \otimes w) = \lambda \cdot A(v) + \mu \cdot A(w)$$

In which we use different symbols to denote scalar multiplication and vector addition in the two vector spaces $(V, \otimes, \oplus)$ and $(W, \cdot, +)$. We only explicitly state this here as it will be useful when going to structures that are more general than vector spaces. In the following we will often not use any symbol for scalar multiplication and denote vector sum by $+$. Linearity is then simply:

$$A(\lambda v + \mu w) = \lambda A(v) + \mu A(w)$$

Given a finite dimensional vector space V we can define its dual V^*, that is the space of functions on V:

$$V^* = \{f | f : V \to F, f \text{ linear}\}$$

Given V and W we can take the tensor product of the two $V \otimes W$ which we can write as

$$V \otimes W = \left\{\sum \alpha_i (v_i \otimes w_i) | \alpha_i \in F, v_i \in V, w_i \in W\right\} / \sim$$

where the equivalence relation is given by:

$$\alpha(v \otimes w) \sim (\alpha v) \otimes w \sim v \otimes (\alpha w)$$
$$(v + v') \otimes w \sim v \otimes w + v' \otimes w$$
$$v \otimes (w + w') \sim v \otimes w + v \otimes w'$$

This simply means that we take the free vector space generated by $V \times W$, that is given by linear combinations of elements (v, w) and then quotient out by the subspace spanned by

$$(\alpha v_1 + \beta v_2, w) - \alpha(v_1, w) - \beta(v_2, w)$$
$$(v, \alpha w_1 + \beta w_2) - \alpha(v, w_1) - \beta(v, w_2)$$

with $\alpha, \beta \in F$. The quotient then exactly satisfies the relations above, in which we denote the class of (v, w) by $v \otimes w$. One can also give a definition by a certain universal property.

Other constructions given two vector spaces V and W are the following:

- the space of linear maps between them $\mathrm{Hom}(V, W)$ which is itself a vector space and satisfies the following relation:

$$\mathrm{Hom}(V, W) \cong V^* \otimes W$$

- Antisymmetric tensors (or 2-forms) which are bilinear maps $\alpha : V \times V \to F$ satisfying $\alpha(v_1, v_2) = -\alpha(v_2, v_1)$.

- More generally k forms, or totally antisymmetric tensor, is a multilinear map $\mu : V \times \cdots \times V \to F$ satisfying

$$\mu(v_1, \ldots, v_k) = \mathrm{sgn}(\sigma)\mu(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

  where $\sigma \in S_k$ is an element of the symmetric group, i.e. the permutation group on k elements. The vector space of such maps is indicated by $\Lambda^k V$.

2.2.2 Functions, 1-forms and higher forms

What we need to study classical mechanics is a vector space equipped with a non-degenerate 2-form $\omega \in \Lambda^2 V$, where V is the phase space. We then call (V, ω) a symplectic vector space.

Denote coordinates in $\mathbb{R}^{2n}$ by $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and take a basis given by $(e_1, \ldots, e_n; f_1, \ldots f_n)$. We can for example take ω to have the following matrix form:

$$J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

Now we can also say a bit more about observables, that is functions on phase space, and their derivatives. Consider a function $A : \mathbb{R}^{2n} \to \mathbb{R}$, its differential is given by

$$dA = \sum \left(\frac{\partial A}{\partial q_i} dq_i + \frac{\partial A}{\partial p_i} dp_i \right)$$

where dq_i is the differential of the coordinate function q_i which is linear and satisfies the following relations:

$$\begin{aligned} dq_i(e_j) &= \delta_{ij} \\ dq_i(f_j) &= 0 \\ dp_i(e_j) &= 0 \\ dp_i(f_j) &= \delta_{ij} \end{aligned}$$

so we can think of the e_i and f_i as column vectors and of the dq_i and dp_i as row vectors. So in general dq_i, dp_i as well as dA are elements in the dual vector space to that of the e_i and f_i. In particular dA is the "row vector version" of the gradient (which is a column vector).

End of lecture 1

Let's stay in the simple example of $\mathbb{R}^n$, with coordinates $(x_1, \ldots x_n)$. Observables then are simply functions $\mathbb{R}^n \to \mathbb{R}$ and we have a basis associated to the coordinates that can be written as $(\partial_{x_1}, \ldots, \partial_{x_n})$ and can be thought of as the tangent vectors in the direction of the coordinates. We then had the differentials $dx_1, \ldots, dx_n$ which we think of as 1-forms, duals to the tangent vectors. We can also take the wedge product of the 1-forms, defined for two differentials as

$$dx_i \wedge dx_j = dx_i \otimes dx_j - dx_j \otimes dx_i$$

for three as

$$\begin{aligned} dx_i \wedge dx_j \wedge dx_k = \; & dx_i \otimes dx_j \otimes dx_k + dx_k \otimes dx_i \otimes dx_j + dx_j \otimes dx_k \otimes dx_i \\ & - dx_i \otimes dx_k \otimes dx_j - dx_k \otimes dx_j \otimes dx_i - dx_j \otimes dx_i \otimes dx_k \end{aligned}$$

which looks complicated but can more simply be written as:

$$dx_1 \wedge dx_2 \wedge dx_3 = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) dx_{\sigma(1)} \otimes dx_{\sigma(2)} \otimes dx_{\sigma(3)}$$

which also suggest how one extends the definition to higher products. This gives a way of obtaining k forms with higher k starting with one forms. In particular by taking wedges of differentials we get a basis for the spaces of k forms.

Given a k form μ one can then contract it with a vector v:

$$i_v \mu = \mu(v, \dots)$$

For example we get

$$i_{\partial_{x_1}} dx_1 \wedge dx_2 = dx_2$$
$$i_{\partial_{x_2}} dx_1 \wedge dx_2 = -dx_1$$

To calculate the second expression one can use the general relation, for η an l form:

$$\mu \wedge \eta = (-1)^{kl} \eta \wedge \mu$$

Some properties of the contraction follow directly:

- $i_v^2 = 0$ which could remind us of the corresponding property for the differential, so we may say that i_v is an anti-derivation,
- $i_v(\mu \wedge \eta) = i_v(\mu) \wedge \eta + (-1)^l \mu \wedge i_v(\eta)$.

Recall that for a function $H : \mathbb{R}^n \to \mathbb{R}$ we can define its differential:

$$dH = \sum_i \frac{\partial H}{\partial x_i} dx_i$$

We can also consider a more general object, a linear combination of differentials (which may not itself be a differential of a function)

$$\alpha = \sum_i \alpha_i(x_j) dx_j$$

We can take *its* differential:

$$d\alpha = \sum_{i,j} \frac{\partial \alpha_i}{\partial x_j} dx_j \wedge dx_i = \sum_{i<j} \left(\frac{\partial \alpha_i}{\partial x_j} - \frac{\partial \alpha_j}{\partial x_i} \right) dx_j \wedge dx_i$$

This should look familiar, in fact if we take a vector field $\vec{a}$ in $\mathbb{R}^3$ then its curl $\nabla \times \vec{a}$ has components given by the coefficients of $dx_j \wedge dx_i$ in the expression above, so we have a correspondence $\nabla \times \vec{a} \leftrightarrow da$ and we should think of a as a 1 form.

2.2.3 Differential Geometry

Up to now we've only worked over $\mathbb{R}^n$ to make things simpler. However in general we may be interested in a theory over a more general space, which we require to be "sufficiently smooth". We will then work with differentiable manifolds, that is a space M which we can cover by open sets U_i which locally look like $\mathbb{R}^n$, i.e. for every U_i we have a map $\phi_i : U_i \to U_i' \subset \mathbb{R}^n$. Then, at a single point we can think of the tangent vectors at that point, these form a vector space usually indicated by T_pM and called the tangent space at point p. Then we can consider functions $H : M \to \mathbb{R}$ and again we can take its differential dH, which we can think of as a linear approximation to H around each point. In particular we have a map $p \mapsto dH_p$ associating to each point, the differential at that point. Explicitly this is a map $M \to \Lambda^1 T_pM$ which we call a 1 form and we can analogously define a k form as a map $M \to \Lambda^k T_pM$.

The tangent spaces at the different points can be collected together to obtain the *tangent bundle* to the manifold M, we write:

$$TM = \coprod_p T_pM$$

which is a first example of a *bundle*.

2.2.4 Back to Hamiltonian Mechanics

Now let us go back to Hamilton mechanics. We had a phase space $\mathbb{R}^{2n}$ which is symplectic, i.e. we have a nondegenerate antisymmetric bilinear form $\omega = \sum dp_i \wedge dq_i$. If we then take a function $H : \mathbb{R}^{2n} \to \mathbb{R}$ and the fact that the symplectic form is nondegenerate tells us that there is a vector field X_H such that

$$dH = \omega(X_H, \cdot) = i_{x_H}\omega$$

In order to find an expression for X_H we can write the left hand side as

$$\sum_i \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right)$$

the right hand side as:

$$\sum_i (dq_i(X_H) \wedge dp_i - dq_i \wedge dp_i(X_H))$$

We can then write X_H generally as

$$X_H = \sum (\alpha_{p_i} \partial_{p_i} + \beta_{q_i} \partial_{q_i})$$

and inserting it into both sides and setting them equal to each other we find α_{p_i} and β_{q_i} and get

$$X_H = \sum \left(\frac{\partial H}{\partial q_i} \partial_{p_i} + \frac{\partial H}{\partial p_i} \partial_{q_i} \right)$$

The Poisson bracket for functions f and g on the phase space can then be defined as

$$\{f, g\} := \omega(X_f, X_g)$$

Let us check that this is the usual expression (1):

$$\begin{aligned} \{f, g\} &= \omega(X_f, X_g) = df(X_g) \\ &= \sum_i \left(\frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial p_i} dp_i \right) \sum_j \left(\frac{\partial g}{\partial q_j} \partial_{p_j} + \frac{\partial g}{\partial p_j} \partial_{q_j} \right) \\ &= \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \end{aligned}$$

In the first expression we used the defining property of X_f: $\omega(X_f, \cdot) = df(\cdot)$.

2.3 Canonical quantization

The idea of canonical quantization is very simple: the Poisson brackets become commutators $\{\cdot, \cdot\} \to [\cdot, \cdot]$

$$\{q_i, p_j\} = \delta_{ij} \to [q_i, p_j] = i\delta_{ij}$$

In field theory the idea is the same but one has to be careful because the coordinates depend on something continuous. In particular we have a coordinate $\phi(x)$ for every point x and the point can vary smoothly. The only formal difference in writing the Poisson brackets (and therefore the commutators) is that the Kronecker deltas become Dirac deltas:

$$\{\phi(x), \pi(y)\} \sim \delta^d(x-y) \rightarrow [\phi(x), \pi(y)] = i\delta^d(x-y)$$

where $\pi(y)$ is the canonical momentum and d is the spacetime dimension. We call these "equal time commutators".

There are some caveats to this simple quantization procedure:

- not every quantum system arises from a classical system, so how are those defined?
- There is an ordering ambiguity since in classical mechanics everything commutes, while in quantum mechanics the chosen ordering matters. For example, to the classical observable xp do we associate the operator $\hat{x}\hat{p}$ or $\hat{p}\hat{x}$?
- constraints: sometimes the lagrangian is independent of $\dot{q}_i$ for some i and then
$$\frac{\partial H}{\partial q_i} = 0$$
becomes a constraint (and not an equation of motion) and then one needs to figure out how to quantize. For example one can constrain the classical system and then quantize or try to do the opposite, but there are also more sophisticated methods.

2.4 Quantum mechanics coupled to electromagnetic field

Consider the lagrangian

$$L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}m\dot{\vec{x}}^2 + \frac{e}{c}\dot{\vec{x}}\vec{A} - e\Phi \tag{2}$$

which is written in terms of the scalar and vector potential. The electric and magnetic field can be written as

$$\begin{aligned}\vec{E} &= -\nabla\Phi - \frac{1}{c}\partial_t\vec{A} \\ \vec{B} &= \nabla \times \vec{A}\end{aligned}$$

We can then consider a gauge transformation which transforms *both* $\vec{A}$ and Φ by

$$\begin{aligned}\vec{A} &\to \vec{A} + \nabla\lambda \\ \Phi &\to \Phi - \frac{1}{c}\partial_t\lambda\end{aligned}$$

for some function $\lambda(\vec{x},t)$. The electric and magnetic fields are invariant under such a transformation:

$$\begin{aligned}\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A} \quad &\to \quad -\nabla\Phi + \frac{1}{c}\nabla\partial_t\lambda - \frac{1}{c}\partial_t\vec{A} - \frac{1}{c}\partial_t\nabla\lambda = \vec{E}' = \vec{E} \\ \vec{B} = \nabla\times\vec{A} \quad &\to \quad \nabla\times\vec{A} + \nabla\times\nabla\lambda = \vec{B}' = \vec{B}\end{aligned}$$

We can then define the canonical momentum

$$p_j = \frac{\partial L}{\partial\dot{x}_j} = m\dot{x}_j + \frac{e}{c}A_j$$

and obtain the Hamiltonian

$$H(\vec{x},\vec{p}) = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\Phi.$$

The Schrödinger equation may be written as

$$i\hbar\partial_t\psi(\vec{x},t) = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\vec{A}\right)^2 + e\Phi\right]\psi(\vec{x},t)$$

However, this equation is no longer gauge invariant. It turns out that gauge invariance is rather important so we would like to keep it, but in order for the Schrödinger equation to be gauge invariant, ψ must also transform:

$$\psi \to \psi' = e^{i\lambda(\vec{x},t)/\hbar c}\psi$$

Let us then check that if ψ satisfies the Schrödinger equation then ψ' satisfies the gauge transformed Schrödinger equation.

End of lecture 2

In order to do so it's useful to introduce a covariant derivative:

$$\mathcal{D}_i = \partial_i - \frac{ie}{\hbar c}A_i$$

which will of course transform under gauge transformation $\mathcal{D}_i \to \mathcal{D}'_i$, in particular we have

$$\mathcal{D}'_i\psi' = e^{i\lambda(\vec{x},t)/\hbar c}\mathcal{D}_i\psi = (\mathcal{D}_i\psi)' \tag{3}$$

This is clear when plugging in the expressions for $\mathcal{D}'$ and ψ':

$$\Big(\partial_i - \frac{ie}{\hbar c}(A_i + \partial_i\lambda)\Big)e^{i\lambda/\hbar c}\psi = e^{i\lambda/\hbar c}(\partial_i - \frac{ie}{\hbar c}A_i)\psi + \partial_i(e^{i\lambda/\hbar c})\psi - \frac{ie}{\hbar c}\partial_i\lambda e^{i\lambda/\hbar c}\psi$$

$$= e^{i\lambda/\hbar c}\mathcal{D}_i\psi + \frac{ie}{\hbar c}\partial_i\lambda e^{i\lambda/\hbar c}\psi - \frac{ie}{\hbar c}\partial_i\lambda e^{i\lambda/\hbar c}\psi = e^{i\lambda/\hbar c}\mathcal{D}_i\psi$$

Using the $\mathcal{D}_i$ we can rewrite the Schrödinger equation as:

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2m}\mathcal{D}_i\mathcal{D}_i + e\Phi\right]\psi$$

and it's now simple to verify that ψ' satisfies the gauge transformed equation:

$$i\hbar\partial_t\psi' = \left[-\frac{\hbar^2}{2m}\mathcal{D}'_i\mathcal{D}'_i + e\Phi'\right]\psi'$$

By using (3) on the right hand side we get

$$\begin{aligned}-\frac{\hbar^2}{2m}\mathcal{D}'_i\mathcal{D}'_i\psi' + e\Phi'\psi' &= -\frac{\hbar^2}{2m}e^{i\lambda/\hbar c}\mathcal{D}_i\mathcal{D}_i\psi + e\Phi e^{i\lambda/\hbar c}\psi - \frac{e}{c}\partial_t\lambda e^{i\lambda/\hbar c}\psi \\ &= e^{i\lambda/\hbar c}i\hbar\partial_t\psi - \frac{e}{c}\partial_t\lambda e^{i\lambda/\hbar c}\psi = i\hbar\partial_t(e^{i\lambda/\hbar c}\psi) = i\hbar\partial_t\psi'\end{aligned}$$

So indeed, in order to preserve gauge invariance, ψ should transform under a gauge transformation as

$$\psi \to e^{i\lambda/\hbar c}\psi$$

This phase is important in general! There are many situations in which it doesn't play a role, since it's a global phase and cancels out. For electromagnetism we say that we have a gauge group $U(1)$, which we can think of as simply a phase, in fact the term $e^{i\lambda/\hbar c}$ is simply a choice of an element in $U(1)$ at every point in space, at every time. The Lie algebra instead represents the group elements infinitesimally close to the identity of the group and in particular for $U(1)$ we have the Lie algebra $\mathbb{R}$.

2.5 Aharonov-Bohm effect and the gauge potential

Consider an infinite solenoid, which generates a uniform magnetic field inside and no magnetic field on the outside. If we then put an electron source to one side of the solenoid and a screen to the other side we observe an interference pattern which changes when the flux of the magnetic field Φ_B changes (there was a drawing in the lecture). This is very surprising since the electrons only pass in an area in which the magnetic field is 0.

In the region with $\vec{B} = 0$, the magnetic field is "pure gauge":

$$\begin{aligned}\vec{B} &= \operatorname{curl}\vec{A} = 0\\ \vec{A} &= \operatorname{grad}\lambda\\ \lambda(\vec{x}) &= \int_{\vec{x}_0}^{\vec{x}} d\vec{s}\vec{A}(\vec{s})\end{aligned}$$

In this situation, the space we're considering is not $\mathbb{R}^2$ (or $\mathbb{R}^3$) but rather $\mathbb{R}^2 \setminus \{\text{disk}\}$ (or $\mathbb{R}^3 \setminus \{\text{cylinder}\}$) and we have loops around the punched out area which are not contractible. In particular we then have

$$\oint_\gamma d\vec{s}\vec{A} = \int_{\Sigma,\partial\Sigma=\gamma} \operatorname{curl}\vec{A}d\vec{f} = \Phi_B$$

We can locally gauge away $\vec{A}$ and the Schrödinger equation simply becomes

$$-\frac{\hbar^2}{2m}\partial_i\partial_i\psi' = i\hbar\partial_t\psi'$$

but then ψ transforms as well

$$\psi(\vec{x}) \to \psi(\vec{x})\exp\left(\frac{ie}{\hbar c}\int_{\vec{x}_0}^{\vec{x}} d\vec{s}\vec{A}(\vec{s})\right)$$

We can then write the wave function of electrons reaching the screen as a linear combination of two wave functions, one for electrons passing above the solenoid and one for those passing below:

$$\begin{aligned}\psi_B(\vec{x}) &= \psi_{I,B=0}e^{\frac{ie}{\hbar c}\int_{\gamma_I}\vec{A}d\vec{s}} + \psi_{II,B=0}e^{\frac{ie}{\hbar c}\int_{\gamma_{II}}\vec{A}d\vec{s}}\\ &= e^{\frac{ie}{\hbar c}\int_{\gamma_I}\vec{A}d\vec{s}}(\psi_{I,B=0} + \psi_{II,B=0}e^{\frac{ie}{\hbar c}(-\int_{\gamma_I}\vec{A}d\vec{s}+\int_{\gamma_{II}}\vec{A}d\vec{s})})\\ &= e^{\frac{ie}{\hbar c}\int_{\gamma_I}\vec{A}d\vec{s}}(\psi_{I,B=0} + \psi_{II,B=0}e^{\frac{ie}{\hbar c}\oint_\gamma\vec{A}d\vec{s}})\end{aligned}$$

where γ is the loop given by the concatenation of γ_{II} and $-\gamma_I$. This is now a relative phase which is actually observable!

2.5.1 A path integral approach

First, given the lagrangian (2) we can write the action as

$$S = \int dt L$$

Then the path integral takes the form

$$Z = \int e^{iS/\hbar}$$

from which we see that we weigh each path γ with a phase

$$\exp\left(i\frac{e}{\hbar c}\int_\gamma \vec{A}d\vec{x}\right)$$

which comes from the $\dot{\vec{x}}\vec{A}$ term in the lagrangian by

$$\int_{t_i}^{t_f} dt\dot{\vec{x}}\vec{A} = \int_\gamma d\vec{x}\vec{A}$$

Some takeaway lessons from the study of the Aharonov Bohm effect are the following:

- ψ is not a function but rather a section of a line bundle and therefore transforms under gauge transformations.
- The gauge group is important: we had $e^{i\chi}$ because we were considering gauge group $U(1)$ whose elements take that form.
- In general, the gauge transformation for A may be written as

$$A \to A - ig^{-1}dg = A - ie^{-i\chi}d(e^{i\chi}) = A + d\chi$$

- Since we're interested in the group elements $e^{i\chi}$ and not in the function χ itself, we then have that $e^{i\chi}$ must be continuous, while χ itself may have jumps.
- The integral of the gauge field around closed paths turns out to be an important quantity, these are called *holonomies* of the gauge field:

$$\oint \vec{A}d\vec{s}$$

2.5.2 Spacetime with a periodic direction

Consider a periodic direction: $x^i \sim x^i + 2\pi R$. What are gauge transformations now? We want $e^{\frac{ie}{\hbar}\lambda}$ to be single valued. In other words, we want to describe a function $S^1 \to U(1)$ which associates a group element to every point in this periodic direction. But recalling that *as a topological space* $U(1) \cong S^1$, we see that we're considering maps $S^1 \to S^1$ and these can have a winding! In particular, we can take a gauge transformation

$$\lambda = \frac{\hbar}{eR} x^i$$

which gives the map $e^{\frac{ie}{\hbar}\lambda}$ that goes around the circle once. However λ is *not* a well defined function on S^1 since $\lambda(0) \neq \lambda(2\pi R)$ for the coordinates 0 and $2\pi R$ which are identified. Nevertheless $e^{\frac{ie}{\hbar}\lambda}$ *is* well defined, so it is a valid gauge transformation. In this case we have the following transformation for A:

$$A_i \to A_i + \partial_i \lambda = A_i + \frac{\hbar}{eR}$$

So A_i and $A_i + \frac{\hbar}{eR}$ are related by gauge transformations and should therefore be identified: so A_i also becomes periodic, but with a different periodicity which in particular scales inversely with R.

This is an example of a *large gauge transformation*, i.e. we have

$$A \to A + \omega$$

for ω a form that is closed but not exact, i.e. $d\omega = 0$ but there does not exist a λ such that $\omega = d\lambda$. (Note that in the example it may seem that we *do* have λ, but we actually don't because it's not a globally well defined function.)

2.6 Magnetic monopoles

Recall Maxwell's equations:

$$\nabla \cdot \vec{E} = \rho_e \qquad \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}_e \qquad \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Note that without sources, i.e. $\rho_e = 0$ and $\vec{j}_e = 0$, they are symmetric under the exchange $(\vec{E}, \vec{B}) \to (\vec{B}, -\vec{E})$. It's also then interesting to consider the possibility of adding a nontrivial ρ_m on the right hand side of the divergence of $\vec{B}$, analogous to ρ_e. In particular we consider a magnetic monopole, that is a configuration with

$$\vec{B} = \frac{g}{4\pi r^2}\vec{e}_r$$

written in spherical coordinates ($\vec{e}_r \sim \partial_r$) on $\mathbb{R}^3 \setminus \{0\}$. If we consider a fixed sphere, we can coordinatize it by dividing it into two patches: the Northern and Southern hemisphere which overlap around the equator. In particular we use the usual spherical coordinates θ and ϕ with $\theta \in [0, \pi)$ for the Northern hemisphere and $\theta \in (0, \pi]$ for the Southern one. On the two charts we can take the potentials

$$\begin{aligned}
\vec{A}_N &= \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\vec{e}_\phi \\
\vec{A}_S &= -\frac{g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}\vec{e}_\phi
\end{aligned}$$

You will see in the exercise sheets on their overlap they are related by a gauge transformation:

$$\vec{A}_N - \vec{A}_S = \nabla\chi$$

with

$$\chi = \frac{g}{2\pi}\phi$$

which is not single valued! We have $\chi(0) = 0$ while $\chi(2\pi) = g$. As anticipated, it turns out that this is not a problem and in fact we see that χ must be so by studying the flux of the magnetic field. This can be written as

$$\begin{aligned}
\Phi_B &= \oint \vec{B}d\vec{f} = \int_N \vec{B}d\vec{f} + \int_S \vec{B}d\vec{f} = \int_{equator} (\vec{A}_N - \vec{A}_S)d\vec{s} = \int \nabla(\lambda)d\vec{s} \\
&= \chi(2\pi) - \chi(0) = g
\end{aligned}$$

which further supports that χ *not* being single valued is in fact the correct choice.

Rather, it is $e^{i\frac{e}{\hbar}\lambda}$ that is supposed to be single valued, meaning

$$\exp\left(i\frac{e}{\hbar}\lambda(2\pi)\right) = \exp\left(i\frac{eg}{\hbar}\right) = 1 = \exp\left(i\frac{e}{\hbar}\lambda(0)\right)$$

and we therefore get

$$eg = 2\pi n\hbar$$

which is called the *Dirac quantization condition* that tells us that if there were even a single magnetic monopole in the universe then that would imply the quantization of the electrical charge, which would be a nice justification for it.

2.7 Mathematical language for gauge theory

Fiber bundles

Fiber bundles are manifolds which have an additional structure, aiming to generalize the product of spaces $B \times F$. As is typical with manifolds, we require this structure *locally* not globally. We therefore have a space E which in general may *not* be written as $B \times F$ but which on small enough patches looks like $U \times F$ where U is a patch on the *base manifold* B. We then have the following data (E, B, F, π), where E is the *total space* which locally looks like $U \times F$, B is the base space and F is the fiber. $\pi : E \to B$, usually called projection, is a surjective map with preimage given by the fiber $F \cong \pi^{-1}(b)$ for any point $b \in B$ which in particular says that $\pi^{-1}(b)$ are diffeomorphic for all $b \in B$.

Example. $S^1 \times I$ in which I is the interval, is a bundle over S^1 which simply looks like a cylinder. There is another bundle over S^1 which is less trivial, the Mobius strip: locally it looks the same, but globally there is a twist.

