Charged Reissner-Nordström Black Holes

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Lightning review:

$R_{\mu\nu}=0$: neutral black hole : Schwarzschild

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega_2$$

Horizon at $r=2M$: regular

$$dr_* = \frac{dr}{f} \qquad r_* = \int \frac{dr}{f} \qquad f = 1-\frac{2M}{r}$$

$$v = t + r_* \qquad\qquad (v, r, \theta, \phi)$$

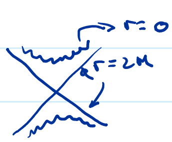
$$ds^2 = -f dv^2 + 2 dr dv + r^2 d\Omega$$: regular at $r=2M$

$f=0$

$g_{\mu\nu}$, $g^{\mu\nu}$ finite, $\det g_{\mu\nu} \neq 0$ finite

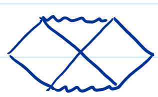
Surface gravity : $\kappa = \frac{1}{2} f' \Big|_{r=2M} = \frac{1}{4M}$

Kruskal extension



$r=0$ is spacelike
not a point but a
moment in time

Penrose :



Charged black holes

$$R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G\, T_{ij}^{EM}$$

$$\nabla_i F^{ij} = 0 \qquad\qquad T_{ij}^{EM} = F_{ik}F_j{}^k - \frac{1}{4}F^2$$

$$I = I_G - \frac{1}{4}\int \sqrt{-g}\, F^2 \qquad\qquad F^2 = F_{ij}F^{ij}$$

What can we expect in the metric?

The electromagnetic field energy will gravitate

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Say $A_t \sim \frac{Q}{r}$

$F_{tr} \sim \frac{Q}{r^2} \qquad T_{ij} \sim \frac{Q^2}{r^4}$

since $\partial^2 g \sim T$

we expect from EM a term $\sim \frac{Q^2}{r^2}$

in the metric

Indeed, solving the Einstein-Maxwell eqns gives

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + Q^2/r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$A_t = -\frac{Q}{r}$$ Reissner-Nordström solution

Mass $= M$ from $-g_{tt} = -1 + \frac{2M}{r} + \dots$

Charge $= Q$ from $A_t = -\frac{Q}{r}$ and Gauss' law

$$\left[\begin{aligned} \text{Charge} &= \frac{1}{4\pi}\int_{S^2} d\vec{s}\cdot\vec{E} \qquad \text{with } F^{tr} = \frac{Q}{r^2} = E^r \\ &= \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, r^2 \frac{Q}{r^2} = Q \end{aligned}\right]$$

<u>Singularity</u>: $r=0$ is a curvature singularity: $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\rho\sigma\tau}R^{\mu\rho\sigma\tau} \xrightarrow{r\to 0} \infty$

[It is a timelike singularity whenever $Q \neq 0$, as we will see.]

<u>Horizons</u>: look for zeroes of $\overset{\text{only if static!}}{g_{tt}}$: $V=0 \Leftrightarrow r^2 - 2Mr + Q^2 = 0$

$r = r_\pm = M \pm \sqrt{M^2 - Q^2}$

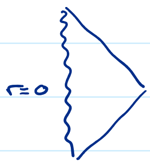
Always assume $M>0$ (negative mass is unphysical)

w.l.o.g. $Q \geq 0$ (in the metric only Q^2 appears)

We'll discuss separately the three cases $M<Q$, $M>Q$, $M=Q$

• $M < Q$: no horizon at real r.

$r = 0$: naked singularity (Timelike)



Notice that for all known charged particles we have $m \ll q$ $\quad q \sim 10^{-1}$

so they're not black holes $\quad m < 10^{-17}$

(Similarly for spin, since $j \sim 1$)

