3D Gravity

martes, 6 de agosto de 2024 11:10

Lower-D gravity?

$$\Phi_N \sim -\frac{GM}{r^{D-3}}$$

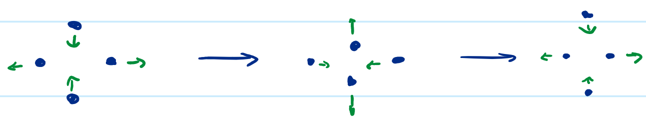
$D < 4$

Gravity is weaker at small r: improved UV behavior
good for QFT fluctuations

not decaying at large r: very sensitive to long distance
May lose black holes!

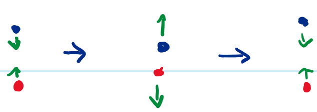
- Gravitational d.o.f.'s: more polarizations in $D > 4$

No propagating d.o.f.'s already in $D=3$:
no room for quadrupolar (shear) motion!



: impossible in $D < 4$

[gauge fields can oscillate in $D=3$
not in $D=2$



]

[scalars in all $D \geq 2$]

Scales and dimensions

In $D=4$ (with $c=1$) GM is a length

In arbitrary D, $(GM)^{1/D-3}$ is a length

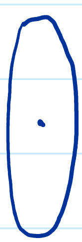
In $D=3$ GM is dimensionless

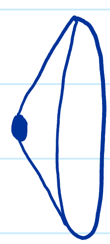
The mass does not determine a length scale,

Thus there can't be a black hole horizon solely determined by M. A particle coupled to gravity creates a conical defect (dimensionless).

[Mass in $D \geq 4$ is measured from "extrinsic curvature defect" of large-radius spheres

M in $D=3$ it's measured from conical angle defect of large-radius circles

$M=0$  $\frac{\Delta \text{ Circle length}}{\Delta \text{ Radius}} = 2\pi$

 $M>0$ $2\pi - \frac{\Delta \text{ Circle length}}{\Delta \text{ Radius}} \propto GM$]

- Need a length scale to have a black hole.

Eg a cosmological radius L, from a cosmological constant $\Lambda \sim 1/L^2$

But having a length scale is necessary, not sufficient: we also need attraction.

Mass in 3D doesn't attract by itself: conical defect is global effect

Mass in 3D doesn't attract by itself; conical defect is global effect

$\Lambda > 0$ won't do: expansion

$\Lambda < 0$ may do: collapse

So we'll consider $\Lambda \sim -1/L^2$. The black holes will have size $\propto L$ and thus will be "of AdS size", ie "large AdS bhs". $\nexists$ small AdS bhs in $D=3$

- <u>Gravity in 3D has no propagating dof's</u>

3D Weyl $= 0$ identically [in 3D Weyl $= 0 \not\Rightarrow$ conf flat; Cotton $= 0 \Leftrightarrow$ conf flat]

Riemann determined by Ricci.

But Ricci is determined by Einstein equations: once a 3D spacetime is required to satisfy the Einstein eqs, there doesn't remain any freedom in it (no grav waves).

With cosmological constant we'll have

$$R_{ij} = -\frac{2}{L^2} g_{ij} \qquad \text{ie Ricci = constant}$$

so then <u>Riemann = constant</u>. If $\Lambda < 0$

$\Rightarrow$ <u>All solutions are locally equivalent to AdS_3</u>

[3D gravity is topological, only boundary dof's]

Can only differ:

(i) in global structure (like plane and cylinder differ)

(ii) in distributional singularities (like plane and cone differ)

Sources: conical points

line sources

So we know the local geometry: AdS_3

and we're going to see how to find

geometries that differ globally.

BTZ '92

BHTZ '93

AdS_3:

We know S^3 is described as a surface in $\mathbb{R}^4$

$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$ in $ds^2 = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$

(unit radius $L=1$)

In order to change $S^3 \to AdS_3$ we flip two signs:

To go Lorentzian (have time) and to have -ve curvature

$$ds^2 = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2$$

$$-x_0^2 - x_1^2 + x_2^2 + x_3^2 = -1$$

Can solve for surface

$$\underbrace{-x_0^2 - x_1^2}_{\substack{\text{polar}\\(t,\,r)}} + \underbrace{x_2^2 + x_3^2}_{\substack{\text{polar}\\(r,\,\phi)}} = -1$$

$x_2 = r \sin\phi$

$x_3 = r\cos\phi$

$x_0 = \sqrt{r^2+1}\,\sin t$: periodic t?

