Euclidean Quantum Gravity and Gravitational Path Integral

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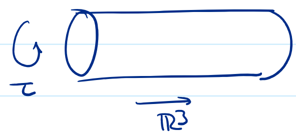
Black holes in imaginary Time

For studying QFT at finite Temperature we consider The QFT in $t \to i\tau$

$$ds^2 = \underbrace{d\tau^2}_{S^1} + \underbrace{dx^2 + dy^2 + dz^2}_{\mathbb{R}^3} \qquad \tau \sim \tau + \beta$$

$S^1 \times \mathbb{R}^3$



KMS condition : correlation functions in QFT at finite Temperature are periodic in imaginary Time

What does This have To do with bhs?

$t = -i\tau$

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega$$

$\tau \sim \tau + \beta$: we're going To see That not all β are allowed

Go near The "Euclidean horizon" at $r = 2M$

$r = 2M + \xi^2 \qquad \xi^2 \ll 2M$

$$ds^2 \simeq \frac{\xi^2}{2M} d\tau^2 + 8M\, d\xi^2 + 4M^2 d\Omega$$

$$8M\,\xi^2 = x^2$$

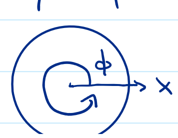
$$ds^2 \simeq +\kappa^2 x^2 d\tau^2 + dx^2 + 4M^2 d\Omega \qquad \kappa = \frac{1}{4M} \text{ : surface gravity}$$

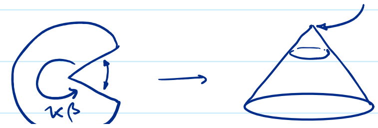
$\rho^2 d\phi^2 + d\rho^2 \qquad x = \rho$

$\kappa\tau = \phi \qquad \tau = \frac{1}{\kappa}\phi$

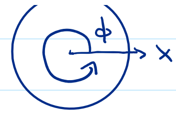
$\tau \sim \tau + \beta \qquad \phi \sim \phi + \kappa\beta$

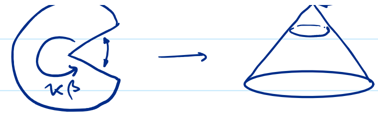
Normally $\phi \sim \phi + 2\pi$. If instead $\kappa\beta < 2\pi$: conical singularity





distributional (delta-fn) curvature





$\kappa\beta = 2\pi$ for absence of conical singularities in the Euclidean geometry

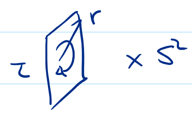
$\Rightarrow \quad \beta = \frac{2\pi}{\kappa} \qquad T = \frac{\kappa}{2\pi} \qquad T = \frac{\hbar\kappa}{2\pi}$: Hawking Temperature

Since the geometry is periodic $\tau \sim \tau + \beta$, all correlation functions of a QFT in this geometry will also exhibit this same periodicity : KMS

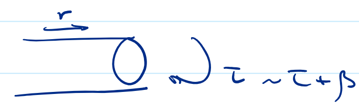
$\Rightarrow$ QFT in a bh background is finite-temperature field theory at $T = T_H = \frac{\kappa}{2\pi}$

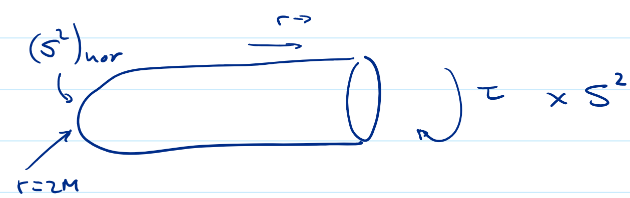
The geometry of Euclidean Schwarzschild:

near $r = 2\pi$



large r $\quad ds^2 \to d\tau^2 + dr^2 + r^2 d\Omega$

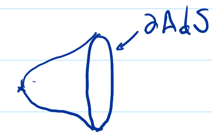




There's nothing behind $r = 2M$
Geometry is completely non-singular and geodesically complete

Topology is $\mathbb{R}^2 \times S^2 \neq S^1 \times \mathbb{R}^3$

This is also valid for black holes in AdS



<u>BH entropy from Euclidean Quantum Gravity</u> (Gibbons + Hawking)

$$Z[\beta] = \int \mathcal{D}g \, e^{-I_E[g]}$$

$I_E[g]$: classical Euclidean action
sum over geometries that are periodic in imaginary time $\Delta\tau = \beta$

The Euclidean gravitational action is

$$I_E = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}\,R - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{h}\,K$$

and it is crucial to include the GHY boundary term.

GPI is ill-defined:
- UV problems: non-renormalizability
- IR problems: unboundedness under conformal transformations (There are ways around this)
- what geometries/topologies/signatures... are allowed?

Evaluate it in the <u>saddle-point approximation</u>:

$$Z[\beta] \approx e^{-I_E[g_{cl}]}$$

g_{cl}: classical solution to the field equations of $I_E[g]$ $\quad \frac{\delta I_E}{\delta g}[g_{cl}] = 0$

with g_{cl} periodic on $\tau \sim \tau + \beta$

This is like a WKB approximation to quantum mechanics, in the semiclassical limit.

There may be more than one saddle contributing. The dominant one is the one with smallest $I_E[g_{cl}]$.