Another simple example is in fact $E = B \times F$ and in the more general case in which that is not true, we say that the bundle has a "twist".

Structure group

Now, consider two open subsets U_i and U_j of B with diffeomorphisms $\phi_{U_i} : \pi^{-1}(U_i) \to U \times F$ which come with inverses $\phi_{U_i}^{-1} : U \times F \to \pi^{-1}(U_i)$. We can then plug in a point $x \in U_i$ and get a map $\phi_{U_i}^{-1}(x, \cdot) : F \to \pi^{-1}(U_i)$. On the intersection of U_i and U_j we can take the composition

$$g_{ij}(x) = \phi_{U_j}(x, \cdot) \circ \phi_{U_i}(x, \cdot) : F \to F$$

for any point $x \in U_i \cap U_j$. This gives us a map $g_{ij} : U_i \cap U_j \to \mathrm{End}(F)$. We often consider fiber bundles with some *structure group* G: that means that we have an action of G on F and the g_{ij} maps actually take values in the gauge group.

End of lecture 3

More bundles and sections

The way to think of a fiber bundle is that we have a space B and to each of its points we associated the fiber F but in general the whole space is not simply $B \times F$ but can be somehow twisted.

If F is a vector space we talk about *vector bundles*. If F is the gauge group itself then we have a *principal bundle*, which is what we have when talking about pure gauge theory. We then also have *associated vector bundles*, in which the fiber V carries a representation of the gauge group G.

So this is the geometrical structure but what are the fields? They are often *sections* of the bundle, that is maps $s : B \to E$ satisfying $\pi \circ s = id_B$. Then the matter fields are sections of the associated bundles and for example the wave function is not simply a function $\mathbb{R}_t \to \mathbb{C}$ but rather a section of an associated bundle with gauge group $U(1)$.

By representations of G on V we mean that to elements of the group we associate a transformation of V, more precisely we have a group homomorphism $G \to \mathrm{Aut}(V)$, where $\mathrm{Aut}(V) = \{\phi \in \mathrm{End}(V), \phi \text{ invertible}\}$, that is invertible maps from V to itself. This is the way we describe symmetries in physics. In particular, when in QFT we have fields with indices, we're simply saying that the fields are sections of an associated bundle in which some representation has been chosen. The vector space which carries the representation of the gauge group is often called the internal symmetry space.

Connection

In differential geometry one usually defines the concept of a connection: we would like a way to compare vectors at tangent spaces in different points but there's no way of "bringing one vector to the tangent space of the other", a connection solves this problem. To do so, consider a principal bundle P, then the tangent space at a point $p \in P$ may be written as $T_pP = T_pV \oplus T_pH$ in which T_pV is the subspace "along the fiber" and T_pH is a "horizontal" subspace. A connection specifies what T_pH is at each point. We can then consider a curve γ in the base space and lift it to a curve γ_P in the total space, which points in the horizontal subspace at each point. One might imagine that then if we take γ to be a loop, then γ_P will also be a loop, but that is *not* true in general, but rather depends on the topology of the manifold and the chosen connection. This is measured by the holonomy of the connection

"Topology" of principal bundles

There are topological invariants called Chern classes. We saw one such invariant when studying the monopole, there we had

$$\frac{1}{2\pi}\oint B = \text{ integer}$$

this is exactly the first Chern class:

$$c_1 = \frac{1}{2\pi}\int F$$

which is related to the winding number of the map $S^1 \to U(1)$ which is an integer. In higher dimension one can then form higher Chern classes which give rise to more topological invariants.

3 Abelian Chern Simons theory

Lagrangian

Abelian Chern Simons theory is a 3d gauge theory with gauge group $U(1)^r$ and for now we restrict to $r = 1$. We write the lagrangian as

$$\mathscr{L}_{CS} = \frac{k}{2\pi} A \wedge dA$$

and then the action as

$$S = \frac{k}{2\pi}\int_M A \wedge dA$$

for some 3d spacetime manifold M.

A is a 1 form, dA is a 2 form and both of these are defined without a metric, so the integral may be done without one and in that sense this theory is *topological*. This is *not* a typical feature for QFTs, for example for Maxwell theory we have $F_{\mu\nu}F^{\mu\nu}$ and so we need a metric to be able to raise and lower indices. Another way of phrasing this is that we can write the same term using differential forms as $F \wedge *F$ and the $*$ operator, the Hodge dual, is defined using a metric. The Maxwell equations then take the form $dF = 0$ and $d * F = j$.

Gauge invariance

Let us check gauge invariance. We start with local gauge transformations, i.e. transformations of the form

$$A \to A + d\lambda$$

Then we have

$$A \wedge dA \to (A + d\lambda) \wedge dA = A \wedge dA + d\lambda \wedge dA = A \wedge dA + d(\lambda \wedge dA)$$

Then, if there are no boundary terms, the theory is invariant under local gauge transformations. What about large gauge transformations? That is we consider

$$A \to A + \omega$$

where ω is closed, $d\omega = 0$ and has integral periods. We then have

$$\int A \wedge dA \to \int (A + \omega) \wedge dA = \int A \wedge dA + \int \omega \wedge dA$$

where the last term is equal to $(2\pi)^2 n$ for some integer n. This is because $\int dA$ is equal to $2\pi n$ for some integer n and the same is true for ω. We then have that S is not invariant but e^{iS} *is* invariant, provided that k is an integer.

Setup for canonical quantization

We now want to canonically quantize. We therefore have to have a time direction and have some "p" and "q" variables. In order to pick a time direction, we consider Chern Simons theory on $\Sigma \times \mathbb{R}_t$ where Σ should be some interesting 2 dimensional manifold. Note that a generic 3d manifold M is *not* of this form, we're therefore restricting our study to the specific class of manifolds that *do*. We then split d and A as follows

$$d = dt\frac{\partial}{\partial t} + \tilde{d}$$
$$A = A_0 + \tilde{A}$$

We then have

$$
\begin{aligned}
S &\sim \int A \wedge dA = \int (A_0 + \tilde{A})(dt\frac{\partial}{\partial t} + \tilde{d})(A_0 + \tilde{A}) \\
&= \int (A_0 + \tilde{A})(\cancel{dt\frac{\partial}{\partial t}A_0} + \tilde{d}A_0 + dt\frac{\partial}{\partial t}\tilde{A} + \tilde{d}\tilde{A}) \\
&= \int (A_0 + \tilde{A})(\tilde{d}A_0 + dt\frac{\partial}{\partial t}\tilde{A} + \tilde{d}\tilde{A}) \\
&= \int \cancel{A_0\tilde{d}A_0} + \cancel{A_0 dt\frac{\partial}{\partial t}\tilde{A}} + A_0\tilde{d}\tilde{A} + \tilde{A}\tilde{d}A_0 + \tilde{A}dt\frac{\partial}{\partial t}\tilde{A} + \cancel{\tilde{A}\tilde{d}\tilde{A}} \\
&= \int A_0\tilde{d}\tilde{A} + \tilde{A}\tilde{d}A_0 + \tilde{A}dt\frac{\partial}{\partial t}\tilde{A}
\end{aligned}
$$

In which only terms with the three different coordinate differentials survive, e.g. $A_0\tilde{d}A_0$ cancels because it has two dts. We can further simplify by observing that

$$\tilde{d}(\tilde{A} \wedge A_0) = (\tilde{d}\tilde{A}) \wedge A_0 - \tilde{A} \wedge \tilde{d}A_0$$

so if boundary terms vanish we can substitute $\tilde{A}\tilde{d}A_0$ with $(\tilde{d}\tilde{A}) \wedge A_0$ or with $A_0\tilde{d}\tilde{A}$. We then get

$$S \sim \int 2A_0\tilde{d}\tilde{A} + \tilde{A}dt\frac{\partial}{\partial t}\tilde{A}$$

We see that A_0 appears with no time derivative, so that gives us the constraint

$$\tilde{F} := \tilde{d}\tilde{A} = 0$$

The second term instead gives us

$$-(A_1dx_1 + A_2dx_2)(\dot{A}_1dx_1 + \dot{A}_2dx_2)dt = -(A_1\dot{A}_2 - A_2\dot{A}_1)dx_1dx_2dt = 2A_2\dot{A}_1dx_1dx_2dt$$

That means that A_1 and A_2 are conjugate variables and the classical phase space is the space of gauge invariant flat connections on Σ, i.e. the space of A_1s and A_2s subject to the constraint $\tilde{F} = 0$, modulo gauge transformations: $\Gamma = \{\tilde{A} | d\tilde{A} = 0\}/\mathcal{G}$ where $\mathcal{G}$ is the group of gauge transformations (including large ones).

These observations allow us to formally interpret the action as follows:

$$S \sim \text{"constraint} + p\dot{q} - H\text{"}, \text{ with } H = 0$$

which tells us that the Hamiltonian is zero! We therefore have no dynamics.

This is a general property of topological theories: H is a component of the stress energy tensor, which is obtained from the action by varying with respect to the metric, so if we have no metric then the stress energy tensor is trivial and we have $H = 0$.

Let's look for example at the case $\Sigma = T^2$. We then have two periodically identified coordinates, which gives us that also the potentials are periodically identified, so the phase space is some quotient of the torus.

Canonical quantization

We have Poisson brackets between "p" and "q":

$$\{A_i(x), A_j(x')\} = \epsilon^{ij}\frac{2\pi}{k}\delta^{(2)}(x - x')$$

Then to canonically quantize we impose the commutation relations:

$$[A_i(x), A_j(x')] = \epsilon^{ij}\frac{2\pi}{k}i\delta^{(2)}(x - x')$$

However, the requirement of gauge invariance makes the procedure more complicated and there are two main approaches. We could take wave functionals which depend on A and then impose the constraint in some form, or we might find gauge invariant quantities on the classical level and then quantize. In other words: quantize and then constrain or constrain first and then quantize. We will follow this second procedure, with the gauge invariant quantities being the holonomies.

End of lecture 4

We could think of the "position" wave functionals $\psi(A)$ on which "momentum" acts as $\frac{\delta}{\delta A}$ and impose a constraint on this space of functionals. This leads to holomorphic quantization which is interesting and has a connection to *conformal* field theory.

Another framework is BV/BRST quantization which has a nice framework for dealing with constraints.

Instead we introduce gauge invariant variables at the classical level and then quantize. The gauge invariant variables characterizing a flat bundle are the holonomies (around nontrivial cycles). For example if $\Sigma = T^2$ is the torus, then we have two types of nontrivial cycles, generated by a and

b which are the loops going around the two directions in the torus. We can then take

$$
\begin{aligned}
q &= \oint_a A \\
p &= \oint_b A
\end{aligned}
$$

We then get the bracket

$$
[q,p] = \Big[\oint_a A, \oint_b A \Big] = i\frac{2\pi}{k}
$$

We can then take the exponentiated observables:

$$
\begin{aligned}
W_q &= e^{iq} \\
W_p &= e^{ip}
\end{aligned}
$$

from which, using BCH we get

$$
\begin{aligned}
W_q W_p &= e^{iq} e^{ip} = e^{i(p+q)} e^{-[q,p]/2} = e^{i(p+q)} e^{+\pi i/k} \\
W_p W_q &= e^{ip} e^{iq} = e^{i(p+q)} e^{-[p,q]/2} = e^{i(p+q)} e^{-\pi i/k}
\end{aligned}
$$

which gives us

$$
W_q W_p = e^{2\pi i/k} W_p W_q \tag{4}
$$

Often the exponential is itself called q, but for the purpose of these lectures we'll use $Q = e^{2\pi i/k}$. (But beware when reading articles about quantum groups/algebras that usually q will be used.)

Now, recall the procedure in quantum mechanics: Poisson brackets $\{q,p\} = 1$ become commutators $[q,p] = i\hbar$, so we now have an abstract algebra with generators q, p with that commutator. We then need to specify what our Hilbert space is and it is simply a representation of the algebra we found. In particular for normal quantum mechanics we find $\mathscr{H} = \mathscr{L}^2(\mathbb{R})$ on which q acts as multiplication by q and p acts as $\frac{1}{i}\frac{\partial}{\partial q}$.

We now want to do the same so we find a representation of this algebra. One representation can be found by picking a basis such that W_q and W_p act as follows:

$$
\begin{aligned}
W_q|n\rangle &= Q^n|n\rangle \\
W_p|n\rangle &= |n+1\rangle
\end{aligned}
$$

for $n = 0, \ldots, k-1$. Let us check that this is indeed a representation, i.e. that it satisfies (4):

$$\begin{aligned} W_p W_q |n\rangle &= Q^n |n+1\rangle \\ W_q W_p |n\rangle &= Q^{n+1} |n+1\rangle = Q W_p W_q |n\rangle \end{aligned}$$

So we get the Hilbert space $\mathscr{H}_{T^2}$ which is finite dimensional of dimension k (and it turns out that this is the only representation). If we took S^2 there's only the trivial bundle so we get again a finite dimensional Hilbert space, but it only has dimension 1.

Normally we have that the Hamiltonian generates time translations, but since $H = 0$ we would have that time evolution from $\Sigma \times \{t_1\}$ to $\Sigma \times \{t_2\}$ is just the identity. If we were to identify the two ends we would get the manifold $\Sigma \times S^1$ which corresponds to taking a trace and therefore gives us $\mathrm{tr}(\mathbb{1}_{\mathscr{H}_\Sigma}) = \dim \mathscr{H}_\Sigma$.

Action of the mapping class group

The mapping class group of a manifold is the group of components of the space of diffeomorphisms from the manifold to itself. More precisely,

$$MCG(M) = \mathrm{Diff}(M) / \text{ diffeotopy}$$

in which $\mathrm{Diff}(M)$ is the set of diffeomorphisms of M and diffeotopy is defined as follows.

Definition 1. Let $f, g : M \to M$ be two diffeomorphisms of M. We say that f and g are diffeotopic if there exists a smooth map $h : [0,1] \times M \to M$ with $h(x, -)$ a diffeomorphism $\forall x \in [0,1]$ such that $h(0,p) = f(p)$ and $h(1,p) = g(p)$ for all $p \in M$.

In other words, in $MCG(M)$ we identify two diffeomorphisms of M is we can smoothly deform one into the other. It turns out that for a topological field theory we have an action of this group on the state spaces, i.e. the mapping class group of Σ acts on $\mathscr{H}_\Sigma$. For example for a diffeomorphism smoothly connected to the identity we get a trivial action since such a diffeomorphism has no effect on the underlying topology.

Instead, if we consider the torus there are diffeomorphims which are not connected to the identity. To see an example consider the torus, seen as R^2/Λ where Λ is the unit lattice. Here we have an S modular transformation which corresponds to an exchange of directions $(p,q) \to (-q,p)$. This then has a nontrivial action on the state space we formulated, which is a form of Fourier transformation since we're exchanging the roles of "position" and "momentum":

$$S|n\rangle = \frac{1}{\sqrt{k}} \sum_{m\in\mathbb{Z}_k} e^{2\pi imn/k}|m\rangle = \frac{1}{\sqrt{k}} \sum_{m\in\mathbb{Z}_k} Q^{mn}|m\rangle$$

And we'd get $S^2 = -1$ just as we expect since that's what also happens if we apply the change of directions twice.

Introducing a source

Consider the Chern Simons lagrangian with an additional source term:

$$\mathscr{L} = \mathscr{L}_{CS} + A_\mu J^\mu$$

Before we had the equations of motion

$$F_{ij} = 0$$

which for the spatial component $\tilde{F}$, which we now call B, reads as

$$B = 0$$

Now, instead, we can write $J^\mu = \begin{pmatrix} \rho \\ \vec{J} \end{pmatrix}$ which gives the following equation for B:

$$B = \frac{2\pi}{k}\rho$$

Then, for the simple case of a point source, i.e. for $\rho \sim \delta(x)$, we get

$$B = \frac{1}{k}\delta^{(2)}(x)$$

It is interesting to study systems of such sources (which we consider particles) because they behave in a rather special way: under exchange of two particles we pick up an Aharonov Bohm phase $e^{2\pi i/k}$! This is an unexpected result as these particles don't satisfy neither Fermi nor Bose statistics, since there under an exchange of identical particles we have

$$\psi_i(x_i)\psi_j(x_j) = \pm\psi_i(x_j)\psi_j(x_i)$$

with + for bosons and − for fermions. Such statistics are satisfied by so called "anyons", which allow for a more general phase under the exchange of identical particles.

This is more interesting when studying Chern Simons with non abelian gauge groups, and there's a good description of such particles in terms of so called modular tensor categories.

This also has applications to knot theory, which is how Chern Simons became very important: Witten computed knot invariants with non abelian Chern Simons theory.

End of lecture 5

4 Axiomatic definition of TQFT

We give here a rigorous definition of a path integral.
First of all, what do we mean by path integral? It's an integral over a space of fields of the following form

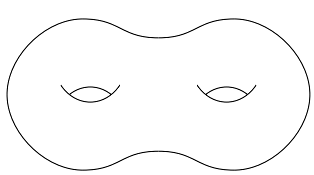
$$\mathcal{Z} = \int \mathcal{D}\phi e^{iS(\phi)}$$

or

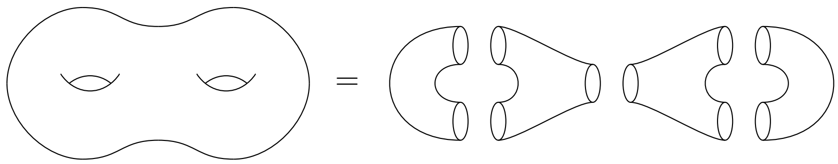
$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S(\phi)}$$

in Euclidean signature. However, we try to formalize this not by making the ingredients in the expression more precise, but rather with an axiomatic framework capturing what the path integral should be. We now spend some time explaining what we mean by this.

Suppose we take a d dimensional TQFT. First of all, it's important to note that one can evaluate the path integral over a closed surface and in this case we just get a number: $S(\phi)$ depends on the fields and presumably comes in the form $\int_M \mathscr{L}$ for some lagrangian $\mathscr{L}$ but we integrate over the manifold and we integrate over all field configurations so in the end we just get a number.

 $\longrightarrow \quad \mathcal{Z} \in \mathbb{C}$

We would like to be able to cut up the manifold M into pieces $M_1, M_2, ...$ (i.e. into manifolds with boundary) and then evaluate the path integral on the pieces and then get back the value for the whole manifold.



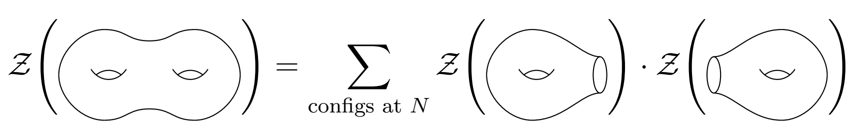
(5)

However, for these pieces what happens at the boundary? Well, if we take the path integral on some manifold M_i with boundary, then the path integral is over the fields which satisfy some boundary conditions:

$$\mathcal{Z}_{M_i}(\text{field config. at } \partial M_i) = \int_{\phi|\partial M_i = ...} \mathcal{D}\phi e^{iS(\phi)}$$

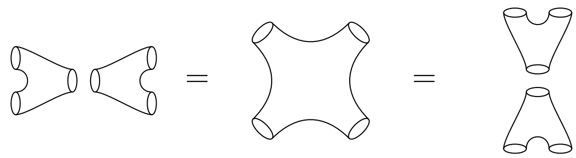
Of course if we take different boundary conditions we in general get different values. So now for a manifold with boundary the path integral is not just a number, but rather a map from the "space of field configurations at the boundary" to numbers.

Suppose we just cut a closed manifold M into two manifolds with boundary M_1 and M_2. We could then glue M_1 and M_2 together and in doing this we must sum over all the field configurations at their common boundary N, so that we get the same result as we'd get if we hadn't cut the manifold M at all (which was just a number and did not depend on field configurations at the common boundary of M_1 and M_2).



This is true in general: gluing manifolds correponds to summing over all field configurations at the common boundary.
In addition, there are different ways of cutting up a manifold into different manifolds with boundary and this cutting and gluing procedure should in the end always give the same result. For example, the middle part in (5) could

also have been decomposed differently:



At the same time, we also have to deal with $d-1$ dimensional manifolds. In Chern Simons we specifically dealt with "time slices" and to those we associated a state space by canonically quantizing on $\Sigma \times \mathbb{R}$. More in general, to $d-1$ manifolds N we would like to associate state spaces (which we may think of as the space of field configurations at the boundary).

So to a $d-1$ dimensional manifold N we should associate a state space $\mathscr{H}_N$ and to a manifold M with boundary $\partial M = N$ we associate a map $\mathscr{H}_N \to \mathbb{C}$.

More in general, it may happen that we have a manifold M whose boundary is given by two disconnected manifolds $\partial M = N_1 \sqcup N_2$. Now, N_1 and N_2 are $d-1$ manifolds, to which we associate spaces $\mathscr{H}_{N_1}$ and $\mathscr{H}_{N_2}$. To M we now associate a map $\mathscr{H}_{N_1} \to \mathscr{H}_{N_2}$ or a map $\mathscr{H}_{N_1} \otimes \mathscr{H}_{N_2} \to \mathbb{C}$.

We then come to this association:

- to $d-1$ manifolds we associate a vector space,
- to d manifolds (bordisms) we associate a linear map,

both on vector spaces and on manifolds we have additional structure:

- the disjoint union of manifolds and the tensor product of vector spaces,
- a "special" $d-1$ dimensional manifold, $\emptyset$, as well as a "special" vector space, $\mathbb{C}$ itself.

But we have different ways of putting together two vector spaces! We could also take the direct sum, why do we take the tensor product? This is the usual thing that we do in quantum mechanics! If we have Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$ characterizing different degrees of freedom a particle may have, then to describe a particle with both degrees of freedom we would have the Hilbert space $\mathscr{H}_1 \otimes \mathscr{H}_2$. In addition, considering M as before and regarding $\mathscr{H}_{N_i}, i = 1, 2$ as the space of boundary configurations of the fields, in order to obtain a value for the path integral we have to set boundary conditions on *all* boundaries N_i (not on one *or* the other) and the result is bilinear in both $\mathscr{H}_{N_i}$. In other words, we get a number for any element in $\mathscr{H}_{N_1} \otimes \mathscr{H}_{N_2}$.

We now come to the formalization of the concept of a TQFT (due to Atiyah and Segal):

Definition 2. A d dimensional TQFT is a symmetric monoidal functor between the category of bordisms and the category of vector spaces:

$$\mathcal{Z} : \mathrm{Bord}_d \to \mathrm{Vect} \tag{6}$$

We will say more about categories and functors but the idea is simply the one given above.
It turns out that this language is actually quite useful. In particular it can be used to talk about the concept of "generalized symmetries" which has been getting a lot of attention lately.

What we described was for theories in which we only care about the topology of our manifolds. In general one can think of manifolds with additional structure, such as orientation, metric, spin structure and so on.

4.1 Categories and functors

Definition 3 (Category)**.** A category $\mathscr{C}$ consists of

- A collection of "objects" $\mathrm{Obj}(\mathscr{C})$,
- for any pair of objects $X, Y \in \mathrm{Obj}(\mathscr{C})$ a set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of "morphisms from X to Y". $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ is usually written as $f : X \to Y$.

with in addition

1. for any object $X \in \mathrm{Obj}(\mathscr{C})$ we have an identity morphism $1_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$, i.e. $1_X : X \to X$,
2. composition of maps: given $f : X \to Y$, $g : Y \to Z$ then there is a map $g \circ f : X \to Z$.

such that:

- the composition is associative,
- if $f : X \to Y$ we have $f \circ 1_X = f$ and $1_Y \circ f = f$.

Example. We have the category of sets, Set, in which $\mathrm{Hom}_{\mathrm{Set}}(X, Y) = \{\text{functions } f : X \to Y\}$. Many examples are simply categories of "sets with extra structure":

1. Category of vector spaces over $\mathbb{C}$, which is indicated by $\mathrm{Vect}_{\mathbb{C}}$ and has
   - $\mathrm{Obj}(\mathrm{Vect}_{\mathbb{C}})$: vector spaces over $\mathbb{C}$,
   - morphisms: linear maps
2. Category of finite dimensional vector spaces (again with linear maps).
3. Category of algebras. We first specify what the objects, the algebras, are. An algebra is a vector space A together with a multiplication $\mu : A \times A \to A$ which is bilinear, i.e.
$$\begin{aligned}\mu(a+b, c) &= \mu(a, c) + \mu(b, c)\\ \mu(\lambda a, b) &= \lambda\mu(a, b)\end{aligned}$$
   or, writing the multiplication as $\cdot$ as usual:
$$\begin{aligned}(a+b)\cdot c &= a \cdot c + b \cdot c\\ (\lambda a)\cdot b &= \lambda(a \cdot b)\end{aligned}$$
   In case μ is associative (i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$) the algebra is called associative. In case there is a unit e (i.e. e satisfies $e \cdot a = a = a \cdot e, \forall a \in A$) the algebra is called unital. For example the $n \times n$ matrices over $\mathbb{C}$ form a unital associative algebra with matrix multiplication, since it is an associative multiplication with unit $e = \mathbb{1}$ the matrix with 1s on the diagonal.

   In order to have a category we also need to specify what the morphisms are: they are algebra homomorphisms, i.e. $\phi : A \to B$ is a linear map satisfying
$$\phi(a \cdot_A \tilde{a}) = \phi(a) \cdot_B \phi(\tilde{a})$$
   in which we specified where we used multiplication in A and where multiplication in B.
4. Category of groups, i.e. a set with a unital associative multiplication $- \cdot - : G \times G \to G$ with inverses, with group homomorphisms as morphisms.

There are however also more abstract examples:

1. Consider a group G. We can assign to it a category BG that has one object $*$ on which "the group acts", i.e. the endomorphisms of $*$ are the group itself:

$$\mathrm{End}_{BG}(*) = \mathrm{Hom}_{BG}(*,*) = G$$

   in which composition is given by the group multiplication. Associativity and existence of a unit morphism then simply come from the associativity and unit of the group G.

