Extrinsic curvature

Consider a cylinder in flat 3D space, namely the surface $r = R$ in

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \,. \tag{1}$$

• Obtain the metric induced on the cylinder surface, h_{ij} (note that it is defined only for indices $i, j = z, \phi$). Observe that it is the metric of 2d flat space, i.e., the same (locally) as the metric on a plane in 3D flat space. Compute the area of this surface, and then its fractional variation in the radial direction,

$$K = n^i \partial_i \log \sqrt{h} \,. \tag{2}$$

This is known as the trace of the extrinsic curvature tensor, or the mean curvature (up to a factor of 2). In other contexts (congruences of curves) it is essentially the same as the expansion θ.

Consider a sphere in flat 3D space, namely the surface $r = R$ in

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,. \tag{3}$$

• Obtain the induced metric and the trace of the extrinsic curvature.

Take a constant-time section of the Schwarzschild geometry, with three-dimensional spatial metric[1]

$$ds^2 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,. \tag{4}$$

Consider now a surface at constant radius $r = R \geq 2M$.

• Compute K. How is this different from the case of spheres in flat space?

• Is any of the spheres at constant r a minimal surface (*i.e.,* $K = 0$)? If so, compute explicitly the area A of this sphere, and show that its Lie derivative along the normal radial direction vanishes (so the surface is extremal) and its second derivative is positive (so it is actually minimal).

• Can you sketch what the geometry (4) looks like? Can you guess why it is called a 'wormhole'?

• Extend the previous analysis to spheres at constant positive r in metrics of the form

$$ds^2 = \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,, \tag{5}$$

for the following cases:

(a) $f(r) = 1 - \frac{r^2}{L^2}$: constant-time section of deSitter space.

(b) $f(r) = \left(1 - \frac{Q}{r}\right)^2$: constant-time section of extremal Reissner-Nordström solution.

[1] As it stands here, this metric is well-defined only for $r > 2M$, although one can extend the manifold to include $r = 2M$, which is at finite proper distance, and then analytically extend it beyond this locus in a maximal way.

Would you call any of these two geometries a 'wormhole'?

- Find a spherical extremal surface in the geometry[2]

$$ds^2 = \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) , \qquad (6)$$

and obtain its area — you are not required to compute the complete extrinsic curvature, nor to find out whether it is a minimal or maximal surface. Can you find a way to reduce this problem to the one in (4)?

[2]Observe that this is conformallly equivalent to $\mathbb{R}^3$.

Action principle and field equations in low-energy string theory

The following action appears as a (partial) description of the dynamics of string theory at low energies

$$I = \frac{1}{16\pi G} \int d^4x\sqrt{-g}\left(R - \frac{1}{2}\nabla_\mu\phi \,\nabla^\mu\phi - 2\Lambda e^\phi\right) . \tag{1}$$

Here Λ is a constant. The massless scalar field ϕ is called the *dilaton*.

Field equations. Vary the action with respect to $g^{\mu\nu}$ and ϕ to obtain the field equations for this theory:

$$R_{\mu\nu} = \frac{1}{2}\nabla_\mu\phi\,\nabla_\nu\phi + \Lambda e^\phi g_{\mu\nu} ,$$
$$\Box\phi = 2\Lambda e^\phi$$

(ignore boundary terms and all issues having to do with Gibbons-Hawking-York terms).

Stress-energy tensor. Obtain the stress-energy tensor $T^{\mu\nu}$ of the scalar ϕ and verify that, when the equation of motion for ϕ is satisfied, the stress-energy tensor is covariantly conserved

$$\nabla_\mu T^\mu{}_\nu = 0 . \tag{2}$$

(These equations are then consistent with the Bianchi identity $\nabla_\mu\left(R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R\right) = 0$, but in the previous result we have not needed to assume that the gravitational equations are satisfied).

Dilaton gravity in string-frame.
(*This is harder*)
For this exercise you need to know something we did not prove in the lectures, namely that if we vary the metric $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$, then

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu v^\mu \tag{3}$$

with

$$v^\mu = -\nabla_\nu \delta g^{\mu\nu} + g_{\nu\rho}\nabla^\mu\delta g^{\nu\rho} \tag{4}$$

(this can be derived using the Palatini identity). We see that $g^{\mu\nu}\delta R_{\mu\nu}$ is a total derivative, which in the Einstein-Hilbert action can be transformed into a boundary term involving derivatives of the boundary metric.

Now show that the field equations of the theory

$$I = \frac{1}{16\pi G} \int d^4x\sqrt{-g}e^{-2\Phi}\left(R + 4\nabla_\mu\Phi\,\nabla^\mu\Phi\right) \tag{5}$$

can be written as

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi = 0 , \qquad \Box\Phi - 2\nabla_\mu\Phi\,\nabla^\mu\Phi = 0 . \tag{6}$$

(Again, you can drop all *total* derivative terms, but be careful). Eq. (5) is known as the *string-frame* action for dilaton gravity.

(*This is harder*) Show that the theory (5) is equivalent to the dilaton-gravity action (1) with $\Lambda = 0$ (this is called the Einstein-frame form of dilaton gravity), *i.e.,* show that, with a suitable transformation of the fields, the equations of one theory can be transformed into the equations of the other theory.

As an observation, notice that in (5) it seems like the kinetic term for Φ has the wrong sign (plus), which one could think would be pathological (a ghost). The fact that it is not pathological is most easily shown by transforming this action through a field redefinition to the form (1), where the kinetic term has the correct sign.