• $M > Q$: There are metric singularities at $r = r_\pm$

Eddington-Finkelstein coordinates

$$dr_* = \frac{dr}{V} \qquad r_* = \int \frac{dr}{V}$$

$$v = t + r_* \qquad (v, r, \theta, \phi)$$

$$ds^2 = -V dv^2 + 2 dr\, dv + r^2 d\Omega \quad : \text{ regular at } r = r_\pm$$

r_+ : outer horizon

r_- : inner horizon

$$\kappa_\pm = \pm \frac{\sqrt{M^2 - Q^2}}{r_\pm^2}$$

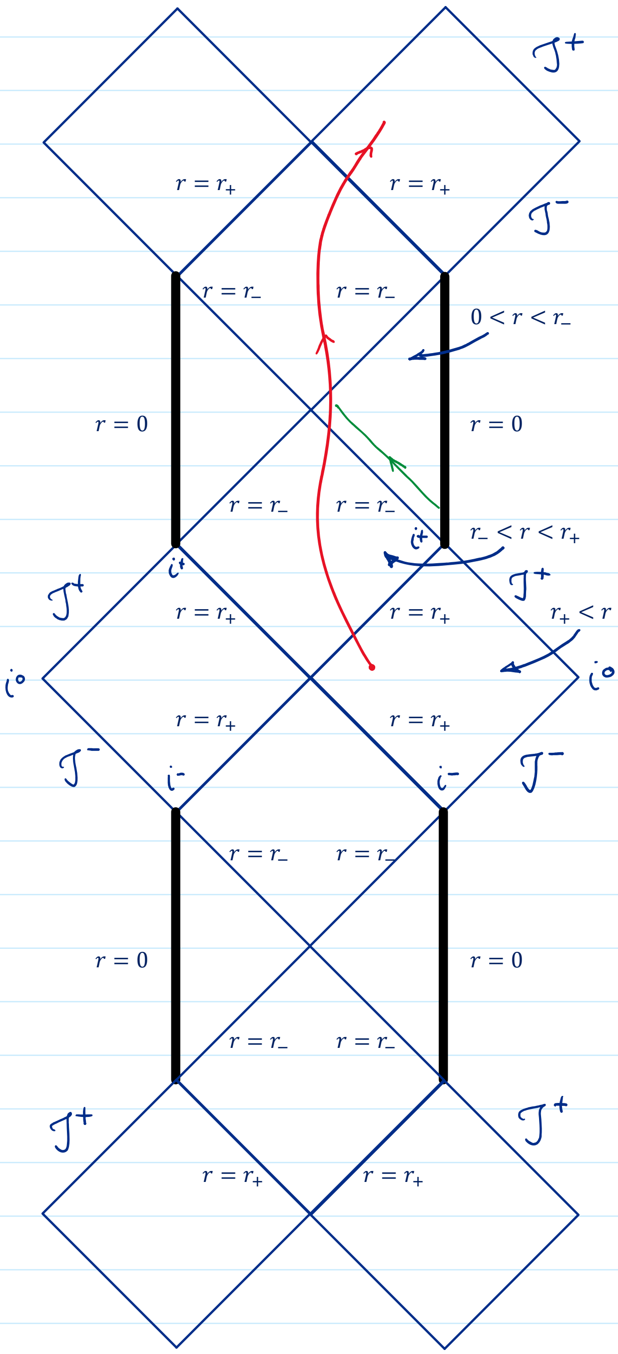
$\kappa_- < 0$ indicates repulsion: one has accelerate to get to it

$r > r_+$: AF region

$r_- < r < r_+$: r timelike, t spacelike

$0 < r < r_-$: r spacelike, t timelike

$r = 0$: timelike singularity

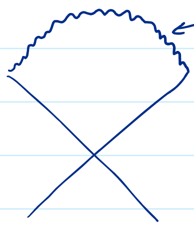


starts in one AF region
and travels to a
different one, emerging
through a white hole

Past r_-, the singularity
at $r=0$ is visible

We expect the inner horizon, to be unstable to

We expect the inner horizon, to be unstable to perturbations, and possibly then the geometry cannot be extended past that singularity.



← singularity. It may have portions that are null or spacelike

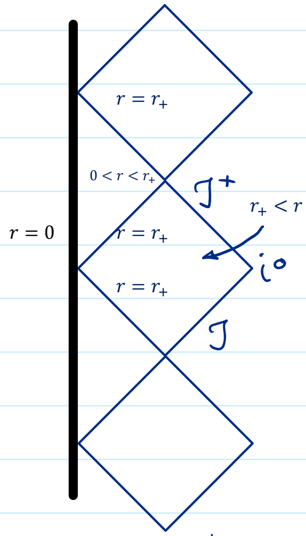
"Strong Cosmic Censorship Conjecture"

There's good evidence that SCC holds, at least when the backreaction of quantum fields is taken into account.