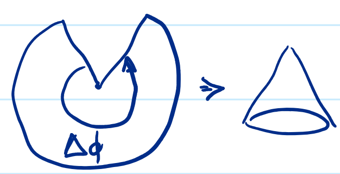
$x_1 = \sqrt{r^2+1}\,\cos t$

No: "unwrap" to global covering space $-\infty < t < \infty$

(but there remains a memory of this periodicity)

$$\Rightarrow \quad ds^2 = -(r^2+1)\,dt^2 + \frac{dr^2}{r^2+1} + r^2\,d\phi^2$$

Take $\phi \sim \phi + \Delta\phi$



If $\Delta\phi \neq 2\pi$: conical singularity at $r=0$

Naked: No horizon

$-g_{tt} > 0 \quad \forall r$

BTZ:

Now solve the surface equation by pairing up the coordinates in a different way, in boost-like fashion:

$$-x_0^2 \overbrace{- x_1^2 + x_2^2}^{\text{boost}} + x_3^2 = -1$$

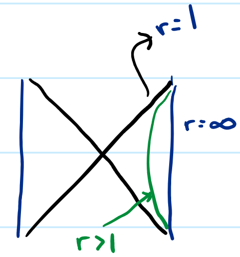
boost

$$x_1 = r\cosh\phi \qquad -x_1^2 + x_3^2 = -r^2$$
$$x_3 = r\sinh\phi$$

$$x_0 = \sqrt{r^2-1}\,\sinh t \qquad -x_0^2 + x_2^2 = r^2-1$$
$$x_2 = \sqrt{r^2-1}\,\cosh t$$

$$\Rightarrow \quad ds^2 = -(r^2-1)\,dt^2 + \frac{dr^2}{r^2-1} + r^2 d\phi^2$$

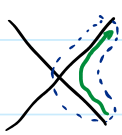
but now in principle $-\infty < t < \infty$
$-\infty < \phi < \infty$



This is Rindler-AdS$_3$

Horizon at $r=1$ is non-compact acceleration horizon.

[This is the AdS version of Rindler space:



$r = 1 + \xi^2/2 \quad \xi^2 \ll 1 \Rightarrow ds^2 \simeq -\xi^2 dt^2 + d\xi^2 + d\phi^2$: 3D Rindler

$= -dT^2 + dX^2 \qquad T = \xi \sinh t$

$X = \xi \cosh t$]

[∂_t and ∂_ϕ are boost vectors

$\partial_t = X_2 \dfrac{\partial}{\partial x_0} + X_0 \dfrac{\partial}{\partial x_2}$ Timelike for $-x_0^2 + x_2^2 > 0$ $\quad r^2-1>0$

$\partial_\phi = x_1 \dfrac{\partial}{\partial x_3} + x_3 \dfrac{\partial}{\partial x_1}$ " " $-x_1^2 + x_3^2 > 0$ " $r^2<0$ "

Spacelike " $-x_1^2 + x_3^2 < 0 \quad r^2 > 0$]

Rindler-AdS$_3$ is just a patch of AdS$_3$.

Rindler-AdS_3 is just a patch of AdS_3.

We now introduce a global difference with AdS_3:

<u>Identify along orbits of ∂_ϕ</u>

$$\phi \sim \phi + \Delta\phi \qquad (\text{ok where } r^2 > 0)$$

$\Delta\phi$ is arbitrary: need not be 2π since ϕ is a boost coordinate, and $r=0$ is hidden behind the horizon (see below for more on $r=0$)

Horizon now is <u>compact</u>: black hole horizon

$$ds^2 = -(r^2-1)dt^2 + \frac{dr^2}{r^2-1} + r^2 d\phi^2 \qquad \text{with } \phi \sim \phi + \Delta\phi$$

is the <u>BTZ black hole</u> with horizon at $r=1$, "area" (length) $A_H = \Delta\phi$

The form above is not the "canonical metric".