2. Given a topological space X, the open subsets $\mathrm{Op}(X)$ form a category in which:

   - Objects: open sets in X
   - morphisms: inclusions, i.e. given U, V open sets, then there is a morphism $U \to V$ if $U \subset V$ (no morphism otherwise). In other words

$$\mathrm{Hom}(U,V) = \begin{cases} \{*\}, & \text{if } U \subset V \\ \emptyset, & \text{otherwise} \end{cases}$$

     where by $*$ we mean the only morphism from $U \to V$. Note that for a strict subset U in V we have a morphism $U \to V$ but not a morphism in the opposite direction $V \to U$.

   This is an example of a partially ordered set and we can always construct an analogous category from such a set.

Now, what do we mean by a map between categories as in (6)? That is the notion of a functor:

Definition 4 (Functor)**.** Given two categories $\mathscr{C}$ and $\mathscr{D}$, a functor $F : \mathscr{C} \to \mathscr{D}$ consists of maps

- $F : \mathrm{Obj}(\mathscr{C}) \to \mathrm{Obj}(\mathscr{D})$
- $F : \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$

such that

$$F(1_X) = 1_{F(X)}, \forall X \in \mathscr{C}$$
$$F(f \circ g) = F(f) \circ F(g)$$

where on the left we have the composition in $\mathscr{C}$ and on the right we have composition in $\mathscr{D}$.

Example. "Forgetful functor". We gave before the definition of an algebra: it's simply a vector space with additional structure. We can therefore think of going back to the vector space and forgetting the additional structure: that's what we mean by the following forgetful functor

$$F : \mathrm{Alg} \to \mathrm{Vect}$$

But vector spaces are just sets with additional structure! So we have another forgetful functor

$$\mathrm{Vect} \to \mathrm{Set}$$

End of lecture 6

Example. Recall the group BG from the examples above. Let us consider a functor $\rho : BG \to \mathrm{Vect}$. To the only object $*$ in BG we associate a vector space $V \in \mathrm{Vect}$ and to every endomorphism $* \to *$ we associate an endomorphism $V \to V$. Now, the functoriality property gives

$$\rho(e) = id_V$$
$$\rho(g \cdot h) = \rho(g) \circ \rho(h)$$

in which in the left hand side we have $g \cdot h$ which is the composition of morphism in BG, i.e. group multiplication of g and h, and on the right we have the composition of linear maps. We also get that

$$\rho(g) \circ \rho(g^{-1}) = \rho(gg^{-1}) = \rho(e) = id_V$$

which implies

$$\rho(g^{-1}) = \rho(g)^{-1}$$

so that the morphisms we get in Vect are also invertible. This is nothing but a representation of the group G!

Let us recall the usual definition: a representation of a group G on a vector space V is a group homomorphism $\tilde{\rho} : G \to \mathrm{Aut}(V)$. The property of being a group homomorphism is the same as the functoriality of the functor above.
In other words, representations of G are the same as functors $BG \to \mathrm{Vect}$.

Example. Is there a category of representations of a group G? In other words, is there a notion of morphisms between representations of G? Yes, that is the notion of an intertwiner: given two representations (ρ_V, V) and (ρ_W, W), an intertwiner between the two is a map $\alpha : V \to W$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "\alpha"'] & W \end{tikzcd}$$

meaning that

$$\alpha \circ \rho_V(g) = \rho_W(g) \circ \alpha$$

(This is what we always mean by commuting diagram: if there are different ways of mapping from one object of the diagram to another then they correspond to the same map.)
This is a "natural" property to ask because once we pick a certain g we could think of first applying the action on V and then going into W with α or first going into W and then applying the action on W and we would hope that the result is the same.

4.2 Monoidal categories

In order to talk about monoidal categories, let us first recall what a monoid is:

Definition 5 (Monoid)**.** A monoid M is a set with an associative multiplication $\cdot : M \times M \to M$ and a unit e such that

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
$$a \cdot e = a = e \cdot a$$

In other words, it's a group without inverses. We usually write associativity in the form above, but for the following it's useful to write it as an equality of maps $M \times M \times M \to M$ in the following way

$$\cdot \circ (\cdot \times id) = \cdot \circ (id \times \cdot)$$

meaning we first multiply the two on the left hand side or the right hand side, and then multiply the two resulting elements. The result should be the same. Also the unity property may be written analogously by comparing maps $M \to M$:

$$(e \cdot -) = id_M = (- \cdot e)$$

where $(e \cdot -)$ is the map $M \to M$ in which we take an element and give the result of its multiplication by e on the left.

This allows us to give a first definition of a (strict) monoidal category.

Definition 6 (Strict monoidal category)**.** A strict monoidal category consists of

- a category $\mathscr{C}$,
- a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$,
- a unit object $\mathbb{1} \in \mathscr{C}$,

satisfying

$$\otimes \circ (\otimes \times id_{\mathscr{C}}) = \otimes \circ (id_{\mathscr{C}} \times \otimes)$$
$$\mathbb{1} \otimes X = X = X \otimes \mathbb{1}, \forall X \in \mathscr{C}$$

(which are the same as the expressions above, replacing $\cdot$ with $\otimes$). In other words $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$.

However, it turns out that this notion is too strict in many situations: it often happens that $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ are *not* the same but they *are* isomorphic. In other words, the functors $\otimes \circ (\otimes \times id_{\mathscr{C}})$ and $\otimes \circ (id_{\mathscr{C}} \times \otimes)$ shouldn't be the *same* functor, but rather *isomorphic* functors via a map $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \forall X, Y, Z \in \mathscr{C}$. We then need a notion of morphism between functors.

Definition 7 (Natural transformation)**.** Given categories $\mathscr{C}$ and $\mathscr{D}$ and functors F, G from $\mathscr{C}$ to $\mathscr{D}$. A natural transformation $\eta : F \to G$ assigns to every $X \in \mathscr{C}$ a component map $\eta_X : F(X) \to G(X)$ with the following compatibility condition with morphisms: given $f : X \to Y$ we get $F(f) : F(X) \to F(Y)$ and $G(f) : G(X) \to G(Y)$, η_X and η_Y such that

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

This turns the the collection of functors from $\mathscr{C}$ to $\mathscr{D}$ into a category $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ with morphisms given by natural transformations.
A natural transformation in which every component map η_X is an isomorphism is called a natural isomorphism.

We can now give a weaker definition of monoidal category.

Definition 8 (Monoidal category)**.** A monoidal category is

- a category $\mathscr{C}$,
- a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$,
- a unit object $\mathbb{1} \in \mathscr{C}$,

with natural isomorphisms

- $\alpha : \otimes \circ (\otimes \times id_{\mathscr{C}}) \to \otimes \circ (id_{\mathscr{C}} \times \otimes)$ called associator,
- $\lambda : \mathbb{1} \otimes - \to id_{\mathscr{C}}$ called left unitor,
- $\rho : - \otimes \mathbb{1} \to id_{\mathscr{C}}$ called right unitor.

In addition the following pentagon and triangle diagrams should commute:

$$
\begin{tikzcd}
 & (X \otimes Y) \otimes (Z \otimes W) \arrow[dr, "\alpha_{X,Y,Z\otimes W}"] & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[ur, "\alpha_{X\otimes Y,Z,W}"] \arrow[d, "\alpha_{X,Y,Z} \otimes id_W"'] & & X \otimes (Y \otimes (Z \otimes W)) \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[rr, "\alpha_{X,Y\otimes Z,W}"'] & & X \otimes ((Y \otimes Z) \otimes W) \arrow[u, "id_X \otimes \alpha_{Y,Z,W}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},Y}"] \arrow[dr, "\rho_X \otimes id_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "id_X \otimes \lambda_Y"] \\
 & X \otimes Y &
\end{tikzcd}
$$

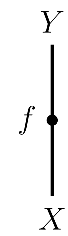
The first part is very similar to before but instead of having equalities, we have natural transformations "witnessing" the unity and associativity property. The pentagon says that moving brackets in different ways give the same result. The triangle is just a compatibility condition between the unitors and associator.

This is the notion of monoidal category that we will need. An example is the tensor product $\otimes$ in $\mathrm{Vect}_{\mathbb{C}}$: there we have natural isomorphisms $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ and $\mathbb{C} \otimes V \cong V \cong V \otimes \mathbb{C}$. In that case we have in addition that $V \otimes W \cong W \otimes V$ which is a "symmetric" structure on the monoidal category. This is analogous to the the commutativity property that a monoid M may have : $m \cdot n = n \cdot m$, or in terms of functions $\cdot = \cdot \circ t$ where t swaps the two entries $(m, n) \mapsto (n, m)$.

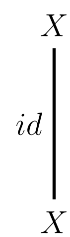
4.3 Graphical calculus

We now introduce the concept of graphical calculus which allows us represent functions diagrammatically. For example a function $f : X \to Y$ for $X, Y \in \mathscr{C}$

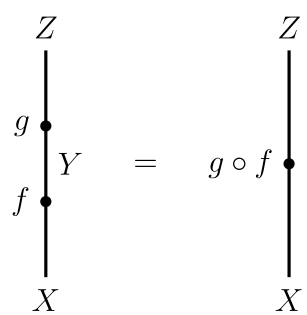
can be represented as



in which we read the picture from bottom to top. The identity morphism is usually represented as a line without a dot:

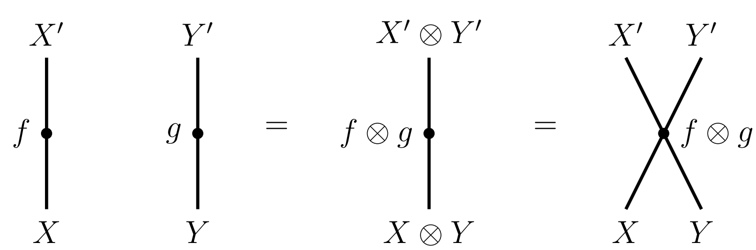


We can then represent the composition of maps as follows ("vertical composition")

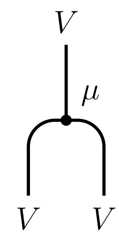


So the way to read these string diagrams is that objects are strands of the diagram and the vertices are morphisms.

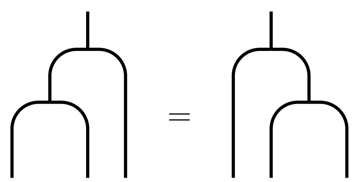
In a monoidal category we can represent tensor products as a merging of lines in the following way ("horizontal composition")



So if we have for example a multiplication $\mu : V \otimes V \to V$ in Vect we can represent it as

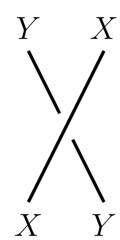


And we can now write associativity as an equality of graphs in the following way

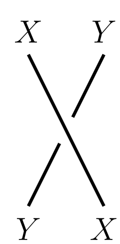


End of lecture 7

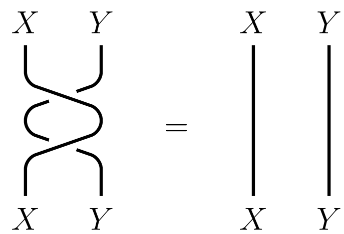
There are some natural structures that one can have on such diagrams. In particular we could have a "braiding", a map $\beta_{X,Y} : X \otimes Y \to Y \otimes X, \forall X, Y \in \mathscr{C}$ which we represent as



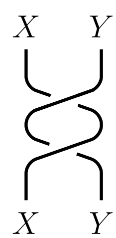
while the inverse of this map is $\beta^{-1}_{X,Y} : Y \otimes X \to X \otimes Y$ which we represent as



Graphically the fact that they're inverses may be seen as



If instead we were to try to compose two braidings after one another $\beta_{X,Y}$ and $\beta_{Y,X}$ then we would have



which in general is *not* the identity. In many cases we require it to be the identity but that's an extra condition.

4.4 Braided and symmetric categories

Now, recall the definition of a natural transformation between functors $F, G : \mathscr{C} \to \mathscr{D}$. This allows us to define the concept of equivalence of categories.

Definition 9 (Faithful, full, isomorphism, equivalence functors)**.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between categories $\mathscr{C}$ and $\mathscr{D}$. F is called

1. faithful if the map

   $$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$$

   is injective,

2. full if the map above is surjective,

3. fully faithful if it's both full and faithful,

4. an isomorphism if there is $G : \mathscr{D} \to \mathscr{C}$ such that

$$F \circ G = id_{\mathscr{D}}$$
$$G \circ F = id_{\mathscr{C}}$$

5. an equivalence if there are $G : \mathscr{D} \to \mathscr{C}$ and natural isomorphisms

$$F \circ G \cong id_{\mathscr{D}}$$
$$G \circ F \cong id_{\mathscr{C}}.$$

   So it's just like the notion of isomorphism above in which we relax the strict equalities to be instead natural isomorphisms.

This is typical in category theory: instead of requiring equalities we require natural isomorphisms. We've seen this motto in action already twice, when passing from a strict monoidal category to a monoidal category and passing from isomorphism of categories and equivalence thereof.

We now come to the concept of a braided category. Recall the comment on commutative monoids at the end of section 4.2, the concepts of braided and symmetric categories are analogues of this for a monoidal category.

Definition 10 (Braided category)**.** Let $(\mathscr{C}, \otimes, \alpha, \lambda, \rho)$ be a monoidal category. Let $\tau : \mathscr{C} \times \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ be the twist functor which sends $(X, Y) \mapsto (Y, X)$ and $(f, g) \mapsto (g, f)$. Then a braided monoidal category is a monoidal category $\mathscr{C}$ with a natural isomorphism

$$\beta : - \otimes - \to \otimes \circ \tau(-, -)$$

with components $\beta_{X,Y} : X \otimes Y \to Y \otimes X$ such that the hexagon diagram below commutes.

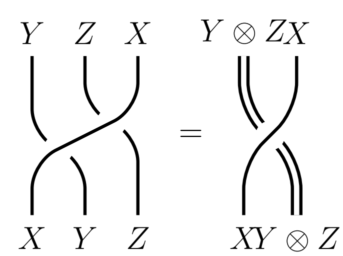
$$
\begin{tikzcd}
& (X \otimes Y) \otimes Z \arrow[dr, "\beta_{X,Y} \otimes id_Z"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "\alpha^{-1}_{X,Y,Z}"] \arrow[d, "\beta_{X,Y\otimes Z}"'] & & (Y \otimes X) \otimes Z \arrow[d, "\alpha_{Y,X,Z}"] \\
(Y \otimes Z) \otimes X & & Y \otimes (X \otimes Z) \arrow[dl, "id_Y \otimes \beta_{X,Z}"] \\
& Y \otimes (Z \otimes X) \arrow[ul, "\alpha^{-1}_{Y,Z,X}"] &
\end{tikzcd}
$$

There is also another hexagon diagram, starting with

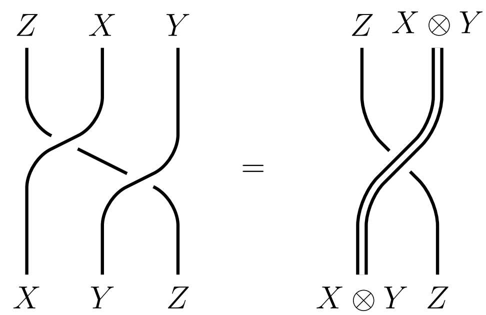
$$
\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[d, "\beta_{X\otimes Y,Z}"'] \\
Z \otimes (X \otimes Y)
\end{tikzcd}
$$

β is called a braiding.

The hexagon diagrams commuting are just a compatibility condition between the associator and the braiding. We may represent them using graphical calculus as follows



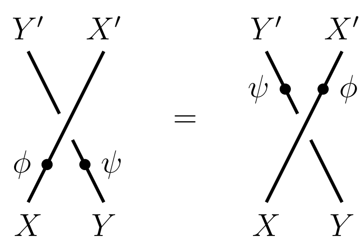
and



Let us make explicit the property of β being a natural transformation. That means that for every X, Y and morphisms $\phi : X \to X'$ and $\psi : Y \to Y'$ then the following diagram commutes

$$\begin{tikzcd} X \otimes Y \arrow[r, "\beta_{X,Y}"] \arrow[d, "\phi\otimes\psi"] & Y \otimes X \arrow[d, "\psi\otimes\phi"] \\ X' \otimes Y' \arrow[r, "\beta_{X',Y'}"] & Y' \otimes X' \end{tikzcd}$$

In terms of diagrams, this is just

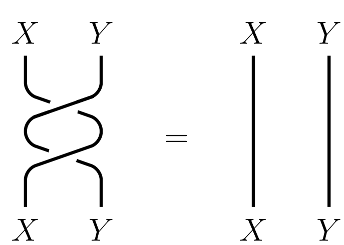


(7)

meaning that it doesn't matter "whether we put the points below or above the braiding".

There is also a notion of symmetric category.

Definition 11 (Symmetric category)**.** A braided monoidal category $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \beta)$ is called symmetric if $\beta_{X,Y} = \beta^{-1}_{Y,X}, \forall X, Y \in \mathscr{C}$. i.e.



In particular, vector spaces form a symmetric monoidal category.

Note that these are additional structures on categories, so now we also want morphisms (functors) which preserve this structure. This is the same as what we did when passing from vector spaces to algebras: we added more structure and we therefore wanted the morphisms to preserve this extra structure.

Definition 12 ((Symmetric) monoidal functor)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be two (symmetric) monoidal categories. A (symmetric) monoidal functor $F : \mathscr{C} \to \mathscr{D}$ is a functor between the two categories that preserves the additional structure. In particular we have

$$F(\mathbb{1}_{\mathscr{C}}) \cong \mathbb{1}_{\mathscr{D}} \qquad\qquad F(A \otimes_{\mathscr{C}} B) \cong F(A) \otimes_{\mathscr{D}} F(B)$$

and the functor should satisfy some compatibility condition with the associators and unitors (and braidings) of the two categories. In particular we have

$$\begin{tikzcd}
F(c) \otimes_{\mathscr{D}} F(c') \arrow[r] \arrow[d] & F(c \otimes_{\mathscr{C}} c') \arrow[d] \\
G(c) \otimes_{\mathscr{D}} G(c') \arrow[r] & G(c \otimes_{\mathscr{C}} c')
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathbb{1}_{\mathscr{D}} \arrow[dl] \arrow[dr] & \\
F(\mathbb{1}) \arrow[rr] & & G(\mathbb{1})
\end{tikzcd}$$

4.5 Duals in a monoidal category

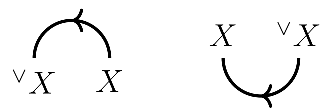
In many categories we have a concept of a dual, like the dual of a vector space. In a general category we can define the dual as follows.

Definition 13 (Left dual)**.** Let $\mathscr{C}$ be a monoidal category. A left dual of an object X is an object ${}^{\vee}X$ together with evaluation and coevalutation maps: $ev_X : {}^{\vee}X \otimes X \to \mathbb{1}$ and $coev_X : \mathbb{1} \to X \otimes {}^{\vee}X$. These maps should satisfy:

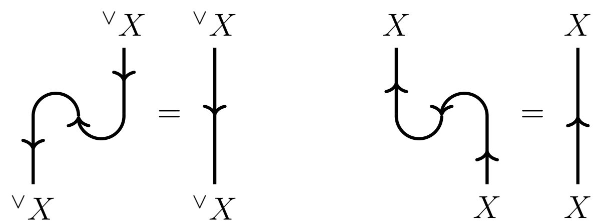
$$\lambda_{{}^{\vee}X} \circ (ev_X \otimes id_{{}^{\vee}X}) \circ \alpha^{-1}_{{}^{\vee}X,X,{}^{\vee}X} \circ (id_{{}^{\vee}X} \otimes coev_X) \circ \rho^{-1}_{{}^{\vee}X} = id_{{}^{\vee}X} \tag{8}$$
$$\rho_X(id_X \otimes ev_X) \circ \alpha_{X,{}^{\vee}X,X} \circ (coev_X \otimes id_X) \circ \lambda^{-1}_X = id_X$$

These equalities are often called ”Zorro moves” or ”snake relations”.

We represent the evaluation and coevaluation maps as

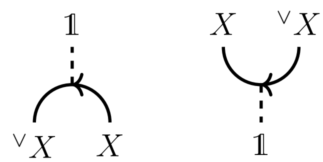


In which the direction is outgoing from X at the bottom and incoming to X at the top and the opposite for $^{\vee}X$. The snake relations are then represented as

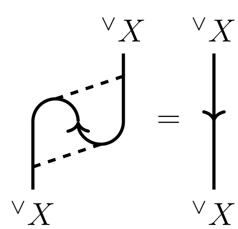


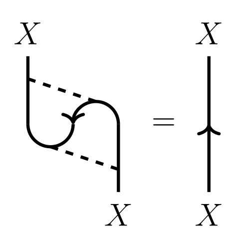
This is a small example of the usefulness of graphical calculus: the complicated looking conditions (8) simply say that we can "straighten lines" from X to X and from $^{\vee}X$ to $^{\vee}X$.

To be more precise, there are also lines that go to the units, although usually they are not drawn. Drawing them as dashed lines, evaluation and coevaluation would be



Then we would draw the snake relations as





in which we explicitly draw the unitors.

Example. Consider $V \in$ vect, where by vect we mean the category of *finite dimensional* vector spaces. Then ${}^{\vee}V = \mathrm{Hom}(V, \mathbb{C})$, but what are the evaluation and coevaluation maps? Consider a basis $\{e_i\}$ of V and the dual basis $\{e_i^*\}$ of ${}^{\vee}V$. We then define *ev* and *coev* as

$$\begin{aligned} ev_V : \quad {}^{\vee}V \otimes V &\to \mathbb{C} \\ \sum_j \phi_j \otimes \theta_j &\mapsto \sum_j \phi_j(\theta_j) \end{aligned}$$

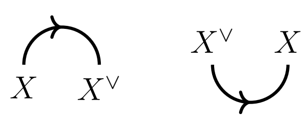
and

$$\begin{aligned} coev_V : \mathbb{C} &\to V \otimes {}^{\vee}V \\ 1 &\mapsto \sum_i e_i \otimes e_i^* \end{aligned}$$

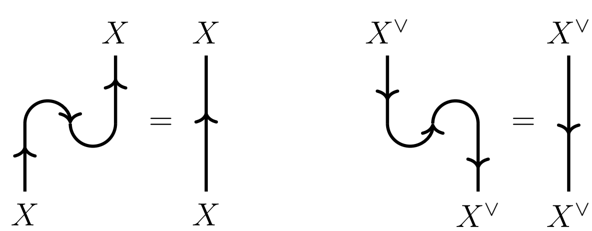
End of lecture 8

In general there's also the analogous definition for a right dual, but for example in Vect they agree.

Definition 14 (Right dual)**.** Let $\mathscr{C}$ be a monoidal category. A right dual of an object X is an object $X^{\vee}$ together with evaluation and coevalutation maps: $\widetilde{ev}_X : X \otimes X^{\vee} \to \mathbb{1}$ and $\widetilde{coev}_X : \mathbb{1} \to X^{\vee} \otimes$ represented as



satisfying the snake relations



Example. Consider the category vect of finite dimensional vector spaces. We already wrote before what the evaluation and coevaluation maps should be. Let us check that they actually make ${}^\vee V = \mathrm{Hom}(V, \mathbb{C})$ into a left dual of V, i.e. that they satisfy (8):

$$\lambda_{{}^\vee V} \circ (ev_V \otimes id_V) \circ \alpha^{-1}_{{}^\vee V, V, {}^\vee V} \circ (id_V \otimes coev_V) \circ \rho^{-1}_{{}^\vee V} = id_{{}^\vee V}$$

So the following composition

$$
{}^\vee V \xrightarrow{\rho^{-1}} {}^\vee V \otimes \mathbb{C} \xrightarrow{id_{{}^\vee V} \otimes coev_V} {}^\vee V \otimes (V \otimes {}^\vee V) \xrightarrow{\alpha_{V,{}^\vee V,V}} ({}^\vee V \otimes V) \otimes {}^\vee V \xrightarrow{ev \otimes id_{{}^\vee V}} \mathbb{C} \otimes {}^\vee V \xrightarrow{\lambda} {}^\vee V
$$

should be the identity. In particular for elements we have:

$$
{}^\vee v \longrightarrow {}^\vee v \otimes 1 \longrightarrow {}^\vee v \otimes \left(\textstyle\sum_j e_j \otimes e_j^*\right) \longrightarrow \textstyle\sum_j ({}^\vee v \otimes e_j) \otimes e_j^* \longrightarrow \textstyle\sum_j {}^\vee v(e_j) \otimes e_j^* \longrightarrow \textstyle\sum_j {}^\vee v(e_j) e_j^* = {}^\vee v
$$

Note that the expression $\sum_j {}^\vee v(e_j)e_j^*$ is exactly ${}^\vee v$ written in components. Indeed, if we wrote ${}^\vee v = \sum_i \mu_i e_i^*$ we would find

$$
{}^\vee v(e_j) = \sum_i \mu_i e_i^*(e_j) = \sum_i \mu_i \delta_{ij} = \mu_j
$$

Example. We could instead consider Vect, which includes *infinite dimensional* vector spaces. We now want to find out whether infinite dimensional vector spaces have duals. Let V be an infinite dimensional vector space. Assume there is ${}^\vee V, ev, coev$. $coev(1)$ is an element in $V \otimes {}^\vee V$ so we can always write it as a *finite* sum

$$
coev(1) = \sum_{i=1}^n v_i \otimes \phi_i
$$

for some $v_i \in V$ and $\phi_i \in {}^\vee V$. Now, the snake relation says

$$
id_V = (id_V \otimes ev_V) \circ (coev_V \otimes id_V)
$$

so applied to a generic $v \in V$

$$
v = (id_V \otimes ev_V)\left(\sum_{i=1}^n v_i \otimes \phi_i \otimes v\right) = \sum_{i=1}^n v_i \phi_i(v)
$$

but that would mean that the v_i form a basis, but that's not possible since they're a finite collection and V is infinite dimensional. That tells us that infinite dimensional vector spaces are *not* dualisable, while finite dimensional ones are both left and right dualisable.