- $M = Q$: <u>extremal</u> RN black hole

$$1 - \frac{2M}{r} + \frac{M^2}{r^2} = \left(1 - \frac{M}{r}\right)^2 \qquad r_+ = r_- = M = Q \quad : \text{double root}$$

$$\kappa_+ = 0 \quad : \text{degenerate horizon}$$



Nowadays we know that the classical extremal black hole is a mathematical artifact of a wrong order of limits: (i) $\hbar \to 0$ (ii) $T \to 0$

If $\hbar \neq 0$, quantum fluctuations of the geometry become uncontrollably large as $T \to 0$

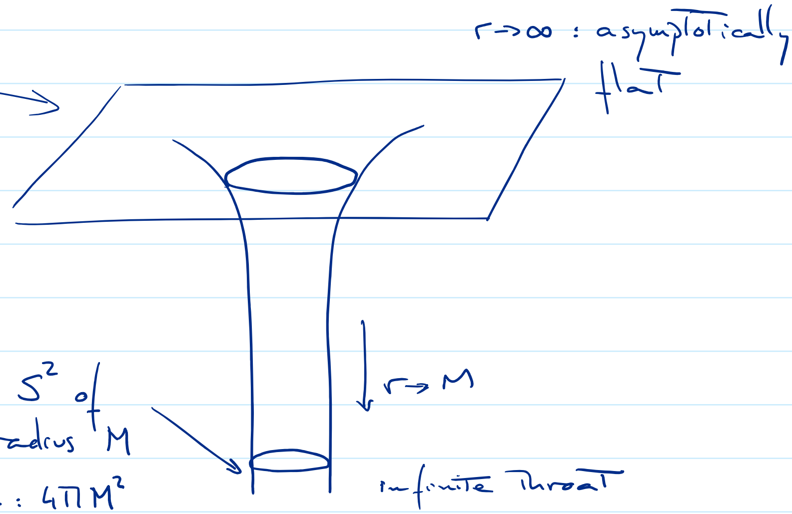
$r = r_+$: event horizon
Cauchy horizon

$r = 0$: Timelike singularity

Proper radial distance to $r=M$ is infinite

$$\int_{r>M}^{M} \frac{dr}{1-\frac{M}{r}} = \int_{r>M}^{M} \frac{r\,dr}{r-M} \;:\; \text{diverges at } r=M$$

$$ds^2\Big|_t = \frac{dr^2}{\left(1-\frac{M}{r}\right)^2} + r^2 d\Omega$$



$$\boxed{\text{This infinite throat is a classical artifact. With quantum effects, the length of the throat has very large variance.}}$$

Near-horizon geometry near extremality

Rewrite the metric by shifting the radial coordinate:

First reparametrize $(M,Q) \to (r_0, q)$

$$M = q + r_0/2 \qquad r_+ = M + \sqrt{M^2-Q^2} = r_0 + q$$

$$Q^2 = q(q+r_0) \qquad M^2 - Q^2 = r_0^2/4$$

Charge $\propto q$ $\quad$ r_0 = non-extremality parameter $\quad$ $r_0 = 0$: extremal

Now shift the radial coordinate

$r = \bar{r} + q$

$$1-\frac{2M}{r}+\frac{Q^2}{r^2}=\frac{(r-r_+)(r-r_-)}{r^2} \quad \frac{(r-q)(r-q-r_0)}{r^2}$$

$$=\frac{\bar r(\bar r-r_0)}{(\bar r+q)^2}=\frac{f(\bar r)}{\left(1+\frac{q}{\bar r}\right)^2}$$

$$f(\bar r)=1-\frac{r_0}{\bar r}$$

In the following, drop the bar $\bar r \to r$

$$ds^2=-\frac{f(r)}{\left(1+\frac{q}{r}\right)^2}dt^2+\left(1+\frac{q}{r}\right)^2\left(\frac{dr^2}{f(r)}+r^2 d\Omega_2\right)$$

This is not the usual area-radius gauge!