Rescale $\phi = \frac{\Delta\phi}{2\pi}\tilde{\phi}$ so $\Delta\tilde{\phi} = 2\pi$

and absorb factors in $r = \frac{2\pi}{\Delta\phi}\tilde{r}$

$$t = \frac{\Delta\phi}{2\pi}\tilde{t}$$

Then

$$ds^2 = -(\tilde{r}^2 - 8GM)d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2-8GM} + \tilde{r}^2 d\tilde{\phi}^2$$

$\tilde{r} = 8GM$

with $8GM = \left(\frac{\Delta\phi}{2\pi}\right)^2$ $\qquad \tilde{\phi} \sim \tilde{\phi} + 2\pi$

M is the black hole mass (from detailed analysis)

$$M = \frac{(D-2)\,\Omega_{D-2}}{16\pi G}\mu \underset{D=3}{=} \frac{\mu}{8G}$$

Notice global AdS_3 has $M = -1/8G$

BHs have $M > 0$.

$-1/8G < M < 0$ are conical singularities (particles)

["M=0 black hole" is mildly singular. It is obtained by identifying along a null boost.]

[$r=0$: null hypersurface : $\partial/\partial\phi$ is null there

no curvature singularity

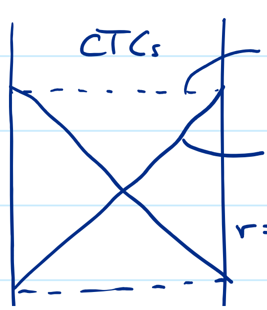
metric can be analytically continued

There are CTCs beyond: points identified along timelike directions (boost)

some null geodesics are incomplete

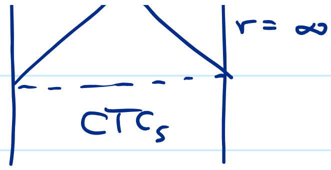
(similar to Lorentzian Taub-NUT and Misner space)

Keep only $r>0$]



$r=0$ but it is actually null, in the ϕ direction

$|\partial/\partial\phi|^2 = 0$



So: $\boxed{\text{AdS}_3 \text{ with points identified along orbits of a boost vector is the BTZ black hole}}$

- Global definition, indep of coordinates!
- Mass is given by period of identification

Other forms of the metric:

from $ds^2 = -(r^2 \pm 1)\, dt^2 + \dfrac{dr^2}{r^2 \pm 1} + r^2 d\phi^2$ AdS$_3$ BTZ

proper radius $d\rho^2 = \dfrac{dr^2}{r^2 \pm 1}$

$+$: $r = \sinh\rho$ $\quad ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2$ AdS$_3$

no zero, no horizon — zero at $\rho = 0$: origin of rotation

$-$: $r = \cosh\rho$ $\quad ds^2 = -\sinh^2\rho\, dt^2 + d\rho^2 + \cosh^2\rho\, d\phi^2$ BTZ

zero at $\rho = 0$ horizon — no zero, finite area horizon

Observe Euclidean $t \to i\tau$ are equivalent $\tau \Leftrightarrow \phi$

Euclidean Thermal AdS_3 and BTZ only differ in the choice of Euclidean Time

Rotation: identify ϕ after a shift in time
(equivalently, along a vector $\partial/\partial\phi + \alpha\, \partial/\partial t$, $|\alpha|<1$)

In this case there are no incomplete geodesics. Excise by hand region w/ CTCs

Easy to find that $\qquad G = 1/8$

$$ds^2 = -\left(r^2 - M + \frac{J^2}{4r^2}\right) dt^2 + \frac{dr^2}{r^2 - M + J^2/4r^2} + r^2\left(d\phi - \frac{J}{2r^2} dt\right)^2$$

$$= -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dt^2 + \frac{r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2\left(d\phi - \frac{r_+ r_-}{r^2} dt\right)^2$$

$$r_+^2 r_-^2 = \frac{J^2}{4}$$

$$r_+^2 + r_-^2 = M$$

Usual pattern: $\quad g_{t\phi} \propto J$

$$-8GM + \frac{J^2}{4r^2}$$

$\hookrightarrow$ centrifugal repulsion

$\exists$ extremal limit $r_+ = r_-$ $\quad M = |J|$

Regular w/ finite area

This is also obtained from identifications along a null boost

The BTZ black hole plays a central role in one of the best understood and most detailedly studied instances of holography: AdS$_3$/CFT$_2$

Both sides of the duality are eminently solvable due to the large amount of symmetry, and allow precise computations and checks.

BTZ also underlies many precise calculations of the Bekenstein-Hawking entropy in string theory, including the first one by Strominger + Vafa.