Example. Let G be a finite group. A finite dimensional representation of G is a pair (V,ρ) in which ρ is a map $\rho : G \to \mathrm{Aut}(V)$. We can then form the dual representation by taking $V^* = \mathrm{Hom}(V,\mathbb{C})$ and we must then specify $\rho^* : G \to \mathrm{Aut}(V^*)$. Then $\rho^*(g)$ is a map $V^* \to V^*$ and we take it to be the map

$$\phi \mapsto (\ \theta \mapsto (\phi \circ \rho(g^{-1})(\theta))\ )$$

This defines a representation in the sense that $\rho^*(gh) = \rho^*(g)\rho^*(h)$.

Now, recall that the representations of a group form a category $\mathrm{Rep}(G)$, with morphisms given by intertwiners. The natural question now is: does this category have duals? Well we can take duals of the underlying vector spaces, but is everything compatible with the additional structure? i.e. do we have a representation on the dual? What are the evaluation and coevaluation maps?

We kind of have clear answers to these questions: take the dual representation above and take the evaluation and coevaluation maps from the vector space dual, i.e.

$$ev_{(V,\rho)} = ev_V \qquad coev_{(V,\rho)} = coev_V$$

However these should be morphisms in $\mathrm{Rep}(G)$, i.e. they should be intertwiners. Note also that in order to be able to talk about duals we need a monoidal structure on $\mathrm{Rep}(G)$, this is given by the tensor product representation: given (V,ρ_V) and (W,ρ_W), we define $(V\otimes W, \rho_{V\otimes W})$ by

$$\rho_{V\otimes W} = \rho_V \otimes \rho_W$$

The unit is the trivial representation $(\mathbb{C}, triv)$. Now, is ev an intertwiner? Let's check, we need

$$ev\Big(\big(\rho_{V^*\otimes V}(g)\big)(\phi\otimes v)\Big) = triv(g)ev(\phi\otimes v) = ev(\phi\otimes v)$$

This is true because

$$ev\Big(\big(\rho_{V^*\otimes V}(g)\big)(\phi\otimes v)\Big) = ev\Big(\big(\rho^*(g)\otimes\rho(g)\big)(\phi\otimes v)\Big) =$$
$$ev\Big(\phi\circ\rho(g^{-1})\ \otimes\ \rho(g)v\Big) = \phi\circ\rho(g^{-1})\Big(\rho(g)v\Big) = \phi(v) = ev(\phi\otimes v)$$

One should then also check that $coev$ is an intertwiner.

4.6 Interlude: what point are we at

Recall that we started talking about category theory right after Equation (4) after giving the definition of a TQFT as a symmetric monoidal functor. So from that point on we've been developing the language necessary to understand that definition. Let's see what point we're at. We stated that a TQFT is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_{n,n-1} \to \mathrm{vect}$$

We understood what is meant by categories and functors. We studied vect in some detail and found that it is a symmetric monoidal category, in detail $(\mathrm{vect}, \otimes, \mathbb{C}, \lambda, \rho, \alpha, \beta)$ (and we also found that objects in vect are dualisable). We have an idea of the concept of a *symmetric monoidal* functor. Instead, we don't know much about the initial category $\mathrm{Bord}_{n,n-1}$. We know that it's a category which has as objects closed $n-1$ manifolds, as morphisms n dimensional "bordisms" between $n-1$ manifolds and as a symmetric monoidal structure the disjoint union. However, we still need to understand better this concept of bordism. In order to get there we start working with some geometric categories.

End of lecture 9

4.7 Geometric categories

The category Top

One example of a geometric category is that of topological spaces. A topological space is a set X together with a topology, i.e. with a notion of open subsets. By a topology we mean a collection of subsets of X

$$\tau = \{U_i \subset X, i \in I\}$$

for some indexing set I with the following properties:

- $\emptyset \in \tau$,
- $\bigcup_{j \in J} U_j \in \tau$ for some indexing set $J \subset I$,
- $\bigcap_{j \in F} U_j \in \tau$ for some *finite* indexing set $F \subset I$.

We call the sets $U_i \in \tau$ the open subsets of X. By a morphism of topological spaces (X, τ) and (X', τ') we mean functions $f : X \to X'$ such that the preimage of open sets are open, i.e. $\forall U_i' \in \tau'$ we have $f^{-1}(U_i') \in \tau$. These are called *continuous maps*. If they are invertible they are called homeomorphisms. Topological spaces, with continuous maps, form the category Top.

The category Man

However in physics we often work with stronger concepts than this: we would like the space to locally look like $\mathbb{R}^n$ which we understand well, i.e. we'd like to work with manifolds, in particular smooth manifolds. By a manifold M we mean that there exists a cover $M = \cup_{i \in C} U_i$ together with homeomorphisms $\phi_i : U_i \to \mathbb{R}^n$. In addition we define the transition maps $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ which are maps between subsets of $\mathbb{R}^n$ and, for smooth manifolds, we impose that their partial derivatives of arbitrary order exist, i.e. that they are smooth.

For example we considered S^2 and we described it as a manifold using two patches, the north and south ones, and one can check that the transition functions are in fact smooth.

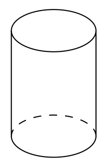
Instead, what are morphisms of smooth manifolds M and M'? Consider a map $f : M \to M'$, with M covered by $U_i, i \in I$ and M' by $U_j', j \in J$ for some indexing sets I and J. We can then "write f in charts" as the following composition

$$\mathbb{R}^n \supset \phi_i(U_i) \xrightarrow{\phi_i^{-1}} U_i \xrightarrow{f} U_j' \xrightarrow{\phi_j'} \phi_j'(U_j') \subset \mathbb{R}^{n'}$$

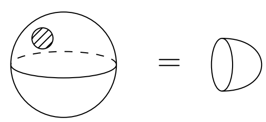
and we call f a smooth map if the composition above is smooth for every $i \in I$ and $j \in J$. Furthermore, we call f a diffeomorphism if it is invertible. Smooth manifolds, together with smooth maps, form a category Man and we will always assume our manifolds are smooth.

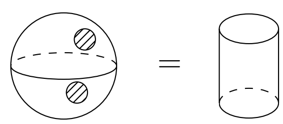
One can also similarly define manifolds with boundary. The difference is that instead of $\mathbb{R}^n$ we use $\mathbb{H}^n := \{x \in \mathbb{R}^n | x_1 \leq 0\}$ which for example in dimension 2 is the lower half plane. For a manifold with boundary we then have an atlas U_i, ϕ_i with $\phi : U_i \to \mathbb{H}^n$ and we adapt the smoothness condition to require only one sided partial derivatives to exist. The boundary ∂M consists of points $p \in M$ such that $\phi_i(p) = (0, \dots)$. In the case that

$\partial M = \emptyset$ we say that M is closed. A simple example is the cylinder which we can write as $I \times S^1$ for some interval $I \subset R$.



More in general, for some arbitrary $n-1$ closed manifold Σ we get an n manifold with boundary $I \times \Sigma$. The disk $D^2 = \{x \in \mathbb{R}^2 | \ |x| \leq 1\}$ is also a manifold, it's boundary is the circle $\partial D^2 = S^1$. One can also obtain a manifold with boundary by starting with a manifold and removing an open subset, for example we can remove open disks from the sphere.





Recall also that in Set there are two different monoidal structures: the cartesian product and the disjoint union. The cartesian product is the set of pairs of elements $X_1 \times X_2 = \{(x_1, x_2) | x_1 \in X_1, x_2 \in X_2\}$. The monoidal unit is then the set $\{*\}$ with a single element, because then $X \times \{*\} \cong X$ since we can just identify the pair $(x, *)$ with x.

Instead, by disjoint union we mean $X_1 \coprod X_2 = (X_1 \times \{1\}) \cup (X_2 \times \{2\})$ and the monoidal unit is the empty set. Why do we use the *disjoint* union?

Example. As an example consider $X_1 = \{v, w, x, y\}$ and $X_2 = \{x, y, z\}$, if we were to take the normal union we would get $X_1 \cup X_2 = \{v, w, x, y, z\}$. However this isn't ideal in the categorical framework: we would like that if we have maps $g_1 : X_1 \to S$ and $g_2 : X_2 \to S$ we also get a map $X_1 \cup X_2 \to S$. Unfortunately, using the union we don't get such a map in general, but only in the case in which X_1 and X_2 agree on their intersection. Another property we want in a categorical framework is to have natural inclusions $X_1 \subset X_1 \cup X_2$ and $X_2 \subset X_1 \cup X_2$ which we *do* have in this case.

We then define $X_1 \coprod X_2 = \{v_1, w_1, x_1, y_1, x_2, y_2, z_2\}$, now we don't have inclusions since $X_1 \not\subset X_1 \coprod X_2$ and $X_2 \not\subset X_1 \coprod X_2$, rather we have inclusion *maps*

$$i_1 : X_1 \to X_1 \coprod X_2$$
$$i_2 : X_2 \to X_1 \coprod X_2$$

and we also solve the problem of the maps: given maps $g_1 : X_1 \to S$ and $g_2 : X_2 \to S$ there exists a unique map $g_1 \coprod g_2 : X_1 \coprod X_2 \to S$ such that the diagram below commutes

$$\begin{tikzcd} & S & \\ X_1 \arrow[ur, "g_1"] \arrow[r, "i_1"'] & X_1 \coprod X_2 \arrow[u, dashed, "\exists!"] & X_2 \arrow[ul, "g_2"'] \arrow[l, "i_2"] \end{tikzcd} \tag{9}$$

This property is called the universal property of the coproduct. Note that to formulate this property we don't use any special property of sets, this means that we can define the coproduct in an arbitrary category the same way! Such definitions, via universal properties, appear quite often when using the language of category theory.

We can define analogous monoidal structures in Top and in Man. We're more interested in the disjoint union, since that's the monoidal structure we'll take on our the category we mostly care about: Bord.

4.8 The category Bord

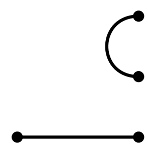
4.8.1 Morphisms: bordisms

We mentioned that in the bordism category, the morphisms will be given by bordisms, which are manifolds themselves, so we need to define what we mean by bordism and then we need a way of "composing" them, i.e. we need a way of gluing manifolds. We start with the definition of bordisms and we'll later see how to "glue" them.

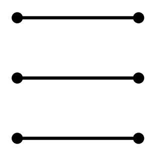
Definition 15 (Bordism)**.** Let E_1 and E_2 be $n-1$ compact manifolds. A bordism $M : E_1 \to E_2$ is an n manifold with boundary such that $\partial M = E_1 \cup E_2$. If such a bordism exists, E_1 and E_2 are called cobordant.

One can also prove that being cobordant is an equivalence relation. We can then take equivalence classes of cobordant manifolds and this leads to the cobordism groups.

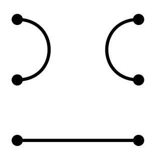
Example. As an example, consider the case $n = 1$, so $n - 1 = 0$. In this case we have that E_1 and E_2 are 0 manifolds, i.e. just collections of points. Instead a bordism M between them is just a collection of lines. For example we see that if $E_1 = \{*\}$ and $E_2 = \{*, *, *\}$ they are cobordant since we have



Also, since it's equivalence relation every manifold is cobordant to itself, for example for E_2 we have

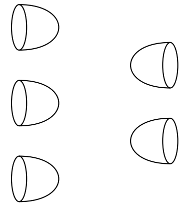


but we also have another bordism from E_2 to itself:

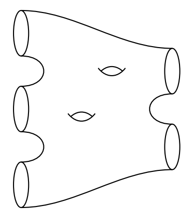


Instead, there is no bordism from one point to two points, this is because any 1 manifold has an even number of boundary points. So in general we see that two collections of points are cobordant if they have the same number of points mod 2.

Example. Consider now the case $n = 2$ and $n - 1 = 1$. Now E_1 and E_2 are collections of circles and a bordism M between them is a 2 manifold, a surface. Here we have that E_1 and E_2 are always cobordant because I can always draw a "cup" for each circle, for example for $E_1 = \coprod_{i=1,2,3} S^1$ and $E_2 = \coprod_{i=1,2} S^1$ we have



But again, as before, that is not the only bordism between the two:



In other words, a circle is cobordant to the empty set, so all collections of circles are cobordant to the empty set and therefore also to each other.

In higher dimensions, it gets much more complicated.

End of lecture 10

4.8.2 Composition: gluing

We now want to define what is meant by composition of cobordisms. What we want to do is to consider collars around the boundaries of two different bordisms and then glue them along the collars. We start by considering gluing in the category Set which is simpler: given sets X_1, X_2 and Y with maps $\theta_i : Y \to X_i$ we can define the "gluing of X_1 and X_2 along Y" as the set

$$X_1 \coprod_Y X_2 := \left(X_1 \coprod X_2\right)/_\sim$$

in which we identify $(x_1, 1) \sim (x_2, 2)$ if $\exists y \in Y$ such that $x_1 = \theta_1(y)$ and $x_2 = \theta_2(y)$. You should think of θ_i giving you a copy of Y inside of both X_1 and X_2 and then $X_1 \coprod_Y X_2$ is exactly the gluing of X_1 and X_2 along the two copies of Y.

This concept of gluing can be phrase more categorically by saying that $X_1 \coprod_Y X_2$ satisfies a universal property (analogous to (9)): for any set S with maps $g_1 : X_1 \to S$ and $g_2 : X_2 \to S$ with $g_1 \circ \theta_1 = g_2 \circ \theta_2$ there exists a unique map $X_1 \coprod_Y X_2 \to S$ such that the following diagram commutes

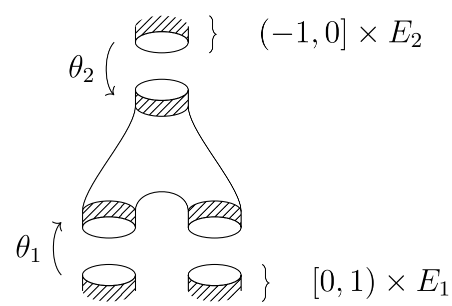
$$\begin{tikzcd}
 & S & \\
X_1 \arrow[ur, "g_1"] \arrow[r, "i_1"'] & X_1 \coprod_Y X_2 \arrow[u, dashed, "\exists!"'] & X_2 \arrow[ul, "g_2"'] \arrow[l, "i_2"] \\
 & Y \arrow[ul, "\theta_1"] \arrow[ur, "\theta_2"'] &
\end{tikzcd}$$

This is just saying that if I have maps $X_1 \to S$ and $X_2 \to S$ which agree on their copies of Y I get a map out of the gluing of X_1 and X_2 along Y.

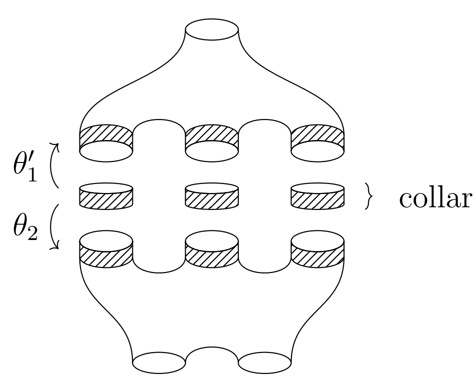
To do the same kind of gluing with bordisms, we introduce the concept of a bordism with collars:

Definition 16 (Bordism with collars)**.** Let E_1 and E_2 be $n-1$ compact manifolds. A bordism with collars from E_1 to E_2 is a triple (M, θ_1, θ_2) given by a manifold M and embeddings $\theta_1 : [0,1) \times E_1 \to M$ and $\theta_2 : (-1, 0] \times E_2 \to M$. In addition, we define the "in" boundary as $(\partial M)_1 := \theta_1(\{0\} \times E_1)$ and the "out" boundary as $(\partial M)_2 := \theta_2(\{0\} \times E_2)$. The boundary of M should be given by $\partial M = (\partial M)_1 \coprod (\partial M)_2$.

So for example for the case $E_2 = S^1$ and $E_1 = S^1 \times S^1$ we have the following bordism with collars:



We can then define the composition of bordisms with collars to simply be the gluing along collars: given bordisms $M : E_1 \to E_2$ and $M' : E_2 \to E_3$ we get a bordism $E_1 \to E_3$ given by $M \coprod_{\text{collar}} M'$, as in the following drawing.



In addition, in the category Bord that we're building, we take the morphisms to be *diffeomorphism classes of* bordisms (with a suitable notion of diffeomorphisms of bordisms). Given a bordism $M : E \to E'$ we may then indicate the morphism as $[M] : E \to E'$ to specify we're taking diffeomorphism classes.

4.8.3 Braiding: twisting cylinders

We now have objects (closed n manifolds) and morphisms (bordisms). In addition we also have a monoidal structure, given simply by the disjoint union. The category we're building also has a braiding which we now want to understand.

Twisted cylinders

We start with the notion of a twisted cylinder. Let E be an $(n-1)$ manifold and consider the cylinder $M = I \times E$, where $I = [0,1]$. Clearly $(\partial M)_1 = \{0\} \times E$ and $(\partial M)_2 = \{1\} \times E$, but in order for M to be a bordism $E \to E$ we need to specify what the maps θ_1 and θ_2 are. Consider

$$\theta_1 : [0,1) \times E \to I \times E \quad \text{sending} \quad (t,p) \to \left(\frac{t}{2}, p\right)$$

$$\theta_2 : (-1,0] \times E \to I \times E \quad \text{sending} \quad (t,p) \to \left(\frac{t}{2} + 1, p\right)$$

This gives the untwisted cylinder.
An alternative is to take θ_1 as above and $\theta_2(t,p) = (\frac{t}{2} + 1, f(p))$ for any diffeomorphism $f : E \to E$. This gives a twisted cylinder C_f. With this notation, the untwisted cylinder is C_{id}.
We could also "twist on both sides", by taking two diffeomorphisms $f_1, f_2 : E \to E$ and taking $\theta_1(t,p) = (\frac{t}{2}, f_1(p))$ and $\theta_2(t,p) = (\frac{t}{2} + 1, f_2(p))$. We then indicate this cylinder by C_{f_1,f_2}. This is not really a different concept since we have $[C_{f_1,f_2}] = [C_{f_1^{-1} \circ f_2}]$, i.e. instead of twisting on both sides, I can twist just on one side by both maps.
One question that now arises is the following: is it possible to get diffeomorphic bordisms when twisting with different maps? The answer is yes, it happens when the maps are pseudodiffeotopic. Two diffeomorphisms $f_1, f_2 : E \to E$ are called pseudodiffeotopic if there is a map $H : [0,1] \times E \to [0,1] \times E$ such that $H(0,p) = (0, f_1(p))$ and $H(1,p) = (1, f_2(p))$. In that case we then get $[C_{f_1,f_2}] = [C_{id}]$. Note that diffeotopic maps are also pseudodiffeotopic.

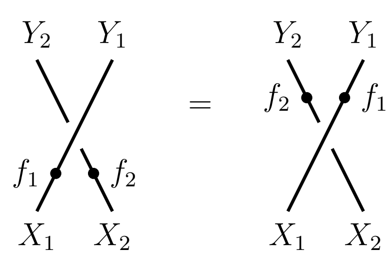
Example. Let's consider the case $\dim E = 1$, in particular $E = S^1$. As maps $S^1 \to S^1$ we can take $f(e^{i\alpha}) = e^{-i\alpha}$ and id. One can show that these are not (pseudo)diffeotopic and that all other maps are (pseudo)diffeotopic to one of those. We therefore get that the only cylinders we can get with S^1 are $[C_{id}]$ and $[C_f]$ for f above. In addition, recalling Definition 1 of the mapping class group, we find $MCG(S^1) \cong \mathbb{Z}_2$.

Braiding in Set

We now come to the braiding. It may seem like a trivial question since we're taking the disjoint union: given closed $(n-1)$ manifolds M and M' their disjoint union is $M \coprod M'$, isn't it the same if I take $M' \coprod M$? Unfortunately no, but we'll see that there is a natural isomorphism, which will be the braiding.
Note that this is nothing new, the same issue arises in the category Set (and in many other symmetric monoidal categories): we have $X_1 \coprod X_2 \neq X_2 \coprod X_1$, since the first is $(X_1 \times \{1\}) \cup (X_2 \times \{2\})$ while the second is $(X_2 \times \{1\}) \cup (X_1 \times \{2\})$. There is however an obvious bijection between the two, given simply by the twist map $tw : X_1 \coprod X_2 \to X_2 \coprod X_1$ which exchanges the labels 1 and 2. This map is also *natural* in the categorical sense:

$$\begin{tikzcd}
X_1 \coprod X_2 \arrow[r, "tw"] \arrow[d, "f_1 \coprod f_2"'] & X_2 \coprod X_1 \arrow[d, "f_2 \coprod f_1"] \\
Y_1 \coprod Y_2 \arrow[r, "tw"'] & Y_2 \coprod Y_1
\end{tikzcd}$$

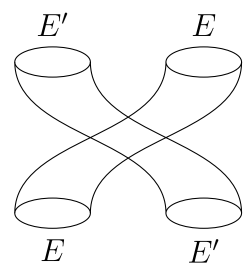
which, as in (7) may also be drawn as



tw is exactly the braiding in Set. In addition, twisting twice is the same as doing nothing, i.e. $tw \circ tw = id$, so Set is not only braided monoidal but also symmetric monoidal.

Braiding in Bord

We now come back to bordism category. Consider the diffeomorphism $tw_{E,E'}: E \coprod E' \to E' \coprod E$. Take its twisted cylinder $C_{tw_{E,E'}}$, this gives a bordism $E \coprod E' \to E' \coprod E$ we can then call this bordism $\beta_{E,E'}$ and this is actually a braiding in our category Bord.



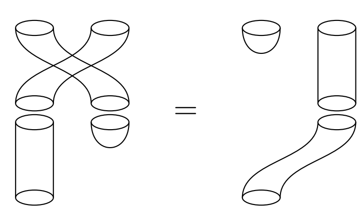
Again, this bordism is natural in the sense that we have the following commuting diagram:

$$\begin{tikzcd} E \coprod E' \arrow[r, "\beta_{E,E'}"] \arrow[d, "M \coprod M'"'] & E' \coprod E \arrow[d, "M' \coprod M"] \\ \tilde{E} \coprod \tilde{E}' \arrow[r, "\beta_{\tilde{E},\tilde{E}'}"'] & \tilde{E}' \coprod \tilde{E} \end{tikzcd}$$

As an example, consider the case $E = S^1, E' = \emptyset, \tilde{E} = S^1, \tilde{E}' = S^1$, with M is a cylinder and M' is the cup, we then have the following diagram

$$\begin{tikzcd} S^1 \coprod \emptyset \arrow[r, "\beta_{S^1,\emptyset}"] \arrow[d, "M \coprod M'"'] & \emptyset \coprod S^1 \arrow[d, "M' \coprod M"] \\ S^1 \coprod S^1 \arrow[r, "\beta_{S^1,S^1}"'] & S^1 \coprod S^1 \end{tikzcd}$$

which is true because of the following drawings, in which on the left we first use the bordism $M \coprod M'$ and then β_{S^1,S^1} and on the right we first have $\beta_{S^1,\emptyset}$ and then $M' \coprod M$.



4.8.4 Putting it all together

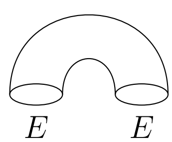
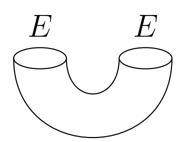
We now finally understand the category of n dimensional bordisms $\text{Bord}_{n,n-1}$, it has the following structure:

- objects: closed $(n-1)$ manifolds,
- morphisms: diffeomorphism classes of bordisms,
- composition given by gluing (identity is $[C_{id}]$),
- unit morphism $1_E := [C_{id_E}]$,
- monoidal structure given by the disjoint union (monoidal unit the empty set, seen as an $(n-1)$ manifold),
- braiding $\beta_{E,E'} := [C_{tw_{E,E'}}]$.

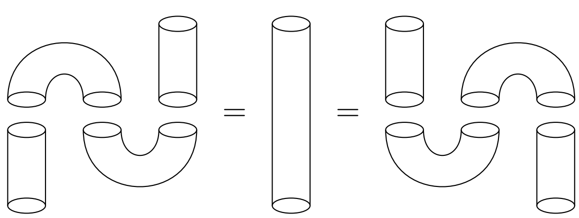
End of lecture 11

4.8.5 Duality in Bord

Do we have duals in $\text{Bord}_{n,n-1}$? Yes! We have $E^\vee = E = {}^\vee E$ because we can take the following bordisms $ev_E : E \coprod E \to \emptyset$ and $coev_E : \emptyset \to E \coprod E$

$ev_E =$  $coev_E =$ 

and the snake relations holds because



(10)

where the cylinder is indeed the unit morphism. Note that as a manifold, the evaluation and coevaluation are just $E \times I$ but we see them as such bordisms by taking appropriate θ_1 and θ_2 maps.

4.9 The oriented bordism category Bord^{or}

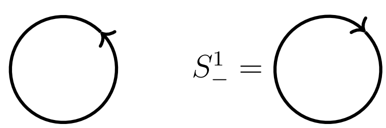
We now want to add more structure to the bordism category. The simplest additional structure to add is an orientation.
We start by reviewing the concept of orientation for vector spaces. Let V be a vector space and consider the set of all its possible bases. There is an equivalence relation on this set given by saying that 2 bases are equivalent if there is a linear map $A : V \to V$ with $\det A > 0$ sending one basis to the other. This gives rise to two equivalence classes of bases and a choice of one such equivalence class is what we call orientation. For example in $\mathbb{R}^3$ the one orientation is given by picking all right handed bases while the other is given by picking left handed bases.

What happens for manifolds? Recall that we can take at each point T_pM, the tangent space at that point, and for each such tangent space we can define an orientation as above. An orientation of a manifold is a smooth choice of orientation at each tangent space. However, not all manifolds have an orientation! It may happen that it's not possible to extend the orientation nicely to the whole manifold. For example, for some given coordinates, we can take the basis of T_pM to be $(\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p)$, but in general this cannot be extended to the whole manifold. For an orientable connected manifold we have one two possible orientations, while for a disjoint union of oriented manifolds we have two possibilities for each component, i.e. there are $2^{\#\text{conn. components}}$ possible orientations. Given a manifold M with an orientation, we define $\overline{M}$ to be the manifold with the opposite orientation (on each component if M is not connected).