Horizon at $r=r_0$. Area $=4\pi(r_0+q)^2$

Extremal limit: $r_0=0$, horizon at $r=0$, finite area

In this case taking $r \ll q$, near-horizon limit:

$$ds^2\simeq-\frac{r^2}{q^2}dt^2+q^2\frac{dr^2}{r^2}+q^2 d\Omega_2$$

$$=q^2\left(\frac{-dt^2+dz^2}{z^2}+d\Omega_2\right)$$

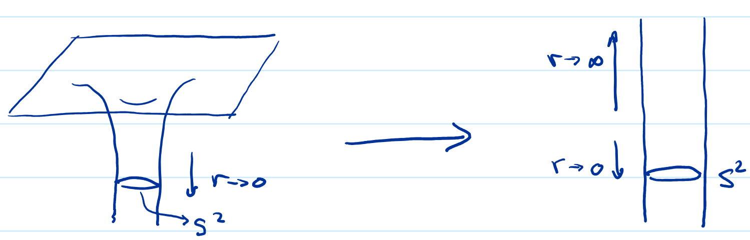
Poincaré-$AdS_2 \times S^2$

$r=\frac{q^2}{z}$

spatial distance $\propto \int \frac{dr}{r}=\log r \xrightarrow{r\to 0} -\infty$

Horizon at $r=0$ $\quad z=\infty$

Boundary (mouth of throat) at $r\to\infty$ $\quad z\to 0$



Now consider near-extremal, near-horizon

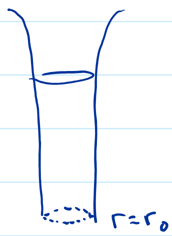
Take $r, r_0 \ll q$

long throat capped by horizon

Take $r, r_0 \ll q$

$$ds^2 = -\frac{r(r-r_0)}{q^2}\,dt^2 + q^2\frac{dr^2}{r(r-r_0)} + q^2\,d\Omega_2$$

long throat capped by horizon



change $r = \frac{r_0}{2}(\tilde r + 1)$ $\quad t = \frac{2q^2}{r_0}\tilde t$. Then

$$ds^2 = q^2\left(-(\tilde r^2 - 1)\,d\tilde t^2 + \frac{d\tilde r^2}{\tilde r^2 - 1} + d\Omega_2\right)$$

$$\underline{\text{Rindler-}AdS_2} \times S^2$$

aka Thermal AdS_2

or $r = r_0\cosh^2\rho/2 \qquad t = \frac{2q^2}{r_0}\tilde t \qquad \rho = 2\,\mathrm{arctanh}\sqrt{\frac{r-r_0}{r}}$

$$ds^2 = q^2\left(-\sinh^2\rho\, d\tilde t^2 + d\rho^2 + d\Omega_2\right)$$

Directly in the extremal case:

$\bar r = r - Q$

$$ds^2 = -\left(1-\frac{Q}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1-\frac{Q}{r}\right)^2} + r^2\,d\Omega = -\frac{1}{H^2}\,dt^2 + H^2\left(d\bar r^2 + \bar r^2 d\Omega\right)$$

$A_t = -\frac{Q}{r} \qquad\qquad A_t = -\frac{1}{H} + \text{const} \qquad H = 1 + \frac{Q}{\bar r}$

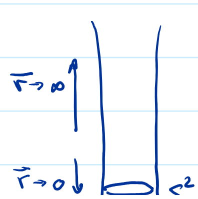
Horizon at $\bar r = 0$: not a point, but a S^2 w/ radius Q
(This is independent of the coordinates)

Near $\bar r = 0 \quad H \simeq \frac{Q}{\bar r}$

$$ds^2 \simeq \underbrace{-\frac{\bar r^2}{Q^2}\,dt^2 + \frac{Q^2}{\bar r^2}\,d\bar r^2}_{AdS_2} + \underbrace{Q^2\,d\Omega_2}_{S^2} \quad : \; AdS_2 \times S^2$$

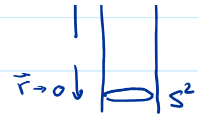
radius Q

Throat



AdS_2 in "Poincaré coordinates" $\bar r = Q$

AdS_2 $\qquad$ S^2



AdS_2 in "Poincaré coordinates" $\frac{r}{Q} = \frac{Q}{z}$

$$ds^2(AdS_2) = Q^2 \, \frac{-dt^2 + dz^2}{z^2}$$

horizon at $z = \infty$

boundary at $z = 0$