BTZ black hole entropy from CFT$_2$

$$ds^2 = -\left(\frac{r^2}{\ell^2} - 8GM\right) dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2} - 8GM} + r^2 d\phi^2$$

(remove ~)

ℓ = AdS$_3$ radius

$$r_H = \ell \sqrt{8GM}$$

$$A_H = 2\pi r_H = 2\pi \ell \sqrt{8GM}$$

$$S = \frac{A_H}{4G} = \frac{\pi \ell}{2}\sqrt{\frac{8M}{G}} = \pi \ell \sqrt{\frac{2M}{G}}$$

We want to reproduce this as the degeneracy

We want to reproduce this as the degeneracy of a CFT_2 with central charge $c = \frac{3\ell}{2G}$ at high energy / Temperature.

There exists a general formula for the degeneracy of states $e^{S_{CFT}}$ of a CFT_2 "Cardy formula"

L = length of spatial circle of CFT_2

$$S_{CFT} = 2\pi \sqrt{\frac{CE}{3} \frac{L}{2\pi}}$$

E = Total energy

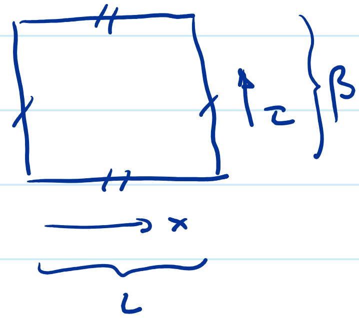
Now plug in here $\begin{cases} E = M \\ L = 2\pi \ell \end{cases}$ $c = \frac{3\ell}{2G}$

$$S_{CFT} = 2\pi \sqrt{\frac{M\ell^2}{2G}} = \pi \ell \sqrt{\frac{2M}{G}} = S_{BH}$$

$$\boxed{S_{CFT} = S_{BH}}$$: it works exactly!

Cardy formula for the degeneracy of CFT_2 states at high energy / Temperature at high energy / Temperature

$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$: compute it from Euclidean path integral on Torus

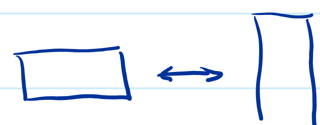


$\tau \sim \tau + \beta$
$x \sim x + L$

$Z(\beta, L)$ is dimensionless, and the only scales are β and L, so it must be that

$$Z(\beta, L) = Z(\beta/L, 1) = Z(\beta/L)$$

Now we use <u>modular invariance</u>, which is a property of CFT_2 at one loop, and which implies that $Z(\beta, L) = Z(L, \beta)$



$$= Z\left(L \times \frac{L}{\beta}, \beta \times \frac{L}{\beta}\right) = Z\left(\frac{L^2}{\beta}, L\right)$$

Then for fixed L we can write

$$Z(\beta) = Z\left(\frac{L^2}{\beta}\right)$$

often one writes $L = 2\pi R$ and then
$Z(\beta) = Z\left(\frac{4\pi^2 R^2}{\beta}\right)$

$$\boxed{Z(\beta) = Z\left(\frac{4\pi^2 R^2}{\beta}\right)}$$

This relates the density of states at high-Temp $\beta \to 0$, to the density at low Temp, $\beta \to \infty$

At low Temperatures, Z is dominated by the vacuum state of the CFT_2 on a circle, with energy

$$E_0 = -\frac{c}{12}\frac{2\pi}{L} = -\frac{\pi c}{6L}$$

(Casimir energy on a circle)

so

$$Z(L^2/\beta) = Z(\tilde{\beta}) \approx e^{-\tilde{\beta} E_0} \qquad \tilde{\beta} = L^2/\beta$$

$$= e^{\frac{L^2}{\beta}\frac{\pi c}{6L}}$$

ie for $\beta \to 0$

$$Z(\beta) \approx e^{\frac{\pi L c}{6\beta}} = e^{-\beta F}$$

$$\Rightarrow \quad F = -\frac{\pi L}{6} c T^2 \qquad T = 1/\beta$$

Using standard Thermodynamics we find that

$$\begin{cases} F = E - TS \\ dF = -S\,dT \\ dE = T\,dS \end{cases}$$

$$\left.\begin{aligned} E &= \frac{\pi L}{6} c T^2 \\ S &= \frac{\pi L}{3} c T \end{aligned}\right\} \Rightarrow \boxed{S = 2\pi\sqrt{\frac{cE}{3}\frac{L}{2\pi}}}$$