In order to talk about the oriented bordism category, we also need the concept of oriented bordism. To be able to define it, we need the notion of orientation preserving and reversing maps.

Example. Consider for example the circle with two different orientations

$S^1_+ =$ $S^1_- =$



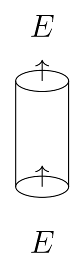
Consider the map $f : S^1 \to S^1$ with $f(e^{i\alpha}) = e^{-i\alpha}$. This map is orientation *reversing* if we view it as a map $S^1_+ \to S^1_+$ (or $S^1_- \to S^1_-$). Instead, it is orientation *preserving* when seen as a map $S^1_+ \to S^1_-$ (or $S^1_- \to S^1_+$).

In general, given oriented manifolds M and N, a diffeomorphism $f : M \to N$ induces an orientation on N. If the induced orientation is the same as the orientation on N we say that f is orientation preserving, otherwise we say it is orientation reversing.

Now, given oriented closed manifolds E_1 and E_2, an oriented bordism $(M, \theta_1, \theta_2) : E_1 \to E_2$ is an oriented manifold M together with orientation preserving embeddings $\theta_1 : [0, 1) \times E_1 \to M$ and $\theta_2 : (-1, 0] \times E_2 \to M$. Note that an orientation on M induces an orientation on ∂M, in particular on both $(\partial M)_1$ and $(\partial M)_2$. By restricting we then have the following diffeomorphisms

$$\begin{aligned} \theta_1|_{\{0\}\times E_1} &: \{0\} \times E_1 \to (\partial M)_1 \\ \theta_2|_{\{0\}\times E_2} &: \{0\} \times E_2 \to (\partial M)_2 \end{aligned}$$

but because of how the product and boundary orientations are defined we have that $\theta_1|_{\{0\}\times E_1}$ is an orientation *reversing* diffeomorphism, while $\theta_2|_{\{0\}\times E_2}$ is an orientation *preserving* diffeomorphism. For example the "vertical" cylinder $M = E \times I$

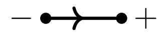


has boundaries $(\partial M)_2 = E$ and $(\partial M)_1 = \overline{E}$, where the opposite orientation on the "in" boundary is important and allows us to view M as a bordism $E \to E$.

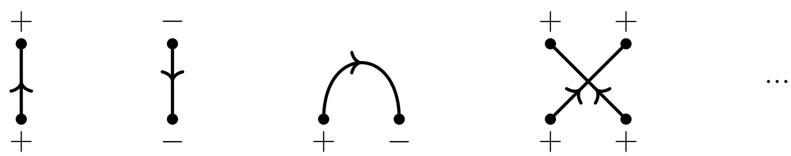
We then get the category $\text{Bord}^{or}_{n,n-1}$ given by:

- objects: oriented $(n - 1)$ manifolds,
- morphisms: orientation preserving diffeomorphism classes of oriented n dimensional bordisms
- unit, monoidal structure etc. inherited from $\text{Bord}_{n,n-1}$.

Example. Consider for example the category $\text{Bord}^{or}_{1,0}$. Now points can have a positive or negative orientation (a label of $+$ or $-$) and generic objects are disjoint unions of points, each with its own label. Instead the bordisms are oriented 1 manifolds with boundary. Note that an oriented line segment induces the following orientations on its boundary points:



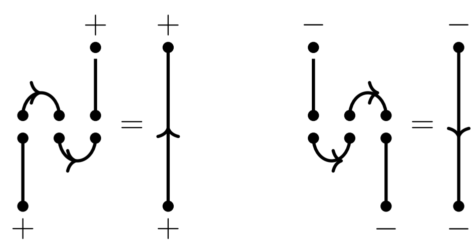
With this in mind, some of the bordisms we get are the following



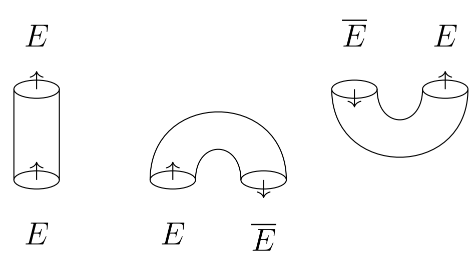
Where we remember that we have to flip the orientation for the incoming manifolds.

We can also see that $\bullet_+$ and $\bullet_-$ are dual to each other, since we have evaluation and coevaluation maps

which satisfy the snake relations



Indeed, in an analogous way, objects $E \in \mathrm{Bord}^{or}_{n,n-1}$ are dualizable: the dual object is $\overline{E}$ with $ev : E \coprod \overline{E} \to \emptyset$ and $coev : \emptyset \to \overline{E} \coprod E$, given again the cylinder $E \times I$, regarded in different ways. The snake relations are satisfied exactly as in (10).



We can then give the following definitions of TQFT (generalizing (6) to a generic destination category) and oriented TQFT.

Definition 17. An n dimensional (closed) TQFT with values in a symmetric monoidal category $\mathscr{C}$ is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_{n,n-1} \to \mathscr{C}$$

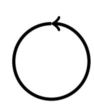
An n dimensional oriented TQFT is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}^{or}_{n,n-1} \to \mathscr{C}$$

As we saw previously, a possible choice for the target category is $\mathscr{C} =$ vect. In that case, to an object $E \in \mathrm{Bord}^{or}_{n,n-1}$ we associate a vector space $\mathcal{Z}(E) \in$ vect which we interpret as a state space. Since we have a functor we also have the following associations:

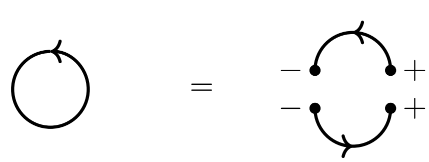
$$\begin{array}{ccc}
\text{bordism } (M, \theta_1, \theta_2) : E_1 \to E_2 & \longrightarrow & \text{linear map } \mathcal{Z}(E_1) \to \mathcal{Z}(E_2) \\
\text{gluing of bordisms} & \longrightarrow & \text{composition of linear maps} \\
E \coprod E' & \longrightarrow & \mathcal{Z}(E) \otimes \mathcal{Z}(E') \\
(E, \overline{E}, ev_E, coev_E) & \longrightarrow & (\mathcal{Z}(E), \mathcal{Z}(E)^{\vee}, ev_{\mathcal{Z}(E)}, coev_{\mathcal{Z}(E)})
\end{array}$$

Example. We can now finally do an explicit calculation with this formalism. Consider a TQFT with values in vect $\mathcal{Z} : \mathrm{Bord}_{n,n-1} \to \mathrm{vect}_{\mathbb{C}}$. Consider S^1 as an oriented bordism from the empty set to itself:



Now, since $\mathcal{Z}$ is a symmetric monoidal functor, we have $\mathcal{Z}(\emptyset) = \mathbb{C}$ and therefore $\mathcal{Z}(\circlearrowleft)$ is a linear map $\mathbb{C} \to \mathbb{C}$ which is just multiplication by a number.

How can we calculate what number it is? Well, that bordism may be seen as a composition of bordism:



Which are just an evaluation and coevaluation (note that they are *not* duality data of a single dual, but rather evaluation of a left dual and coevaluation of a right dual, however since we're in a symmetric category this is not an issue). This means that they are also sent to evaluation and coevaluation in vect, which we know. Since $\mathcal{Z}$ is a functor and $\bullet_+$ is an object, we have that $\mathcal{Z}(\bullet_+)$ is a vector space, which we call V. Then $V^* = \mathcal{Z}(\bullet_-)$ and we can calculate the

Let us then call $V^* = \mathcal{Z}(\bullet_+)$ and calculate $\mathcal{Z}(\circlearrowright)$ as the composition

$$\mathbb{C} \xrightarrow{coev} V^* \otimes V \xrightarrow{ev} \mathbb{C}$$

which is simply

$$1 \longrightarrow \textstyle\sum_i e_i^* \otimes e_i \longrightarrow \textstyle\sum_i e_i^*(e_i) = \dim V$$

so $\mathcal{Z}(\circlearrowright) = \dim V$.

End of lecture 12

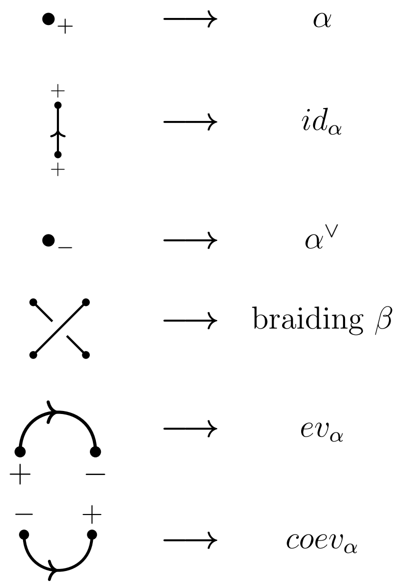
4.10 Classification of TQFTs

4.10.1 Oriented 1d TQFTs

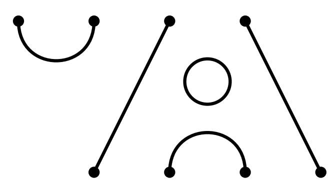
Let us now consider a general category $\mathscr{C}$ and consider in detail a generic 1d TQFT $\mathcal{Z} : \mathrm{Bord}_{1,0}^{or} \to \mathscr{C}$. Recall that the category $\mathrm{Bord}_{1,0}^{or}$ has:

- objects: collections of points, with orientation given by a labeling by $+$ or $-$,
- morphisms: 1d oriented bordisms which are given by oriented line segments and S^1s.

Functoriality then gives us the following associations between things in $\mathrm{Bord}_{1,0}^{or}$ and in $\mathscr{C}$:



where $\alpha \in \mathscr{C}$ is an object which must be dualizable since it's the image of a dualizable object α with duality data $(\alpha^\vee, ev_\alpha, coev_\alpha)$. Note that once $\alpha = \mathcal{Z}(\bullet_+)$ is fixed, all of the other associations are fixed and also generic bordisms such as



may be written as compositions of those. This is a very important observation and tells us that the TQFT is fully determined by the element to which $\mathcal{Z}(\bullet_+)$ is mapped. We have thus classified *all* 1d TQFTs into a generic category $\mathscr{C}$
.

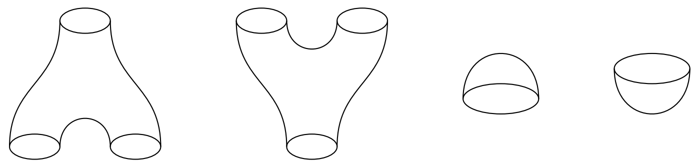
4.10.2 Oriented 2d TQFTs

The 2d oriented bordism category $\mathbf{Bord}^{or}_{2,1}$

We now wish to classify oriented 2d TQFTs and here we have slightly more structure. Recall that $\mathrm{Bord}^{or}_{2,1}$ has

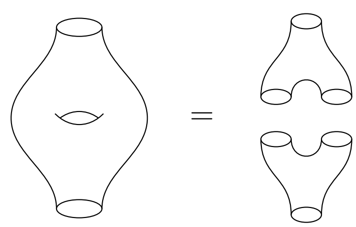
- objects: oriented circles and their disjoint unions,

- morphisms: compositions and disjoint unions of the following bordisms



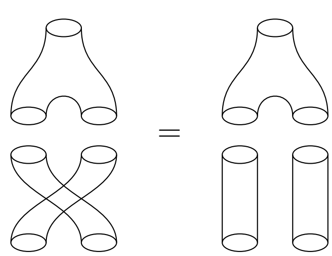
  and of twisted cylinders $[C_f]$ for any diffeomorphism $f : E \to E$ for a given object E.

We call the basic bordisms above respectively product, coproduct, counit, unit and the reason for this naming will become clear soon. Some examples of twisted cylinders were $[C_{id}]$ on the single circle or $[C_{tw}]$ for two circles, which gave us the braiding since any permutation $g \in S_n$ of n circles comes from this basic one. As an example of generic bordisms being decomposable into the basic bordisms above, we have the following decomposition:

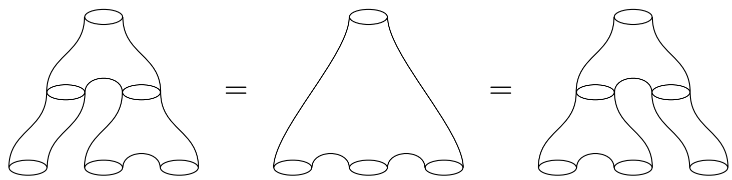


In order to prove rigorously that any 2d bordism can be constructed from the basic bordisms above one first has to find a classification of 2 dimensional oriented manifolds with boundary and then one can find an algorithmic method to obtain any oriented 2d manifold with boundary from the ones above. This will be explored in more detail in the next exercise sheet.

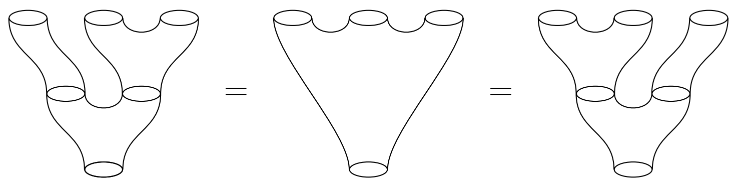
In the set of morphisms we have the following relations:



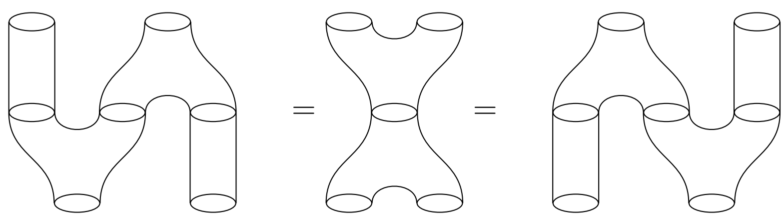
but also



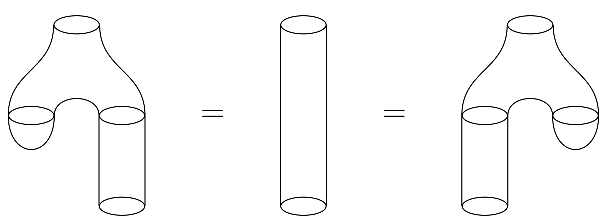
and

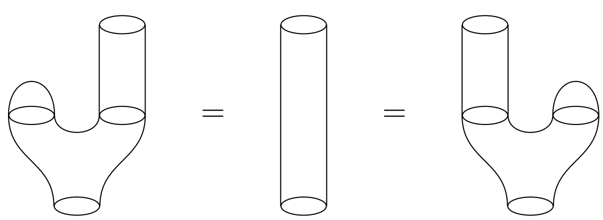


but more interestingly



We also have relations including the cup and cocup bordisms:



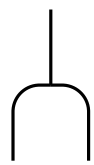


It's important to note these relations, because they should be preserved in the target category of the TQFT.

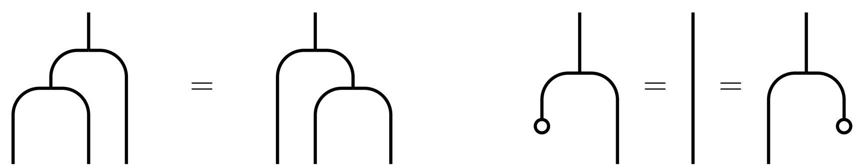
Frobenius algebra

Let us consider such properties in a generic category $\mathscr{C}$.

Definition 18. An associative unital algebra object is an object $A \in \mathscr{C}$ with morphisms:

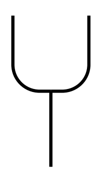
$\mu =$  $\eta =$ 

called product and unit, such that

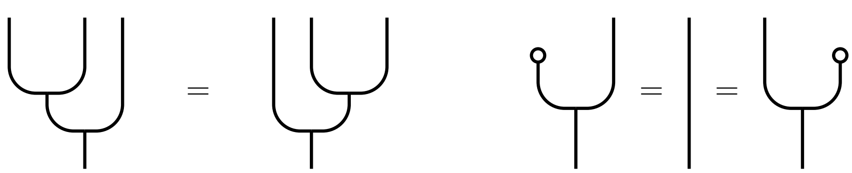


Note that we're in a generic category! $\mathscr{C}$, and therefore also A, could be anything! The name however is quite suggestive, in fact an associative unital algebra object in vect is a vector space with the properties above, and that is exactly the usual notion of an associative unital algebra. We won't always specify associative and unital but it will be implied unless otherwise stated.

Definition 19. A (coassociative counital) coalgebra object is an object $A \in \mathscr{C}$ with morphisms:

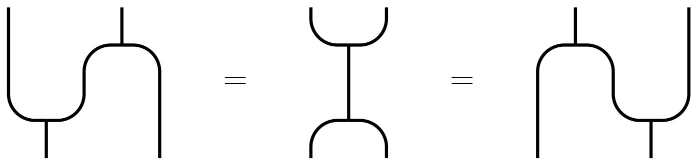
$\Delta =$  $\epsilon =$ 

called coproduct and counit, such that

 (11)

We can now put these together, with an additional relation.

Definition 20. A Frobenius algebra $(A, \mu, \eta, \Delta, \epsilon)$ is both an algebra and coalgebra such that

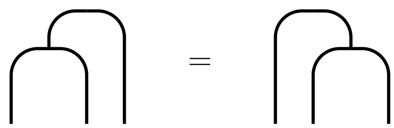


(12)

Note that we didn't come up with these relations arbitrarily: they are exactly the relations that we had in $\mathrm{Bord}_{2,1}$ above!

There is also another, equivalent, definition of a Frobenius algebra.

Definition 21. A Frobenius algebra is an algebra (A, μ, η) together with a nondegenerate $k : A \otimes A \to \mathbb{1}_{\mathscr{C}}$ which is also invariant, i.e.

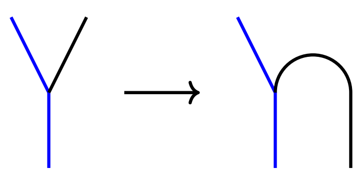


Let us recall the notion of degenerate map for finite dimensional vector spaces. Let A be a vector space and $k : A \otimes A \to \mathbb{C}$, we get a map $A \to A^*$ by sending $a \to k(a, -)$ and the nondegeneracy condition is that this map be an isomorphism. We could define degeneracy as an analogous condition in a generic category $\mathscr{C}$, but instead it's useful to see it differently: for arbitrary $Y, Z \in \mathrm{vect}$, k gives us an isomorphism $\mathrm{Hom}(Z, Y \otimes A) \to \mathrm{Hom}(Z \otimes A, Y)$ since we have

$$\begin{aligned}\mathrm{Hom}(Z, Y \otimes A) &\cong Z^* \otimes Y \otimes A \cong Z^* \otimes A \otimes Y \cong Z^* \otimes A^* \otimes Y \\ &\cong (Z \otimes A)^* \otimes Y \cong \mathrm{Hom}(Z \otimes A, Y)\end{aligned}$$

where the third isomorphism uses the isomorphism $A \to A^*$ we found above. We define degeneracy in a generic category by this isomorphism.

Definition 22 (Nondegenerate pairing)**.** Let $\mathscr{C}$ be a monoidal category and $A \in \mathscr{C}$ a dualizable object. Then a map $k : A \otimes A \to \mathbb{1}_{\mathscr{C}}$ (a *pairing* on A) is called *nondegenerate* if the induced map $\mathrm{Hom}_{\mathscr{C}}(Z, Y \otimes A) \to \mathrm{Hom}_{\mathscr{C}}(Z \otimes A, Y)$ given by



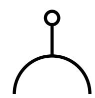
is an isomorphism $\forall Z, Y \in \mathscr{C}$.

Note that, in addition, we get that the pairing k is part of duality data manifesting A as its own dual, i.e. there exists $\gamma : \mathbb{1}_{\mathscr{C}} \to A \otimes A$ such that (A, k, γ) are duality data for A being (left and right) dual to itself.

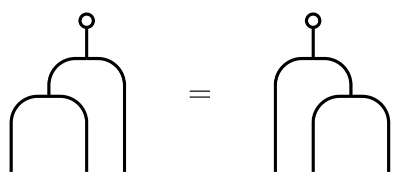
Since we gave two definitions of Frobenius algebras, we should prove that they are in fact equivalent.

Proof.

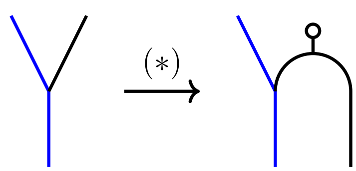
($\underline{1 \implies 2}$) We start by proving that the first definition implies the second one. We should therefore define the pairing $k : A \otimes A \to \mathbb{1}_{\mathscr{C}}$ and the obvious way to define it is by $k = \epsilon \circ \mu$:

$k :=$  (13)

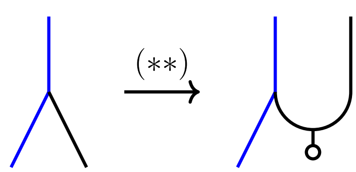
We then automatically have that k is invariant by associativity of μ:



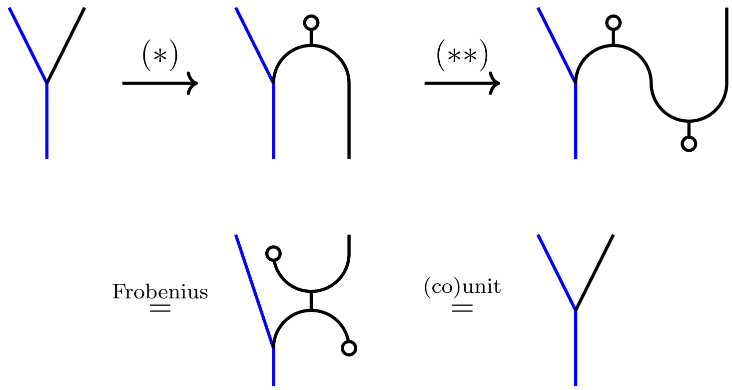
We now need to show that the map



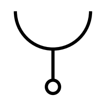
from $\mathrm{Hom}_{\mathscr{C}}(Z, Y \otimes A)$ to $\mathrm{Hom}_{\mathscr{C}}(Z \otimes A, Y)$ is invertible. To do this we provide an explicit inverse $\mathrm{Hom}_{\mathscr{C}}(Z \otimes A, Y) \to \mathrm{Hom}_{\mathscr{C}}(Z, Y \otimes A)$:



Let us check that it is in fact an inverse by first applying one and then the other



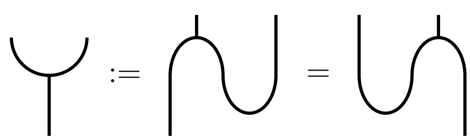
At first, it looks pretty different from what we started with, but using the Frobenius relations and then the properties of unit and counit we can go back to the original diagram. Note, in addition, that thanks to the Frobenius relations we have that

$\gamma :=$  (14)

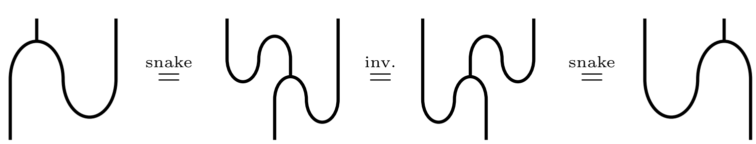
is the coevaluation that with k witnesses A as dual to itself. This concludes the first part.

End of lecture 13

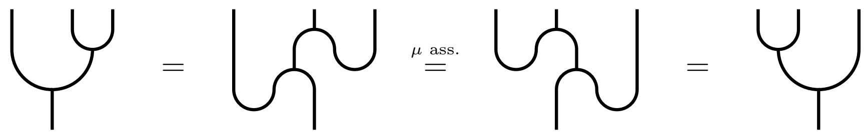
(<u>$2 \implies 1$</u>) We now want to show that the second definition implies the first. To do this we start with an algebra (A, μ, η) with a nondegenerate invariant form k (thanks to which we also have a coevaluation γ) and we want to find coproduct and counit maps. We define $\Delta : A \to A \otimes A$ by

$\Delta =$  (15)

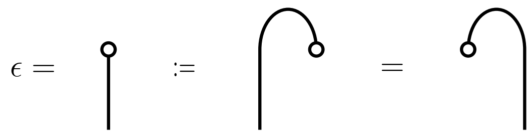
Note that the last equality is true by invariance:



Δ is coassociative thanks to the associativity of μ:

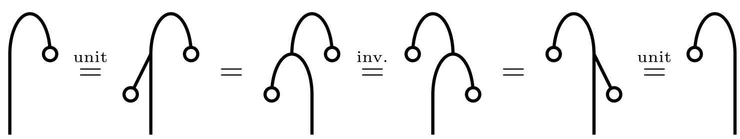


in which we use the two equivalent definitions of Δ. We can then define the counit by

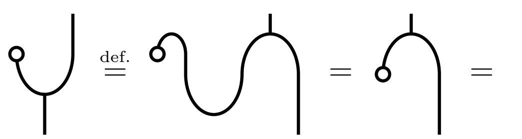


(16)

and again the second equality follows from invariance:



and it *is* in fact a counit by



(counit on the other side is checked analogously). $\square$

We can then think of a Frobenius algebra as an object having maps $\mu, \eta, \Delta, \epsilon, k, \gamma$ with the relations (13), (14), (15) and (16) above.
Let us consider an example to understand the concept of a Frobenius algebra better.

Example. Consider a finite group G and its group algebra $\mathbb{C}G$. The regular representation of G is the vector space $V = \mathbb{C}^{|g|}$ with basis $\{e_g | g \in G\}$ with the representation $\rho_{reg} : G \to \mathrm{Aut}(V)$ given by $\rho_{reg}(h)e_g = e_{hg}$. This is indeed a representation since

$$\rho_{reg}(h_1 h_2)e_g = e_{h_1 h_2 g} = \rho_{reg}(h_1)e_{h_2 g} = \rho_{reg}(h_1)\rho_{reg}(h_2)e_g$$

Now, we can equip $\mathbb{C}^{|g|}$ with an algebra structure inherited by the group operation:

$$e_g \cdot e_h := e_{gh}$$

One often writes g instead of e_g and writes the vector space as $\mathbb{C}G = \{\sum_g f_g g | g \in G, f_g \in \mathbb{C}\}$. We can then write the product of generic elements as

$$(f \cdot \tilde{f}) = (\sum_g f_g g) \cdot (\sum_{\tilde{g}} \tilde{f}_{\tilde{g}} \tilde{g}) = \sum_{g,\tilde{g}} f_g \tilde{f}_{\tilde{g}} (g \cdot \tilde{g}) = \sum_h \Big(\sum_g f_g \tilde{f}_{g^{-1}h} \Big) h \qquad (*)$$

Now consider the space of functions $G \to \mathbb{C}$, i.e. $\mathcal{O}(G) = \{f | f : G \to \mathbb{C}\}$. We then have an algebra homomorphism $\mathbb{C}G \to \mathcal{O}(G)$ given by

$$\begin{aligned} \sum f_g g &\mapsto f : G \to \mathbb{C} \text{ with } f(g) = f_g \\ \cdot &\mapsto * \end{aligned}$$

where the operation $*$ in $\mathcal{O}(G)$ can be read off from the last part of (*) and is called the convolution product

$$(f * \tilde{f})(h) = \sum_g f_g \tilde{f}_{g^{-1}h}.$$

We can then define a coalgebra structure by the coproduct

$$\begin{aligned} \Delta : \mathbb{C}G &\to \mathbb{C}G \otimes \mathbb{C}G \\ g &\mapsto \sum_h gh \otimes h^{-1} \end{aligned}$$

and the counit

$$\begin{aligned} \epsilon : \mathbb{C}G &\to \mathbb{C} \\ \sum f_g g &\mapsto f_e. \end{aligned}$$

Now, we should check the relations in (11). The counitality property is given by

$$(id \otimes \epsilon) \circ \Delta = (\epsilon \otimes id) \circ \Delta$$

which is true since

$$g \longrightarrow \textstyle\sum_h gh \otimes h^{-1} \xrightarrow{id \otimes \epsilon} g$$

$$g \longrightarrow \textstyle\sum_h gh \otimes h^{-1} \xrightarrow{\epsilon \otimes id} g$$

and for coassociativity we have:

$$g \longrightarrow \sum_l gl \otimes l^{-1} \longrightarrow \sum_{l,\hat{l}} (gl\hat{l}) \otimes \hat{l}^{-1} \otimes l^{-1}$$

$$g \longrightarrow \sum_h gh \otimes h^{-1} \longrightarrow \sum_h gh \otimes \sum_{\hat{h}} h^{-1}\hat{h} \otimes \hat{h}^{-1}$$

the final two results are the same since by relabeling $h = l\hat{l}$ and $\hat{h} = l$ in the second expression we get $\sum_{l,\hat{l}} (gl\hat{l}) \otimes (l\hat{l})^{-1} l \otimes l^{-1}$. We can now check the Frobenius relations (12):

$$g \otimes h \longrightarrow g \otimes \sum_{\tilde{h}} h\tilde{h} \otimes \tilde{h}^{-1} \longrightarrow \sum_{\tilde{h}} gh\tilde{h} \otimes \tilde{h}^{-1}$$

$$g \otimes h \longrightarrow gh \longrightarrow \sum_{\tilde{h}} gh\tilde{h} \otimes \tilde{h}^{-1}$$

$$g \otimes h \longrightarrow (\sum_l gl \otimes l^{-1}) \otimes h \longrightarrow \sum_l gl \otimes l^{-1}h$$

where again, by relabeling $l = h\tilde{h}$ in the last expression we obtain the same expression. We then have that $(\mathbb{C}G, \mu, \eta, \Delta, \epsilon)$ is a Frobenius algebra.

$\mathbb{C}G$ can also be equipped with different additional structure, for example we can take the coproduct $\tilde{\Delta}(g) = g \otimes g$ with counit $\tilde{\epsilon}(g) = 1$. This does *not* give a Frobenius algebra structure, but rather a Hopf algebra structure.

Note that, just as we proved above, multiplication and counit together give a bilinear form $(-,-) := \epsilon \circ \mu : \mathbb{C}G \otimes \mathbb{C}G \to \mathbb{C}$:

$$(f, \tilde{f}) = \epsilon(f \cdot \tilde{f})$$

which is invariant by associativity of the product

$$(f\hat{f}, \tilde{f}) = \epsilon(f \cdot \hat{f} \cdot \tilde{f}) = (f, \hat{f}\tilde{f})$$

It is also nondegenerate, i.e. if $(f, \tilde{f}) = 0, \forall \tilde{f}$ then $f = 0$. This is true since if $f \neq 0$ then $\exists g$ such that $f_g \neq 0$ and then we can pick $\tilde{f} = f_g g^{-1}$ which has $(f, \tilde{f}) \neq 0$.

We can then also define a coevaluation map *coev* $:= \Delta \circ \eta : \mathbb{C} \to \mathbb{C}G \otimes \mathbb{C}G$ sending $1 \to e \to \sum_h h \otimes h^{-1}$ which together with the bilinear form satisfies the snake relations:

$$g \longrightarrow g \otimes \sum_h h \otimes h^{-1} \longrightarrow \sum_h gh \otimes h^{-1} \longrightarrow g$$

Conversely, we can see that starting with the *coev* we can define a coproduct:

$$g \longrightarrow g \otimes \sum_h h \otimes h^{-1} \longrightarrow \sum_h gh \otimes h^{-1}$$

which is exactly the coproduct we started with!

This Frobenius algebra structure that we introduced gives us the classification of 2d TQFTs: a 2d TQFT $\mathcal{Z} : \mathrm{Bord}_{2,1}^{or} \to \mathscr{C}$ is entirely determined by the choice of a Frobenius algebra object $A \in \mathscr{C}$ to which the circle is sent. In order to understand better the concept of classification of TQFTs it makes sense to introduce a category of TQFTs.

End of lecture 14

The category of TQFTs

It's important to note that the TQFTs together also form a category. The objects are the TQFTs but in order to have a category we need a notion of maps between them and it turns out it's actually something we know already. Recall that a TQFT is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_{n,n-1}^{or} \to \mathscr{C}$$

into a symmetric monoidal category $\mathscr{C}$. Then what should maps between TQFTs be? Well we had a notion of map between functors in Definition 7: it's a natural transformation! In particular, consider two TQFTs $\mathcal{Z}, \tilde{\mathcal{Z}} : \mathrm{Bord}_{n,n-1}^{or} \to \mathrm{vect}$. A natural transformation $\eta : \mathcal{Z} \to \tilde{\mathcal{Z}}$ is then labeled by objects $E \in \mathrm{Bord}_{n,n-1}^{or}$

$$\eta(E) : \mathcal{Z}(E) \to \tilde{\mathcal{Z}}(E)$$

and the $\eta(E)$ are compatible with morphisms $M : E \to E'$:

$$\begin{tikzcd}
\mathcal{Z}(E) \arrow[r, "\mathcal{Z}(M)"] \arrow[d, "\eta(E)"] & \mathcal{Z}(E') \arrow[d, "\eta(E')"] \\
\tilde{\mathcal{Z}}(E) \arrow[r, "\mathcal{Z}(M)"] & \tilde{\mathcal{Z}}(E').
\end{tikzcd}$$

In addition we have that the natural transformation preserves the structure:

$$\begin{tikzcd}
\mathcal{Z}(E_1) \otimes \mathcal{Z}(E_2) \arrow[r] \arrow[d, "\eta_{E_1} \otimes \eta_{E_2}"] & \mathcal{Z}(E_1 \coprod E_2) \arrow[d, "\eta_{E_1 \coprod E_2}"] \\
\tilde{\mathcal{Z}}(E_1) \otimes \tilde{\mathcal{Z}}(E_2) \arrow[r] & \tilde{\mathcal{Z}}(E_1 \coprod E_2).
\end{tikzcd}$$

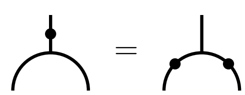
Recall that Definition 7 also gave the notion of the functor category which allows us to define the category of TQFTs: it is simply $\mathrm{Fun}(\mathrm{Bord}^{or}_{n,n-1}, \mathscr{C})$. It turns out that this particular functor category automatically has more structure.

Theorem 23. *The category of symmetric monoidal natural functors* $\mathrm{Fun}(\mathrm{Bord}^{or}_{n,n-1}, \mathscr{C})$ *is a groupoid, i.e. all morphisms are invertible.*

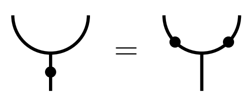
The problem of classifying n dimensional TQFTs then turns into the that of finding another category which is equivalent to the category of TQFTs $\mathrm{Fun}(\mathrm{Bord}^{or}_{n,n-1}, \mathscr{C})$. In particular, for the 2 dimensional case, the informal statement that a TQFT is entirely determined by a choice of Frobenius algebra corresponds to an equivalence of categories between $\mathrm{Fun}(\mathrm{Bord}^{or}_{2,1}, \mathscr{C})$ and the groupoid of commutative Frobenius algebras in $\mathscr{C}$.

Let us then define the notion of morphisms between Frobenius algebras, in order to get that category. A morphism of Frobenius algebras A, A' is a linear map $\psi : A \to A'$ which as usual preserves the additional structure, i.e. it is compatible with

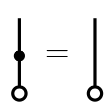
- the product: $\psi \circ \mu = \mu'(\psi \otimes \psi)$



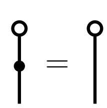
- the coproduct:



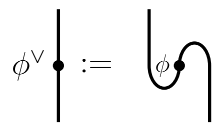
- the unit:



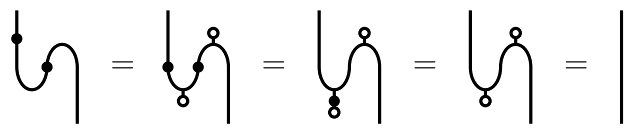
- the counit:



However, we mentioned above that it should be a groupoid, why is that? We can always construct an inverse by taking the dual map! Recall that given two dualizable objects X and Y and a map between them $\phi : X \to Y$ one can define the dual map $\phi^\vee : Y^\vee \to X^\vee$ as



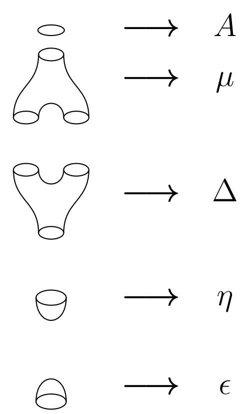
So in particular from a Frobenius algebra map $\psi : A \to A'$ we can define its dual map which we call $\psi^\vee$. It is in fact its inverse because for $\psi \circ \psi^\vee : A' \to A'$ we have:



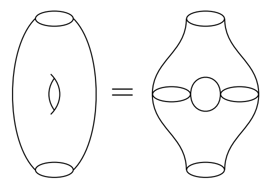
and analogously one can show that $\psi^\vee \circ \psi = id_A$. We therefore have that the category of Frobenius algebras forms a groupoid.

"In the same way" we can give an explicit inverse for the natural transformations in the context of symmetric monoidal categories with dualizable objects. In particular for a natural transformation η between functors $F, G : \mathscr{C} \to \mathscr{D}$ we have that, given a dualizable object $X \in \mathscr{C}$, η_X is invertible with inverse again given by the dual map $\eta_X^\vee$. Observing that objects in $\mathrm{Bord}^{or}_{n,n-1}$ are dualisable then leads to 23.

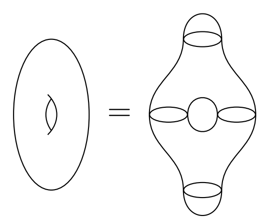
Let us now spell out in what sense a choice of commutative Frobenius algebra entirely determines a TQFT. Given a commutative Frobenius algebra $(A, \mu, \eta, \Delta, \epsilon)$, a TQFT is entirely specified by the following associations:



This allows us to evaluate the functor on any bordism, by simply decomposing it into the basic bordisms above. For example for the bordism



by using the expressions at the end of this section, we have $e_i \mapsto \frac{1}{\alpha_i} e_i \otimes e_i \mapsto \frac{1}{\alpha_i} e_i$. Instead for a torus



we have

$$1 \longmapsto \sum_i e_i \longmapsto \sum_i \frac{1}{\alpha_i} e_i \otimes e_i \longmapsto \sum_i \frac{1}{\alpha_i} e_i \longmapsto \sum_i \frac{1}{\alpha_i} \alpha_i = \dim A$$

and more in general for a genus g surface we'll see in the tutorial that one gets $Z_g = \sum_i \alpha_i^{-g+1}$.

End of lecture 15

Example. Consider the group algebra $\mathbb{C}G$. Recall that it's possible to construct maps such that it's also a Frobenius algebra. However it was *not* commutative and for 2d (closed) TQFTs we require a commutative Frobenius algebra.

To fix this, we can instead pass to its algebra of functions $\mathcal{O}(G)$ and take its center $Z(\mathcal{O}(G)) = \{f \in \mathcal{O}(G) | f * f' = f' * f, \forall f' \in \mathcal{O}(G)\}$. Then if we consider a class function[1] ϕ we have

$$(f*\phi)(g) = \sum_{h \in G} f(h)\phi(h^{-1}g) = \sum_{h \in G} f(h)\phi(gh^{-1}) = \sum_{\tilde{h} \in G} f(\tilde{h}^{-1}g)\phi(\tilde{h}) = (\phi * f)(g)$$

[1]Recall that a class function is a function $\phi : G \to \mathbb{C}$ such that $\phi(h^{-1}gh) = \phi(g)$

in which we used that ϕ is a class function in the third step. So class functions commute with all other functions, but is the converse also true? Consider $f \in Z(\mathcal{O}(G))$, then in particular we have $f * \delta_{h^{-1}} = \delta_{h^{-1}} * f$ where $\delta_{h^{-1}}$ is the delta function:

$$\delta_{h^{-1}}(g) = \begin{cases} 1 & \text{if } h^{-1} = g \\ 0 & \text{else} \end{cases} .$$

We then have $f(gh) = f(hg)$ since

$$f(gh) = \sum_{\tilde{h}} f(\tilde{h})\delta_{h^{-1}}(\tilde{h}^{-1}g) = f*\delta_{h^{-1}g} = \delta_{h^{-1}g}*f = \sum_{\tilde{h}} \delta_{h^{-1}}(\tilde{h})f(\tilde{h}^{-1}g) = f(hg)$$

and picking $h = kg^{-1}$ we have

$$f(gkg^{-1}) = f(k)$$

which is exactly the statement that f is a class function. We then have that class functions are exactly those in the center. In addition, from representation theory, we have that a basis for the space of class functions is given by the characters[2] of irreducible representations. Recall that a representation is called irreducible if there are no subrepresentations, i.e. $\nexists U \subset V$ such that $\rho(g)(u) \in U \; \forall g \in G, u \in U$. In addition, for finite groups we have that the number of conjugacy classes is the same as the number of inequivalent irreducible representations.

In particular, characters of irreducible representations give an *orthonormal* basis on $Z(\mathcal{O}(G))$ with the Hermitian inner product given by

$$\langle f_1, f_2 \rangle = \sum_g \frac{f_1^*(g) f_2(g)}{|G|}$$

which we may take as the pairing to get a Frobenius algebra. In particular, for irreducible representations (V_i, ρ_i) this gives

$$\langle \chi_{\rho_i}, \chi_{\rho_j} \rangle = \begin{cases} 1 & \text{if } \rho_i \cong \rho_j \\ 0 & \text{else} \end{cases}$$

In addition characters of irreducible representations satisfy

$$\chi_{\rho_i} * \chi_{\rho_j} = \begin{cases} 0 & \text{if } \rho_i \neq \rho_j \\ \frac{|G|}{\dim V_i} \chi_{\rho_i} & \text{if } \rho_i \cong \rho_j \end{cases}$$

[2]Consider a representation (V, ρ), its character $\chi_\rho : G \to \mathbb{C}$ is defined as $\chi_\rho(g) := \mathrm{tr}_V \, \rho(g)$ and is a class function by the cyclicity of the trace.

so we can define $e_i = \frac{|G|}{\dim V_i}\chi_{\rho_i}$ which gives a basis of idempotents

$$e_i * e_j = \delta_{ij} e_j$$

which has unit $e = \sum e_i$. Note that to get the pairing above we must choose the counit as

$$\epsilon(e_i) = 1$$

so that

$$\epsilon(\chi_{\rho_i} * \chi_{\rho_j}) = \langle \chi_{\rho_i}, \chi_{\rho_j} \rangle = 1.$$

Note that the choice of ϵ above was not the only possibility: given a Frobenius algebra A with known algebra structure, the only additional data needed to specify the Frobenius structure are the maps $\epsilon(e_i) = \alpha_i$. These α_i should be nonzero in order for the pairing to be nondegenerate. In particular, in the case that we have a basis of idempotents $e_i \cdot e_j = \delta_{ij} e_j$, then the unit is $e = \sum_i e_i$ and if the counit is $\epsilon(e_i) = \alpha_i$ then the comultiplication is given by $\Delta(e_i) = \frac{1}{\alpha_i} e_i \otimes e_i$.

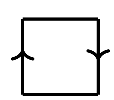
4.11 Unoriented 2d TQFTs

We now move to unoriented 2d TQFTs. We of course still have all the structure we had before, but now we also need to worry about unorientable manifolds.

Recall that the mapping class group of the circle (without orientation) is $\mathbb{Z}_2$ with generator the map σ given by complex conjugation (which of course has $\sigma^2 = id$). This gives rise to a nontrivial cylinder $[C_{S^1,\sigma}]$ which gives a bordism $S^1 \to S^1$ that is *not* the identity. This means that in the target category there should also be a (nontrivial) map $A \to A$, where $A = \mathcal{Z}(S^1)$ is the image of the circle.

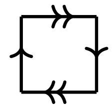
Now consider the unorientable bordisms:

- the Möbius band



  is a nonorientable surface with boundary S^1, meaning it may be seen as a bordism $\emptyset \to S^1$ which, just like the cup, selects an element $\theta \in \mathcal{Z}(S^1)$;

- the projective plane $\mathbb{R}P^2$, is a closed surface which may be obtained from the Möbius band by gluing in a disk;
- the Klein bottle which can be obtained analogously to the Möbius band but with a further identification:



It may also be obtained by using the twisted cylinder or by gluing together two Möbius bands. These two possibilities give a relation between morphisms.

If we then pass to the target category we have Frobenius algebras but now with additional structure, which is due to Turaev and Turner (2005).

Definition 24. An extended Frobenius algebra is a Frobenius algebra $(A, \mu, \eta, \Delta, \epsilon)$ together with an involution $\phi : A \to A$ (i.e. $\phi^2 = id$) and an element $\theta \in A$ such that:

1. $\phi(\mu(\theta, v)) = \mu(\theta, v), \forall v \in A$,
2. $\mu(\phi \otimes id)\Delta(\epsilon) = \theta^2$.

A morphism between extended Frobenius algebras $\psi : A \to A'$ must preserve the extra structure: $\psi(\theta) = \theta'$ and $\psi \circ \phi = \phi' \circ \psi$.

End of lecture 16

5 TQFTs from physics

5.1 Ideas for obtaining TQFTs

It may seem like we have by now gone quite far from physics since, aside from the path integral perspective, it feels like we've mostly just studied mathematical structures. We are however not that far from physics and as an example there is a way of formulating ordinary quantum mechanics as something similar to a TQFT.

To do this we may consider the source category $\text{Bord}_{1,0}^{vol}$, which is related to $\text{Bord}_{1,0}$, and has:

- only one object, a point;
- morphisms given by intervals I_t of any "length" $t \in \mathbb{R}_{>0}$;
- composition given by gluing: $I_t * I_{t'} = I_{t+t'}$.

So basically we just add the notion of "length" to the ordinary bordism category in which segments of different lengths were identified because they're diffeomorphic. Now, in the case where $\mathscr{C} = \mathrm{Hilb}$, the category of Hilbert spaces with unitary transformations as morphisms, what does a functor

$$\mathrm{Bord}_{1,0}^{vol} \to \mathrm{Hilb}$$

look like? There's a unique object $*$ in $\mathrm{Bord}_{1,0}^{vol}$ to which we assign a Hilbert space V and to morphisms I_t we associate some unitary map $U_t : V \to V$ and functoriality tells us that $U_t \circ U_{t'} = U_{t+t'}$. This is exactly what time evolution is in quantum mechanics! From quantum mechanics we also know that U_t may be written as $U_t = e^{iHt}$ for some hermitian operator H, which we call Hamiltonian. This point of view tells us that the structure of quantum mechanics has to do with the geometry of the line. In general this example is useful both in underlying the value of this "functorial" perspective and, in addition, TQFTs allow us to obtain a certain class of such theories.

A general idea for obtaining a TQFT is the following. A *non* topological field theory should define a functor

$$\mathrm{Bord}_{n,n-1}^{str} \to \mathscr{C}$$

from some *structured* bordism category. You could then imagine forgetting the additional structure and that gives a functor $\mathrm{Bord}_{n,n-1}^{str} \to \mathrm{Bord}_{n,n-1}$. There is then a particular class of structured TQFTs which factor through $\mathrm{Bord}_{n,n-1}$ in the sense that there exists a TQFT $\mathrm{Bord}_{n,n-1} \to \mathscr{C}$ such that

$$\begin{tikzcd}
\mathrm{Bord}_{n,n-1} \arrow[dr, dashed] & \\
\mathrm{Bord}_{n,n-1}^{str} \arrow[u] \arrow[r] & \mathscr{C}
\end{tikzcd}$$

In other words, a TQFT may obtained from an ordinary QFT $\mathrm{Bord}_{n,n-1}^{str} \to \mathscr{C}$ which does *not* depend on the additional structure str. Note that for example setting *str* to be a choice of Lorentzian metric on the bordisms and choosing $\mathscr{H}$ as the target category corresponds, at least formally, to a standard QFT.

So if we go back to the example above, which quantum mechanical systems arise from 1d TQFTs in this manner? Since in $\mathrm{Bord}_{1,0}$ line segments of all lengths are identified, we need

$$U_t = U_{t'}, \forall t, t'$$

which implies $U_t = 1, \forall t$ which in turn corresponds to $H = 0$. So quantum mechanical theories which come from TQFTs are those with trivial Hamiltonian.

This now gives another idea for obtaining a TQFT: we may obtain it as the zero energy sector of a physical QFT. Given a physical field theory

$$\tilde{Z} : \mathrm{Bord}^{str}_{n,n-1} \to \mathrm{Hilb}$$

then we can try to define a TQFT by setting $Z(X)$ to be subspace of $\tilde{Z}(X)$ given by the zero eigenspace of H. Here, H is defined by cylinders $X \times [0,1]$, since $\tilde{Z}(X \times [0,1])$ gives us a unitary operator on $\tilde{Z}(X)$ which again gives a Hamiltonian. The idea then is that a state of Z is a vacuum state of $\tilde{Z}$, in other words, the vacuum degeneracy of some full theory is captured by a topological field theory.
Note that this procedure just tells us what to do on objects, but we also need to say something about the morphisms. In addition, it's not clear what it means for H to be zero, since energy is defined up to some constant. So there are some technical difficulties. We can then restrict to a certain class of theories were this construction may be made precise by taking $\tilde{Z}$ to be a supersymmetric QFT, although this is *not* the only construction that one can do.

5.2 Some interesting target categories

Before jumping into supersymmetric QFTs, we start by considering some interesting examples of target categories $\mathscr{C}$ that show up in the physics context. To begin with, we require that $\mathscr{C}$ have a monoidal structure because we require a notion of putting together subsystems. Then some examples are:

- vector spaces with the tensor product, which we often considered in examples;

- Hilbert spaces, also with the tensor product, which is more appropriate for quantum mechanical aspects;
- sets, with the Cartesian product;
- symplectic manifolds, again with the cartesian product. Note that for this category we require symplectomorphisms, i.e. the morphisms should preserve the symplectic form. These manifolds can be thought of as phase spaces and a TQFT valued in this category gives something which resembles classical mechanics. So from this point of view, classical mechanics is different from quantum mechanics because we choose different target categories, and the reason we do this is that the notion of a state is different.

  What should duals be in this category? Well, duals should have something to do with time reversal, which for the phase space corresponds to adding a relative $-$ sign between position and momentum, so the dual to a symplectic manifold is the same symplectic manifold but with opposite symplectic form.
- Representations of a group G, again with tensor product. This is just a decoration of Vect or Hilb by the information of some symmetry G.
- G graded vector spaces for a group G, which are vector spaces V with a decomposition labelled by G:

  $$V = \bigoplus_{g \in G} V_g$$

  An important example is $G = \mathbb{Z}$ and we then have that objects in $\mathrm{Vect}^{\mathbb{Z}}$ are given by

  $$V = \bigoplus_{i \in \mathbb{Z}} V_i$$

  This is actually very similar to taking $\mathrm{Rep}(U(1))$ by decomposing into irreducible representations, since irreducible representations of $U(1)$ are labelled by integers. $\mathrm{Vect}^{\mathbb{Z}}$ is however a different category from $\mathrm{Rep}(U(1))$ because we take a different symmetric monoidal structure. The tensor product is given by

  $$V \otimes W = \bigoplus_{i,j \in Z} V_i \otimes W_j = \bigoplus_{k \in Z} \left[\bigoplus_{i+j=k} V_i \otimes W_j \right] = \bigoplus_{k \in Z} (V \otimes W)_k$$

where at the end we make the $\mathbb{Z}$ grading explicit. For the braiding there are however two possibilities:

$$V \otimes W \longrightarrow W \otimes V$$

$$v \otimes w \longmapsto w \otimes v$$

or

$$V \otimes W \longrightarrow W \otimes V$$

$$v \otimes w \longmapsto (-1)^{i \cdot j} w \otimes v$$

for $v \in V_i$ and $w \in W_j$. Rep$(U(1))$ has the first one, while for $\mathrm{Vect}^{\mathbb{Z}}$ we take the second one. We may be interested in taking the second one because of exchange statistics, where the index would be counting the fermion number, and we want that swapping fermions gives a minus sign. So for a theory with fermions we should map into $\mathrm{Vect}^{\mathbb{Z}}$.

Another variant is to take $\mathbb{Z}_2$ instead of $\mathbb{Z}$. Then we have a decomposition into two vector spaces:

$$V = V_+ \oplus V_-$$

which are called even and odd. Here it's like the situation above but in which we only remember the fermion parity and not the integer fermion number. This category is called "super vector spaces", SVect. There is also an obvious symmetric monoidal functor $\mathrm{Vect}^{\mathbb{Z}} \to \mathrm{SVect}$ which only remembers the fermion number modulo 2:

$$V_+ = \bigoplus_{i \text{ even}} V_i$$

$$V_- = \bigoplus_{i \text{ odd}} V_i$$

There is yet another example which is not obviously related to physics. Consider $\mathbb{Z}$ graded vector spaces

$$V = \bigoplus_{i \in \mathbb{Z}} V_i,$$

these can be decorated with additional data. An important example is that of having a map $d : V \to V$ of degree $+1$, called differential or boundary map, satisfying $d^2 = 0$ meaning that we have maps

$$... \longrightarrow V_{i-1} \longrightarrow V_i \longrightarrow V_{i+1} \longrightarrow ...$$

but any composition of consecutive maps is just zero. This gives rise to the category of cochain complexes. In a sense this additional structure may be thought of as making vector spaces more interesting in order to be able to "model" more complicated spaces. In other words, a vector space is just "flat", with the only important characteristic being its dimension, while a chain complex is more structured and may be used to describe properties of general spaces. More precisely, if we take a space X, I can imagine gluing X together out of "cells" of dimension n (each cell being homeomorphic to an n dimensional disk) and I can think of the i cells as generating V_i and the gluing gives a differential $V_i \to V_{i+1}$ so we obtain a cochain complex.

The $d^2 = 0$ property is very important because it implies $im(d) \subseteq ker(d)$ and therefore allows us to take the following quotient:

$$H^k(C, d) = \frac{\ker(d : C^k \to C^{k+1})}{\mathrm{im}(d : C^{k-1} \to C^k)}$$

This is called the cohomology of the cochain complex (C^*, d). Note that the Hs have a grading, so the cohomology of a cochain complex is itself a graded vector space $H^\bullet$ (additionally, it may also be seen as a cochain complex with zero differential).

End of lecture 17

5.3 Supersymmetric quantum mechanics

Consider a super quantum system, i.e. a super Hilbert space $\mathscr{H}$ together with an even Hermitian operator H on this space called the Hamiltonian. The even and odd subspaces of $\mathscr{H}$ correspond to bosonic and fermionic states. Note that a map of super vector spaces V, W is called even if it maps the V_+ to W_+ and V_- to W_-, i.e. even to even and odd to odd. It is instead called odd if it maps the V_+ to W_- and V_- to W_+, i.e. even to odd and odd to even. We require the Hamiltonian to be even because we'd like to be able to assign an energy to systems that are purely bosonic (or fermionic) and for a bosonic (or fermionic) state $|\psi\rangle$ to be an eigenvector of H we need that H maps the bosonic (and fermionic) part of $\mathscr{H}$ to itself. We can then restrict to a certain class of quantum systems (in this section quantum systems will usually be super even if not specified).

Definition 25. A quantum system is called *supersymmetric* if the Hamiltonian admits a fermionic "square root", i.e. there exists an operator Q of odd parity such that

$$\{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q = H \tag{17}$$

and

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0. \tag{18}$$

i.e. $Q^2 = 0 = (Q^\dagger)^2$.

An immediate observation is that for a supersymmetric quantum system we have

$$QH = Q(QQ^\dagger + Q^\dagger Q) = QQ^\dagger Q = HQ$$

and the same works for $Q^\dagger$. So Q and $Q^\dagger$ both commute with H

$$\begin{aligned} [Q, H] &= 0 \\ [Q^\dagger, H] &= 0 \end{aligned} \tag{19}$$

In other words, Q and $Q^\dagger$ are not just square roots of the Hamiltonian, but they're also *symmetries* of the Hamiltonian.

Another immediate and important property is that H must have positive spectrum since

$$\begin{aligned} \langle\psi|H|\psi\rangle &= \langle\psi|QQ^\dagger + Q^\dagger Q|\psi\rangle = \langle\psi|QQ^\dagger|\psi\rangle + \langle\psi|Q^\dagger Q|\psi\rangle \\ &= (Q^\dagger|\psi\rangle)^2 + (Q|\psi\rangle)^2 \geq 0 \end{aligned}$$

with equality if and only if $Q|\psi\rangle = 0 = Q^\dagger|\psi\rangle$. We have then solved one of the issues we had since we now have a definition of a zero energy state: it is a state that is annihilated by both Q and $Q^\dagger$, in other words it's a state invariant under the symmetry generated by Q and $Q^\dagger$, a supersymmetric state. Then the zero sector is just picked out as the supersymmetry invariant subspace of the total Hilbert space.

Representations of the supersymmetry algebra

Note that one can consider abstract operators $Q, Q^\dagger, H$ satisfying (17), (18) and (19) and these form a super Lie algebra. The details of the structure of a super Lie algebra aren't essential for now, but it's important to note that $\mathscr{H}$ carries a representation of this super Lie algebra. A natural question to ask then is what are the irreducible representations of this super Lie algebra? Well, we start by noticing that we can restrict to an energy eigenspace because if $H|\psi\rangle = E|\psi\rangle$ then also $HQ|\psi\rangle = QH|\psi\rangle = EQ|\psi\rangle$ and the same for $Q^\dagger|\psi\rangle$. Then consider the energy eigenspace with energy $E \geq 0$. If $E = 0$ then $Q|\psi\rangle = 0$ and $Q^\dagger|\psi\rangle = 0$, so we just have the trivial representation. For $E > 0$ there are potentially four states: $|\psi\rangle, Q|\psi\rangle, Q^\dagger|\psi\rangle, QQ^\dagger|\psi\rangle$. We don't need $Q^\dagger Q|\psi\rangle$ since it's a linear combination of the others

$$Q^\dagger Q|\psi\rangle = H|\psi\rangle - QQ^\dagger|\psi\rangle = E|\psi\rangle - QQ^\dagger|\psi\rangle.$$

However, the states above may not all be nonzero. Consider the case $Q^\dagger|\psi\rangle = 0$, this gives a two dimensional representation, spanned by $|\psi\rangle, Q|\psi\rangle$, where the actions of Q and $Q^\dagger$ are given by

$$\begin{aligned} Q &: |\psi\rangle \mapsto Q|\psi\rangle, & Q|\psi\rangle &\mapsto 0 \\ Q^\dagger &: |\psi\rangle \mapsto 0, & Q|\psi\rangle &\mapsto E|\psi\rangle \end{aligned}$$

What if instead neither $Q|\psi\rangle$ nor $Q^\dagger|\psi\rangle$ vanishes? Consider $span(|\psi\rangle, QQ^\dagger|\psi\rangle)$ and apply $Q^\dagger$ to a generic element:

$$\begin{aligned} Q^\dagger(a|\psi\rangle + bQQ^\dagger|\psi\rangle) &= aQ^\dagger|\psi\rangle + bQ^\dagger QQ^\dagger|\psi\rangle) = aQ^\dagger|\psi\rangle + bHQ^\dagger|\psi\rangle) \\ &= (a + bE)Q^\dagger|\psi\rangle \end{aligned}$$

so we see that states with $a = -bE$ are sent to zero by $Q^\dagger$. This then means that this representation splits.

$\mathscr{H}$ is therefore a sum of irreducible representations of the following types:

1. "long" ones, with $E > 0$, given by $span(|\psi\rangle, Q|\psi\rangle)$, one bosonic, one fermionic,

2. "short" (vacuum) representations, with $E = 0$, spanned by a single vector $|\psi\rangle$ satisfying $Q|\psi\rangle = Q^\dagger|\psi\rangle = 0$ and $|\psi\rangle$ where be either bosonic or fermionic.

The zero energy eigenspace is then simply made up of some bosonic short representations and some fermionic ones. An important quantity then is $n_+ - n_-$ which is sometimes called the index of the quantum system, where n_+ is the number of bosonic representations and n_- are the fermionic ones.

An example

Are there any nontrivial examples of such systems? Consider the following, a single particle moving in 1d with one bosonic and one fermionic internal state, in other words we have $\mathscr{H}_+ = L^2(\mathbb{R})$ and $\mathscr{H}_- = L^2(\mathbb{R})$ so we have something in both parities and so Q and $Q^\dagger$ may act nontrivially. Note that this describes real physical systems, such as a hydrogen atom which may be ionized or not. We may also write $\mathscr{H}$ as $L^2(\mathbb{R}) \otimes C^{1|1}$ where $C^{1|1}$ is simply a shorthand notation for a super vector space with a copy of $\mathbb{C}$ both in even and odd parity, whereas by $L^2(\mathbb{R})$ we mean that it's concentrated in even parity. An element $\mathscr{H}$ may be written as

$$\begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix}$$

and an operator may be represented in matrix form with its even part on the diagonal and odd part on the opposite diagonal

$$\begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix}$$

and we can take Q and $Q^\dagger$ to be

$$Q = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 & 0 \\ \hat{p} & 0 \end{pmatrix}, \qquad Q^\dagger = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 & \hat{p} \\ 0 & 0 \end{pmatrix}$$

We then have

$$\begin{aligned} \{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q = \frac{1}{2m}\left(\begin{pmatrix} 0 & 0 \\ 0 & \hat{p}^2 \end{pmatrix} + \begin{pmatrix} \hat{p}^2 & 0 \\ 0 & 0 \end{pmatrix} \right) = \frac{1}{2m} \begin{pmatrix} \hat{p}^2 & 0 \\ 0 & \hat{p}^2 \end{pmatrix} \\ = \frac{\hat{p}^2}{2m} \otimes id_{C^{1|1}} \end{aligned}$$

What are the short representations here? A bosonic state is of the form

$$\begin{pmatrix} \psi(x) \\ 0 \end{pmatrix}$$

and $Q^\dagger \begin{pmatrix} \psi(x) \\ 0 \end{pmatrix} = 0$ is automatic. Instead

$$Q \begin{pmatrix} \psi(x) \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ \hat{p}\psi(x) \end{pmatrix}$$

which vanishes if and only if $\psi(x)$ is a constant. However, this is not normalizable. The same happens if we start with a fermionic state. This is a sign that the topology of the space where the particle is moving is relevant.

End of lecture 18

5.4 1d twist: TQFT in cochain complexes from supersymmetric quantum mechanics

5.4.1 Weakly topological theory

We mentioned previously some examples of different target categories that we may choose for a TQFT. In particular we had that SVect is relevant if fermions are present, in $\mathrm{Vect}^{\mathbb{Z}}$ the grading would represent some fermion number or "ghost number". Instead choosing the category of cochain complexes turns out to be useful for describing gauge symmetry.

Example. As an example of this we can consider electrodynamics. In electrodynamics we have physical degrees of freedom $\vec{E}$ and $\vec{B}$ which fit together into the field strength two form $F_{\mu\nu}$ as $B_i = \epsilon_{ijk}F_{jk}$ and $E_i = F_{0i}$. Importantly though, F is constrained by $d * F = 0$ and $dF = 0$ which are respectively the equations of motion and the constraints (Bianchi identities). So the space of fields is not simply the space of all Fs, but rather the space of Fs subject to the constraints above. It's then useful to write $F = dA$ where A is unconstrained. However we just shifted the problem since now A is not unique since $A + d\lambda$ is also a possibility. We should therefore somehow identify As which differ by the differential of some function. This is exactly what cohomology does! We can say that we have a cochain complex with A in degree 0 and the gauge parameters in degree -1 with differential given by the usual de Rham differential:

$$\begin{array}{ccc} \underline{-1} & & \underline{0} \\ \\ \Omega^0 = \{\lambda\} & \xrightarrow{d} & \Omega^1 = \{A\} \end{array}$$

There is also another cochain complex that one can write for this model. We could have directly put F in degree 0 and put the differential as the map going out from degree 0, in this way taking cohomology gives us the Fs such that $dF =$ which again is what we want.

$$\begin{array}{ccccc} \underline{0} & & \underline{1} & & \underline{2} \\ \Omega^2 = \{F\} & \xrightarrow{d} & \Omega^3 & \xrightarrow{d} & \Omega^4 \end{array}$$

Previously we said that a theory is topological if it factors through $\mathrm{Bord}_{n,n-1}$. We can now call that *strictly* topological and if we're mapping into cochain complexes we can have a weaker notion. A theory with gauge invariance, i.e. valued in Ch, is "weakly" topological if the dependence on the additional structure is gauge exact. In other words we have a strictly topological field theory if there exists the functor represented by the dotted line

$$\begin{tikzcd} \mathrm{Bord}_{n,n-1} \arrow[dr, dashed] & \\ \mathrm{Bord}^{str}_{n,n-1} \arrow[u] \arrow[r, "\tilde{\mathcal{Z}}"'] & \mathscr{C} \end{tikzcd}$$

and it is instead weakly topological if $\mathscr{C} = Ch$ and we only get such a functor once we take cohomology

$$\begin{tikzcd} \mathrm{Bord}_{n,n-1} \arrow[dr, dashed] & \\ \mathrm{Bord}^{str}_{n,n-1} \arrow[u] \arrow[r, "H^*(\tilde{\mathcal{Z}})"'] & Ch \end{tikzcd}$$

In quantum mechanics we have $\mathrm{Bord}_{1,0}$ which is not very interesting or we had $\mathrm{Bord}^{vol}_{1,0}$ in which the intervals are assigned a length. Then a strict QM is a single Hilbert space $\mathcal{Z}(pt)$. Instead, a nontopological QM is again a single Hilbert space, but now we have to associate something to the intervals and this will be a one parameter group $U_t = \mathcal{Z}(I_t)$ of unitary transformations. Equivalently $U_t = e^{iHt}$ for some Hermitian operator H called the Hamiltonian. This system is strictly topological if and only if $H = 0$.

5.4.2 Weakly topological QM

If instead we map into cochain complexes, what would a weakly topological QM be? To start with, a QM in cochain complexes is

- a cochain complex C^* of Hilbert spaces,
- a Hamiltonian $H : C^* \to C^*$ of degree 0, since we should be able to have homogeneous eigenvalues; H should also be compatible with the differential in the sense that $[H, d] = 0$.

Now what is the interpretation of the cochain complex and of its cohomology? The idea is that not all the states in the Hilbert space are physical, just as in the electrodynamics examples not all Fs are allowed. We define the physical states to be those that are closed, i.e. $|\psi\rangle$ is physical if and only if $d|\psi\rangle = 0$. Additionally there are states which are "trivially physical", i.e. $d\phi$ is physical but this is simply a consequence of $d^2 = 0$. We then impose an equivalence condition by saying that two physical states are *the same* if they differ by an exact state, just like in electrodynamics A and $A + d\lambda$ are identified.
Taking cohomology then simply corresponds to taking nontrivial physical states!

The condition $[H, d] = 0$ above can then be justified by a physical argument: if $|\psi\rangle$ is a physical state then $H|\psi\rangle$ shouls also be physical

$$d(H|\psi\rangle) = Hd|\psi\rangle - [H, d]|\psi\rangle = [H, d]|\psi\rangle$$

which is true if we set $[H, d] = 0$. Additionally trivial states $|\psi\rangle = d|\phi\rangle$ should still be trivial after applying H:

$$H\psi = Hd\phi = dH|\phi\rangle + [H, d]|\phi\rangle = dH|\phi\rangle$$

Now, topological means $H = 0$. Instead what should weakly topological be? We simply require the relation above only on physical states and on the right we don't necessarily get 0, but we should get something trivial in cohomology. So we a weakly topological theory if

$$H|\psi\rangle = d|\phi\rangle$$

for all physical states ψ.

It turns out that this corresponds to the statement that H is a commutator

$$H = [d, \tilde{d}]$$

for some degree -1 operator $\tilde{d}$. Indeed choosing H of that form we get

$$H|\psi\rangle = d\tilde{d}|\psi\rangle + \tilde{d}d|\psi\rangle = d\tilde{d}|\psi\rangle$$

which is in fact exact. So a QM valued in cochain complexes is weakly topological if and only if its Hamiltonian can be written as a commutator.

But that is precisely the algebra of supersymmetric quantum mechanics! For now though we're dealing with very different theories. In supersymmetric quantum mechanics $Q, Q^\dagger$ are physical symmetries and the Hilbert space is a representation the supersymmetry algebra $H = [Q, Q^\dagger]$, but it is *not* a cochain complex, we don't have a differential or a gauge invariance.

5.4.3 Topological twist

There is however an obvious way of getting a QM valued in cochain complexes from a supersymmetric QM: consider Q as the differential. This involves a change of interpretation of the theory: we now consider as physical states the ones that are annihilated by Q, so the supersymmetry preserving states. We found in addition that the states that are annihilated by Q are exactly the vacuum states of theory. We're therefore restricting to the vacuum sector of the supersymmetric theory.

Starting with a nontopological theory valued in SVect we have then obtained a (weakly) topological theory valued in cochain complexes! This procedure of obtaining a topological theory is called *topological twisting*.

Example. Recall that at the end of the last section we wrote an example of a supersymmetric QM theory, it was a particle moving in $\mathbb{R}$ with an internal state (either bosonic or fermionic). We can give this theory an integer grading by putting H_+ in degree 0 and H_- in degree 1. Then $\deg Q = 1$ and $\deg Q^\dagger = -1$ which is the grading of the differentials we had above. This results in something closely related to our gauge field example. To see this we may write a generic element

$$\begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix} \in \mathscr{H}$$

as $\psi(x) + \phi(x)dx$ where dx carries degree 1. Then $\mathscr{H}$ can be identified as a graded vector space with $\Omega^\bullet(\mathbb{R})$, the space of forms on $\mathbb{R}$. Now, the action of Q is given by

$$Q(\psi(x) + \phi(x)dx) = \frac{1}{i\sqrt{2m}}\phi'(x)dx$$

So $Q = \frac{1}{i\sqrt{2m}}d$ where d is the usual exterior derivative! So not only is $\mathscr{H}$ isomorphic to $\Omega^\bullet(\mathbb{R})$ as a graded vector space, but it also has the same differential! So our Hilbert space is the space of forms on $\mathbb{R}$ and therefore the space of physical states is the de Rham cohomology of $\mathbb{R}$ (the space the particle moves in) with L^2 coefficients.

So the topological twist of a particle with an internal degree of freedom moving on the real line gives us the cochain complex $\Omega^\bullet(\mathbb{R})_{L^2}$ as a Hilbert space and once we take cohomology we get a topological theory.

This theory generalizes in an immediate way to SQM of a particle moving on a space X (with Riemannian metric, to define p^2): we can take the Hilbert space to be $\Omega^\bullet(X)_{L^2}$ with inner product

$$(\alpha, \beta) = \int_\gamma \bar{\alpha} \wedge *\beta$$

with supersymmetry generator $Q = d$, the de Rham operator and $Q^\dagger = d^\dagger$ adjoint with respect to the L^2 inner product. We then have that the Hamiltonian is given by $H = [d, d^\dagger]$ which is simply the Laplacian on X.

End of lecture 19

5.5 Topological twist in arbitrary dimension

What we did was a 0+1 dimensional version of the general construction known as topological twisting. Let us know consider the construction more generally. In a d dimensional QFT, H is the timelike component of the Lorentz vector P_μ. Because of spin and statistics, in order to have something like

$$\{Q, Q\} = P_\mu$$

we need the set of supercharges $\{Q\}$ to transform together in a spinor representation of the Lorentz group. In general picking a Q with $Q^2 = 0$ breaks Lorentz symmetry. To solve this we first modify the action of the Lorentz symmetry on the theory by mixing it with some other symmetry. In particular we can change the Lorentz symmetry in such a way that one of the Qs is then a scalar with respect to this twisted Lorentz symmetry. We can then use this Q as differential to get a theory valued in cochain complexes. If we find that the P_μ are exact, we get a TQFT.

5.5.1 Aside: closed and exact operators

When is an operator exact? By operator on a cochain complex of vector spaces V^* we mean an element of $\mathbf{Hom}(V^*, V^*)$ and in order to have a notion of exactness $\mathbf{Hom}(V^*, V^*)$ should itself be a cochain complex. Let us explain how this works in general for two cochain complexes V^* and W^* and the maps between them $\mathbf{Hom}(V^*, W^*)$. First of all note that this is *not* the space of morphisms from V^* to W^*, $\mathrm{Hom}(V^*, V^*)$, since those are only the degree preserving maps which commute with the differential. Instead we define the *inner Hom* as $\mathbf{Hom}(V^*, W^*) = (V^*)^\vee \otimes W^*$ which essentially means that we also allow maps that shift the degree. The grading is given by how much a map shifts the degree, i.e. if $f : V^* \to W^*$ acts as $f : V^i \to W^{i+k}$ then we call f a degree k map. We also have a differential on $\mathbf{Hom}(V^*, W^*)$ given by

$$d\phi = d_V \circ \phi - (-1)^{|\phi|} \phi \circ d_W$$

So the name inner Hom represents the fact that it is in a way the space of maps from V^* to W^* but in addition it is itself an object in the category of cochain complexes, just like V^* and W^*.
In particular, if we go back to the case $W^* = V^*$ we have that the differential $d_{op.}$ of a map $\phi : V^* \to V^*$ is given by

$$d_{op.}\phi = d \circ \phi - (-1)^{|\phi|} \phi \circ d$$

which is just the (graded) commutator of ϕ and d! So on operators we can represent the differential as $[d, \cdot]$ and an operator is exact if it is the bracket of the differential with another operator. Now, for a QM in cochain complexes we found that it is topological if $H = [d, \tilde{d}]$ for some $\tilde{d}$. We can now phrase this as follows: a QM in cochain complexes is topological if its Hamiltonian is exact. Above we also had the corresponding statement in higher dimension: a QFT in cochain complexes is topological if its momentum vector is exact.

5.5.2 2d twist

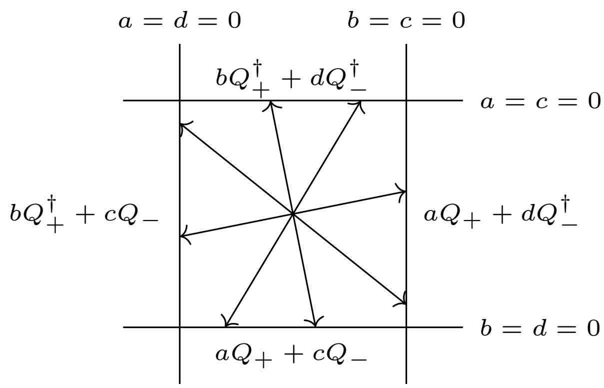
1+1 dimension (Euclidean). $P = \{P_0, P_1\}$ is acted on by the Lorentz symmetry SO(2), the circle group. P decomposes into two eigenstates $P_0 \pm iP_1 =: P_\pm$ of "spin" ± 1. An example of a susy algebra, add square roots of e.g. P_+ just as in QM. So $P_+ = [Q_+, Q_+^\dagger]$ and for spin to be consistent we require that both Q_+ and $Q_+^\dagger$ are spin $\frac{1}{2}$. To get topological supercharges need to also add Q_- and $Q_-^\dagger$. What are the square zero supercharges now? A general fermionic element of this algebra

$$Q = aQ_+ + bQ_+^\dagger + cQ_- + dQ_-^\dagger$$

We'd like to find now what kind of element squares to zero:

$$Q^2 = 2[Q,Q] = 4ab[Q_+, Q_+^\dagger] + 4cd[Q_-, Q_-^\dagger] = 4[abP_+ + cdP_-]$$

So a square zero Qs corresponds to solutions to $ab = 0$ and $cd = 0$. We can represent the solutions in the following diagram



where the lines correspond to solutions in which two of the four parameters are set to zero. Instead at intersection points three of the parameters are set to, for example in the top left we have $a = c = d = 0$ so we just have $bQ_+^\dagger$. The arrows instead represent the action of taking the dagger. Now, which of these solutions give rise to TQFTs? Well we need the the translation generators $P_\pm$ to be exact with respect to the Q we choose. We see that for example for the top left intersection this is *not* possible, since there is no operator $\mathcal{O}$ for which we can get $P_\pm = [bQ_+^\dagger, \mathcal{O}]$. This is the same for all other intersection points. Instead for every other point it works! We then have two equivalence classes of possibilities: we can take $bQ_+^\dagger + dQ_-^\dagger$ or we can take $bQ_+^\dagger + cQ_-$ (and their respective adjoints). These give rise to two equivalence classes of 2d topological twists, called the B model and the A model.

Suppose we pick the B twist with $Q_B = Q_+^\dagger + Q_-^\dagger$. Recall that we need this to be a scalar, so we can change the Lorentz symmetry in order to get this. We have

$$P_+ = [Q_+, Q_+^\dagger]$$
$$P_- = [Q_-, Q_-^\dagger]$$

where the $P_\pm$s have spin ± 1 and under the usual Lorentz symmetry $Q_\pm$s have spin $\pm\frac{1}{2}$. If we were to change the Lorentz symmetry then we could say that $Q_+^\dagger$ has spin s and Q_+ has spin $1-s$, analogously we assign r to $Q_-^\dagger$ and $-1-r$ to Q_-. If we pick $s = r = 0$ then $Q_+^\dagger$ and $Q_-^\dagger$ are scalars and so is Q_B. Instead, Q_+ has spin 1 and Q_- has spin -1 so they transform together as a vector after the twist.

This is what generally happens: we have a scalar Q and we need a relation of the form $P_\mu = [Q, \tilde{Q}_\mu]$.

So twisting essentially consists of two steps: picking a square zero operator in order to get a differential and changing the Lorentz symmetry in such a way as to have that the differential chosen is a scalar.

5.5.3 An example: 2d sigma model

We now want to consider an example where we can apply the ideas above. In particular we can get a 2d theory in a way similar to what we did in the SQM example.

For the SQM example of a particle moving in a Riemannian manifold X with an internal degree of freedom, we just focused on the Hilbert space. We now shift perspective and think about the fields. There we had a single bosonic field $\phi \in \mathrm{Map}(\mathbb{R} \to X)$ representing the position of the particle in X as a function of time (in general such a theory of maps into a space is called a sigma model). Instead, how do we represent the fermionic field? Well it should represent some infinitesimal deformation around the path of the particle, so we should the tangent bundle $T_{\mathbb{C}}X$ of the manifold X. However that's too much because we only need the tangent spaces at the points along the path, this corresponds to taking the pullback bundle $\phi^* T_{\mathbb{C}}X$. Then the fermionic field is given by $\psi \in \Gamma(\phi^* T_{\mathbb{C}}X)$.

Correspondingly, a 2d theory should have $\phi \in \mathrm{Map}(\Sigma, X)$ as a bosonic field, for some surface Σ. However, we now have the 2d supersymmetry algebra, which should have 4 supercharges, i.e. 4 "natural" operators on the Hilbert space $\Omega^\bullet(X)$.

One way to do this is to take X to be a complex manifold. We then get a decomposition of $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$. Also the differential forms become more interesting

$$\Omega^n(X) = \bigoplus_{p+q=n} \Omega^{p,q}(X)$$

where in $\Omega^{p,q}(X)$ we have p dzs and q $d\bar{z}$s. Then the usual de Rham operator is written as $d = \partial + \overline{\partial}$, with ∂ and $\overline{\partial}$ satisfying $\partial^2 = 0, \overline{\partial}^2 = 0, \{\partial, \overline{\partial}\} = 0$. We can think of them as $\partial = Q_+, \overline{\partial} = Q_-$. Given a metric on X we can then define their adjoints $\partial^\dagger$ and $\overline{\partial}^\dagger$, which also satisfy $(\partial^\dagger)^2 = (\overline{\partial}^\dagger)^2 = \{\partial^\dagger, \overline{\partial}^\dagger\} = 0$. So we have four operators on the Hilbert space $\Omega^\bullet(X)$! There is a problem though: in general

$$[\partial, \overline{\partial}^\dagger] \neq 0$$

The reason for this is that the adjoint is defined by using the metric, which knows nothing about the complex structure. If we instead use a so called Kähler metric, which has a compatibility condition with the complex structure, then the bracket above is zero, thanks to the Kähler identities. Such a manifold is called a Kähler manifold. In that case we also have $[\partial, \partial^\dagger] = [\overline{\partial}, \overline{\partial}^\dagger] = \Delta$, the normal laplacian.

So we've found a representation of the 2d supersymmetry algebra!

Let us know simply state the result one gets when twisting this theory. Let X be a Kähler manifold and consider the B twist of this sigma model. Then the bosonic fields are given by $\phi \in \mathrm{Map}(\Sigma, X)$, the fermionic fields instead split up into some which are vector like

$$\rho \in \Gamma(\Sigma, \phi^* T^{1,0}X) \otimes \Omega^1(\Sigma)$$

and some which are scalar

$$\eta \in \Gamma(\Sigma, \phi^* T^{0,1}X) \otimes \Omega^0(\Sigma)$$

After twisting Q_B becomes the operator $\overline{\partial}$ and the theory assigns (roughly) $(\Omega^{\bullet,\bullet}, \overline{\partial})$ to the circle.

This example skips over some details, but the general idea is that it's possible to write down a theory with an action of the susy algebra which is not too different from what we did in SQM. In particular we get the B model, which is a TQFT labelled by a Kähler manifold.

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5.6 More on supersymmetric quantum mechanics

The Hamiltonian H and its "square roots" $Q, Q^\dagger$ have the structure of a *super* Lie algebra since we're now dealing with super vector spaces.

Definition 26. A super Lie algebra is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a "bracket" map $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which

- is graded antisymmetric
$$[x, y] = -(-1)^{|x||y|}[y, x]$$
where $|x| = i$ for $x \in \mathfrak{g}_i$,
- satisfies the graded Jacobi identity
$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|z||y|}[z, [x, y]] + (-1)^{|y||x|}[y, [z, x]] = 0$$

Note that the antisimmetry property becomes $[a, b] = [b, a]$ for a and b in $\mathfrak{g}_1$. In this case we will usually write $\{a, b\}$ instead of $[a, b]$. The notation is suggestive of the fact that if $\mathfrak{g}$ is a space of matrices then $[\cdot,\cdot]$ will be the commutator and $\{\cdot,\cdot\}$ will be the anticommutator.

In super quantum mechanics we then have the super lie algebra spanned by $Q, Q^\dagger, H$ with $|H| = 0$ and $|Q| = |Q^\dagger| = 1$ satisfying

$$\begin{aligned} &[H, Q] = 0 &\qquad &\{Q, Q\} = 0 \\ &[H, Q^\dagger] = 0 & &\{Q^\dagger, Q^\dagger\} = 0 \\ &\{Q, Q^\dagger\} = H & & \end{aligned}$$

We have representations on $\mathbb{Z}_2$ graded Hilbert spaces $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ and in the representation, brackets become commutators/anticommutators.

The super Lie algebra above is called the $\mathcal{N} = 2$ supersymmetry algebra for spacetime dimension $d = 1$. The number 2 is simply due to the fact that we have two odd generators.

In the above super Lie algebra there's not really a distinction between Q and $Q^\dagger$ since they have the same brackets. However, it's possible to add an even generator to the super Lie algebra which commutes with H and which distinguishes between Q and $Q^\dagger$:

$$\begin{aligned} [F, Q] &= Q \\ [F, Q^\dagger] &= -Q^\dagger \\ [F, H] &= 0 \end{aligned}$$

Such an operator, which commutes with the Hamiltonian (or more in general with the spacetime translation generators) and acts nontrivially on the Qs is called R symmetry.

5.6.1 Supersymmetric quantum mechanics with a potential

We can now consider supersymmetric quantum mechanics with a potential.

Example. Start with the harmonic oscillator bosonic system

$$H_B = \frac{1}{2}p^2 + \frac{1}{2}x^2 = a^\dagger a + \frac{1}{2}$$

with $a, a^\dagger$ given by

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(x + ip) \\ a^\dagger &= \frac{1}{\sqrt{2}}(x - ip) \end{aligned}$$

We have commutation relations $[x, p] = i$ and $[a, a^\dagger] = 1$. Energy eigenstates may then be written as

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

where $|0\rangle$ satisfies $a|0\rangle = 0$ and we have

$$H_B|n\rangle = \Big(n + \frac{1}{2}\Big)|n\rangle$$

The Hilbert space $\mathscr{H}_B$ is then generated by the $|n\rangle$. We now want a supersymmetric version of this, so we do the following steps: extend $\mathscr{H}$ to also have odd degrees of freedom, then find Q and $Q^\dagger$ such that the supersymmetry algebra is represented.

Let us find the odd analog of the construction above and we can then put the two together. By mimicking the a and $a^\dagger$, we can take the odd (fermionic) degrees of freedom $\psi, \psi^\dagger$ which have brackets

$$\begin{aligned} \{\psi, \psi\} &= 0 \\ \{\psi^\dagger, \psi^\dagger\} &= 0 \\ \{\psi, \psi^\dagger\} &= 1 \end{aligned}$$

Then let the fermionic Hamiltonian be

$$H_F = \frac{1}{2}[\psi^\dagger, \psi] = \psi^\dagger \psi - \frac{1}{2}$$

We can then again pick $|0\rangle$ to be annihilated by ψ and we get the two dimensional representation spanned by $|0\rangle$ and $\psi^\dagger|0\rangle$. In this basis we get a matrix representation for ψ and $\psi^\dagger$:

$$\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \psi^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We then have

$$\begin{aligned} H_F|0\rangle &= -\frac{1}{2}|0\rangle \\ H_F(\psi^\dagger|0\rangle) &= \psi^\dagger\psi\psi^\dagger|0\rangle - \frac{1}{2}\psi^\dagger|0\rangle = \psi^\dagger|0\rangle - \frac{1}{2}\psi^\dagger|0\rangle = \frac{1}{2}\psi^\dagger|0\rangle \end{aligned}$$

The Hamiltonian in this basis then takes the form

$$H_F = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

As a super vector space, the Hilbert space is now $\mathbb{C}^{1|1}$ with the even part spanned by $|0\rangle$ and the odd one by $\psi^\dagger|0\rangle$.

We can now combine the two systems:

$$\mathscr{H} = \mathscr{H}_B \otimes \mathbb{C}^{1|1}$$

with the Hamiltonian

$$H = H_B \otimes \mathbb{1} + \mathbb{1} \otimes H_F$$

Let us now see what happens on basis vectors:

$$\begin{aligned} H|n\rangle \otimes |0\rangle &= (H_B \otimes \mathbb{1} + \mathbb{1} \otimes H_F)(|n\rangle \otimes |0\rangle) = \left(n + \frac{1}{2}\right)|n\rangle \otimes |0\rangle - \frac{1}{2}|n\rangle \otimes |0\rangle \\ &= n|n\rangle \otimes |0\rangle \\ H|n\rangle \otimes \psi^\dagger|0\rangle &= \left(n + \frac{1}{2}\right)|n\rangle \otimes |0\rangle + \frac{1}{2}|n\rangle \otimes |0\rangle = (n+1)|n\rangle \otimes |0\rangle \end{aligned}$$

We then find that states $|n+1\rangle \otimes |0\rangle$ and $|n\rangle \otimes \psi^\dagger|0\rangle$ have eigenvalue $n+1$, so all positive eigenvalues are degenerate. We can then realize a representation of the supersymmetry algebra by taking $Q = a^\dagger\psi$ and $Q^\dagger = \psi^\dagger a$. We then have $[H, Q] = 0 = [H, Q^\dagger]$ and

$$\{Q, Q^\dagger\} = \{a^\dagger\psi, \psi^\dagger a\} = \{a^\dagger\psi, \psi^\dagger\}a - \psi^\dagger[a^\dagger\psi, a] = a^\dagger a + \psi^\dagger\psi$$

which is (up to a factor of $\frac{1}{2}$) the Hamiltonian we started with!

This example may not seem so interesting since it's basically a double copy of the harmonic oscillator which is well known. It is however very useful because we can now generalize to an arbitrary potential: note that $x = \frac{d}{dx}\left(\frac{1}{2}x^2\right)$, so if in the Hamiltonian we took an arbitrary polynomial potential $h(x)$, we can change the $a, a^\dagger$ by taking

$$\begin{aligned} A &= \frac{1}{\sqrt{2}}(h'(x) + ip) \\ A^\dagger &= \frac{1}{\sqrt{2}}(h'(x) - ip) \end{aligned}$$

We then define $Q = A^\dagger\psi$ and $Q^\dagger = \psi^\dagger A$ as above and we can also *define* H to be $\{Q, Q^\dagger\}$. Let us then compute what H is.

$$H = \{A^\dagger\psi, \psi^\dagger A\} = \{A^\dagger\psi, \psi^\dagger\}A - \psi^\dagger[A^\dagger\psi, A] = A^\dagger A - \psi^\dagger\psi[A^\dagger, A]$$

Now using $p = -i\partial_x$ we we have

$$2A^\dagger A = (h'(x) - \partial_x)(h'(x) + \partial_x) = (h')^2 - \partial_x^2 - (\partial_x h') + h'\partial_x = (h')^2 - \partial_x^2 - h''$$

and $[A^\dagger, A] = -h''$. Then the Hamiltonian is

$$H = \frac{1}{2}(-\partial_x^2 + (h')^2) + h''\left(\psi^\dagger\psi - \frac{1}{2}\right)$$

which we may write in matrix form as

$$H = \frac{1}{2}\begin{pmatrix} -\partial_x^2 + (h')^2 - h'' & 0 \\ 0 & -\partial_x^2 + (h')^2 + h'' \end{pmatrix}$$

so we have that $H|_{\mathscr{H}_+}$ and $H|_{\mathscr{H}_-}$ look like non susy Hamiltonian operators, with different potentials, but just like for the harmonic oscillator they have the same spectrum!

Q and $Q^\dagger$ take the matrix form

$$Q = \psi^\dagger(\partial_x + h') = \begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + h' & 0 \end{pmatrix}$$
$$Q^\dagger = (-\partial_x + h')\psi = \begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + h' & 0 \end{pmatrix}$$

We can now consider the ground states, i.e. the states satisfying $Q\psi = Q^\dagger\psi = 0$. Consider the generic state

$$\phi = f_1(x)|0\rangle + f_2(x)\psi^\dagger|0\rangle$$

with $f_1(x), f_2(x) \in \mathscr{H}_B$. For a ground state we then have

$$\Big(\frac{d}{dx} + h'\Big) f_1(x) = 0 = \Big(-\frac{d}{dx} + h'\Big) f_2(x)$$

Solving these differential equations gives $f_1(x) = e^{-h(x)}$ and $f_2(x) = e^{h(x)}$, but the normalizibility of these states depends on the function h:

1. if $h \to \pm\infty$ for $x \to \pm\infty$ or $h \to \mp\infty$ for $x \to \pm\infty$ then both f_1 and f_2 are divergent.
2. if $h \to \infty$ for $x \to \pm\infty$ then only $e^{-h(x)}$ is renormalizable and $\phi = e^{-h(x)}|0\rangle$ is the bosonic ground state.
3. if instead $h \to -\infty$ for $x \to \pm\infty$ then only $e^h(x)$ is renormalizable and $\phi = e^{h(x)}\psi^\dagger|0\rangle$ is the fermionic ground state.

If we then consider the Witten index we have

$$n_+ - n_- = \text{tr } (-1)^F e^{-\beta H} = \begin{cases} 0, & 1. \\ +1, & 2. \\ -1, & 3. \end{cases}$$

The Witten index is then determined by the asymptotics of h and robust under perturbations of h that do not change that.

5.7 2d Topological twist

5.7.1 Setup: 2d supersymmetry algebra and its R symmetries

Now the supersymmetry algebra is an extension of the Poincarè algebra by odd generators such that we obtain a super Lie algebra with the Poincarè algebra in the even part. Now we have that H generates translations in time and P generates translations in space. We can take light cone coordinates $x_\pm$:

$$ds^2 = dt^2 - dx^2 = 2dx^+dx^-$$

and in these coordinates we have that a Lorentz boost takes the form

$$\Lambda = \begin{pmatrix} e^\eta & 0 \\ 0 & e^{-\eta} \end{pmatrix} = \exp\left(\eta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

Where we see that the two dimensional representation of the 2d Lorentz group $SO(1,1)$ decomposes into two one dimensional representations. We can then take $H+P$ and $H-P$ as the Hamiltonian from the one dimensional case, i.e. we can find Q_+ and Q_- such that

$$H + P = \{Q_+, Q_+^\dagger\}$$
$$H - P = \{Q_-, Q_-^\dagger\}$$

and all the other brackets are 0:

$$\{Q_+, Q_-\} = \{Q_+, Q_-^\dagger\} = \{Q_+^\dagger, Q_-\} = \{Q_+^\dagger, Q_-^\dagger\} = 0 \tag{20}$$
$$\{Q_\pm, Q_\pm\} = \{Q_\pm^\dagger, Q_\pm^\dagger\} = 0$$

We choose these combinations because $H+P$ and $H-P$ generate translations of the lightcone coordinates. This super Lie algebra is called $\mathcal{N} = (2,2)$ supersymmetry algebra. Analogously in supersymmetric quantum mechanics we had the $\mathcal{N} = 2$ supersymmetry algebra because we had two odd operators Q and $Q^\dagger$. Now that we have two sets of two odd operators we call it the $\mathcal{N} = (2,2)$ supersymmetry algebra.

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Just like for $d = 1$ we could also add an R symmetry, an even generator which commutes with the spacetime translation generators and has nontrivial brackets with the Qs, we can also do it here. In particular here it's possible to add two additional generators:

$$[F_V, Q_\pm] = -Q_\pm \qquad [F_A, Q_\pm] = \mp Q_\pm$$
$$[F_V, Q_\pm^\dagger] = Q_\pm^\dagger \qquad [F_A, Q_\pm^\dagger] = \pm Q_\pm^\dagger$$

So in a sense F_V sees whether there is a dagger, while F_A sees whether there's a $+$ or a $-$.

There is a Lie algebra automorphism μ defined on generators as as

$$\mu(Q_-) = Q_-^\dagger \qquad \mu(Q_-^\dagger) = Q_- \qquad \mu(F_V) = F_A \qquad \mu(F_A) = F_V$$

and acts as the identity on the other generators. For it to be a Lie algebra automorphism we should check that $\mu([\cdot,\cdot]) = [\mu(\cdot), \mu(\cdot)]$:

$$\mu[F_V, Q_+] = \mu(-Q_+) = -Q_+ = [F_A, Q_+] = [\mu(F_A), \mu(Q_+)]$$
$$\mu[F_V, Q_-] = \mu(-Q_-) = -Q_-^\dagger = [F_A, Q_-^\dagger] = [\mu(F_A), \mu(Q_-)]$$

We then say that two $\mathcal{N} = (2,2)$ supersymmetric quantum field theories are *mirror* to each other if they are equivalent as quantum field theories and the isomorphism acts as that automorphism on generators.

5.7.2 Topological twist

We can instead consider the Euclidean setting, in which we introduce a new coordinate x^2 replacing the time direction by $x^0 = -ix^2$ so, as Lorentz symmetry, instead of $SO(1,1)$ we get $SO(2) \cong U(1)_E$ (E for Euclidean). If the system has a $U(1)_R$ symmetry (either vector or axial symmetry) then topological twisting refers to a change in the Lorentz symmetry: we have symmetry group $U(1)_E \times U(1)_R$ and now as Lorentz symmetry we can pick a *different* subgroup $U(1)_{E'} \subset U(1)_E \times U(1)_R$ instead of $U(1)_E$ (in particular the diagonal subgroup). There are 2 inequivalent possibilities called A twist and B twist, depending on whether we take the vector or axial R symmetry respectively. Now, depending on which twist we do there are different operators which become scalars after the twist (note that since we change what Lorentz symmetry is, we also change what scalars, spinors and so on are):

$$Q_A = Q_+^\dagger + Q_-$$
$$Q_B = Q_+^\dagger + Q_-^\dagger$$

Q_A becomes a scalar upon the A twist and the same is true for Q_B with the B twist. In addition, because of the brackets of the Qs (20) we have that if we consider a representation on a Hilbert space we get $Q^2_{A/B} = 0$, since in the representation $\{Q_{A/B}, Q_{A/B}\} = 2Q^2_{A/B} = 0$. We can therefore think of it as a differential and take its cohomology. Recall from the supersymmetric quantum mechanics case that this corresponds to taking the ground states of the supersymmetric theory.

5.7.3 Physical operators

It's also interesting to study what happens to operators. Let Q be our differential, so $Q = Q_A$ or Q_B depending on which twist we do. As usual, on operators we take the differential to be $\{Q, \cdot\}$. To get physical operators we would then take cohomology.

So now let us consider an even Q closed operator $\mathcal{O}(x^+, x^-)$ inserted at the spacetime point (x^+, x^-), i.e. $[Q, \mathcal{O}(x^+, x^-)] = 0$. We can understand its dependence on the point by studying its derivatives, or equivalently the bracket with the translation generators $P_\pm = H \pm P$. To perform calculations we now consider the B twist and use $Q = Q_B$ although it works analogously for the A twist.

$$\begin{aligned}
\Big(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\Big)\mathcal{O}(x^+, x^-) &= [P_\pm, \mathcal{O}] = [\{Q_\pm, Q^\dagger_\pm\}, \mathcal{O}] \\
&= \{[Q_\pm, \mathcal{O}], Q^\dagger_\pm\} + \{Q_\pm, [Q^\dagger_\pm, \mathcal{O}]\} \\
&= \{[Q_\pm, \mathcal{O}], Q^\dagger_\pm\} - \{Q_\pm, [Q^\dagger_\mp, \mathcal{O}]\} \\
&= \{[Q_\pm, \mathcal{O}], Q^\dagger_\pm\} + \{[Q_\pm, \mathcal{O}], Q^\dagger_\mp\} - [\{Q_\pm, Q^\dagger_\mp\}, \mathcal{O}] \\
\\
&= \{Q_B, [Q_\pm, \mathcal{O}]\}
\end{aligned}$$

where in the second line we use the Jacobi identity

$$[\{Q_\pm, Q^\dagger_\pm\}, \mathcal{O}] - \{[Q_\pm, \mathcal{O}], Q^\dagger_\pm\} - \{[Q^\dagger_\pm, \mathcal{O}], Q_\pm\} = 0$$

in the third line we used that $\mathcal{O}$ is Q_B closed

$$[Q_B, \mathcal{O}] = 0 = [Q^\dagger_+, \mathcal{O}] + [Q^\dagger_-, \mathcal{O}]$$

in the fourth line we used Jacobi again as

$$-\{[Q^\dagger_\mp, \mathcal{O}], Q_\pm\} + \{[\mathcal{O}, Q_\pm], Q^\dagger_\mp\} + [\{Q_\pm, Q^\dagger_\mp\}, \mathcal{O}] = 0$$

and finally in the last line we read out Q_B from the first two brackets and the third one is zero since $\{Q_\pm, Q_\mp^\dagger\} = 0$. The result is a bracket of Q_B with something and this is the definition of Q_B exact. This means that the action of $P_\pm$ on $\mathcal{O}$ is cohomologically trivial. In other words, translations do not change the cohomology class, so once we take cohomology the theory is entirely topological!

5.7.4 Nonlocal operators

The result above was for a scalar operator and it may be formulated as follows: the de Rham differential of a scalar physical observable $\mathcal{O}^{\{}(0)\}$ is the bracket of Q with some 1 form operator $\mathcal{O}^{\{}(1)\}$

$$d\mathcal{O}^{(0)} = \{Q, \mathcal{O}^{(1)}\}$$

where $d\mathcal{O}^{(0)} = \partial_{x^+}\mathcal{O}^{(0)}dx^+ + \partial_{x^-}\mathcal{O}^{(0)}dx^-$ and for the Q_B case we found

$$\mathcal{O}^{(1)} = [Q_+, \mathcal{O}]dx^+ + [Q_-, \mathcal{O}]dx^-$$

It turns out that this result can be generalised: there exists also a 2 form operator $\mathcal{O}^{\{}(2)\}$ such that

$$d\mathcal{O}^{(1)} = [Q, \mathcal{O}^{(2)}]$$

Putting these results together we get what are called the topological descent relations:

$$\begin{aligned} 0 &= [Q, \mathcal{O}^{(0)}] \\ d\mathcal{O}^{(0)} &= \{Q, \mathcal{O}^{(1)}\} \\ d\mathcal{O}^{(1)} &= [Q, \mathcal{O}^{(2)}] \\ d\mathcal{O}^{(2)} &= 0 \end{aligned}$$

where the additional first and last expressions are respectively the statement that $\mathcal{O}^{(1)}$ is physical, i.e. Q closed, and that in dimension 2 there are no 3 forms.

But now these 1 and 2 form operators allow us to define additional nonlocal observables, for example if we take a 1 cycle γ with $\partial\gamma = \emptyset$ then we have

$$[Q, \int_\gamma \mathcal{O}^{(1)}] = \int_\gamma [Q, \mathcal{O}^{(1)}] = \int_\gamma d\mathcal{O}^{(0)} = 0$$

and analogously for 2 cycles Σ with $\partial\Sigma = \emptyset$ we have

$$[Q, \int_\Sigma \mathcal{O}^{(2)}] = \int_\Sigma [Q, \mathcal{O}^{(2)}] = \int_\Sigma d\mathcal{O}^{(1)} = 0$$

So for any γ and Σ we now have the physical nonlocal observables given by $\int_\gamma \mathcal{O}^{(1)}$ and $\int_\Sigma \mathcal{O}^{(2)}$.

However, since we've built a topological theory, we expect that if we deform a 1 cycle into another 1 cycle, the two should give the same observable. This is in fact true, consider two 1-cycles γ and γ' such that there is a 2 dimensional surface B with $\partial B = \gamma - \gamma'$, then

$$\int_\gamma \mathcal{O}^{(1)} - \int_{\gamma'} \mathcal{O}^{(1)} = \int_{\partial B} \mathcal{O}^{(1)} = \int_B d\mathcal{O}^{(1)} = [Q, \int_B \mathcal{O}^{(2)}]$$

which is trivial in cohomology.

5.7.5 Correlation functions as pairings

Now, if we actually constructed a TQFT we should have a Frobenius algebra structure, how do we get it? Consider physical operators ϕ_i, i.e. $[Q, \phi_i] = 0, \phi_i \in H^0(Q)$. We can then define a threefold pairing by taking the three point function

$$c_{ijk} = \langle \phi_i \phi_j \phi_k \rangle$$

which depends only on the cohomology classes of ϕ_i, ϕ_j, ϕ_k as we'll show below. If we choose one of the ϕ_i to be the identity we get the pairing of our Frobenius algebra

$$k_{ij} = \langle \phi_i \phi_j \rangle = c_{ij0}$$

Note that a priori the correlation functions depend on the insertion points, however we don't specify them because when we take cohomology the dependence disappears. It's not clear that it's nondegenerate but for the class of theories we consider it is true and we therefore get a Frobenius algebra.

Now let us check that the correlation functions are in fact independent of the choice of the representative of the cohomology class. This corresponds to showing that if we insert a cohomologically trivial element $[Q, \lambda]$ into a correlation function then we always get 0. To show this we assume that the vacuum is invariant under supersymmetry, i.e. annihilated by the generator $Q|0\rangle = 0$. This implies

$$\langle [Q, \lambda] \rangle = 0$$

while we would like the following to be true

$$\langle [Q, \lambda] \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = 0$$

This would be true if we could move the Q to the right past all the operators and we can only do that if $[\mathcal{O}_i, Q] = 0$. But that is exactly the condition that $\mathcal{O}_i$ are physical observables! We then get that correlation functions of physical observables only depend on their cohomology classes.

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