Once we have computed the partition function w/ canonical boundary conditions (fixed temperature), we can apply conventional thermodynamics to it:

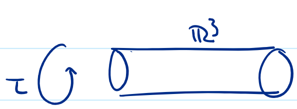
$$F = -\frac{1}{\beta}\log Z \approx \frac{1}{\beta} I_E \qquad \text{(using the saddle-point result)}$$

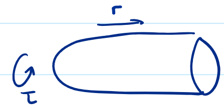
$$E = -\frac{\partial \log Z}{\partial \beta} \approx \partial_\beta I_E$$

$$S = -(\beta\partial_\beta - 1)\log Z \approx (\beta\partial_\beta - 1)\, I_E$$

For AF spaces w/ $\tau \sim \tau + \beta$, we have at least two different saddle point geometries:

Euclidean flat space $S^1 \times \mathbb{R}^3$



Euclidean Schw black hole $\quad S^2 \times \mathbb{R}^2$ 

$\times S^2$

↳ horizon is a fixed point of Euclidean time evolution

When computing $I_E[g_{cl}]$, the GHY term is crucial. E.g. for a vacuum solution only the boundary term contributes.

The action I_E is infinite, since space is infinite. To deal with this:

- First, regularize w/ cutoff at $r=R$ (large radius)
- Subtract $\quad I_E - I_E^0 \quad I_E^0$ action of a reference background state ($\sim$ vacuum)

  Ensure that boundary conditions are the same for $g_{\mu\nu}$ and $g^0_{\mu\nu}$.

  $g_{\mu\nu}$: Euc Schw with $\beta = 8\pi M$

  $g^0_{\mu\nu}$: Eucl Mink

  $\Rightarrow$ finite $\quad I_E - I_E^0 \approx \beta F(\beta)$ for Schw bh

This can indeed be carried out and one finds $F = \frac{\beta}{16\pi}$

so, using the expressions above, $E = \frac{\beta}{8\pi} = M$ $\quad$ which are correct.

$$S = 4\pi M^2 = \frac{A_H}{4}$$

In AdS, it is possible to perform the subtraction without reference to a background. One introduces "boundary counterterms", which are of the form $\int_{\partial M} \sqrt{h}\, F(g, \nabla g \ldots)$

↳ local intrinsic invariants of the bdy geometry

eg $\sqrt{h}$, $\sqrt{h}\, R(h)$, ...

(In AF, counterterms are not local invariants).

There is a general argument* to show that in any Einstein-Hilbert theory of gravity (w/ any matter), the GPI yields an entropy

There is a general argument to show that for any Einstein-Hilbert theory of gravity (w/ any matter), the GPI yields an entropy

$$S = \frac{A_H}{4G\hbar}$$

In this sense, there's a <u>very direct relation between Einstein gravity</u> (ie the EHGHY action) <u>and $S \propto A_H$.</u> "Matter = Geometry"

* For obtaining only the entropy we can take a shortcut, which avoids the regularization and subtraction.

- For <u>any</u> field in <u>flat space</u> if $\partial/\partial\tau$ is a Killing symmetry

$$I_E = \int d\tau\, d^3x\, \mathcal{L} = \beta \int d^3x\, \mathcal{L} = \beta H \qquad H: \text{hamiltonian}$$

$$S = (\beta \partial_\beta - 1) I_E = 0$$

This means that the <u>classical</u> contribution to the entropy in QFT is zero. Entropy in QFT arises at <u>one-loop</u> level

entropy is a sum over microstates = Trace over states

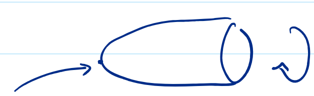
$\sum_{\text{states}}$ $\rightarrow$ ~ # of states of the theory

This result is valid for any Euclidean spacetime where $\partial/\partial\tau$ generates a symmetry in which $\partial/\partial\tau$ acts without fixed points

$\Rightarrow \quad I_E \propto \beta \quad \Rightarrow \quad S = 0$

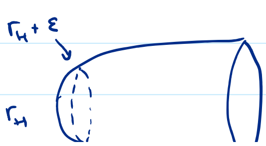
- When there's a black hole horizon (static — can be extended to stationary)

$\frac{\partial}{\partial\tau}$ has a fixed point at the Euclidean horizon

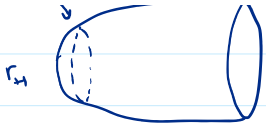


AT the horizon the Hamiltonian evolution degenerates

Split the integration of the action into two regions: near the horizon, and the rest
(cap) (cylinder)



The contribution to I_E from $r > r_H + \varepsilon$ is $\propto \beta$ $\Rightarrow$ This region doesn't contribute to S



The contribution to I_E from $r > r_H + \varepsilon$ is $\propto \beta$ $\Rightarrow$ This region doesn't contribute to S (it would contribute to F and to E, and is the region that requires regularization and subtraction, but S is unaffected by that)

So we only need to compute the Euclidean gravitational action:

$$I_E = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}\,R - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{h}\,K$$

in the cap region.

For a solution of $R_{\mu\nu}=0 \Rightarrow R=0$: The bulk term vanishes.

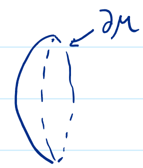
Actually, the bulk contribution in the cap vanishes in general, even with matter, since the volume (and any bulk magnitude) vanishes in the limit $\varepsilon\to 0$.

Only the boundary GHY term will contribute to the entropy.

$K = \frac{1}{\sqrt{h}}\, n^\mu\partial_\mu\sqrt{h}$ $\quad$ n^μ : normal to the boundary

$h_{\mu\nu}$: metric induced at bdry

$$I_E^{(\varepsilon)} = -\frac{1}{8\pi G}\, n^\mu\partial_\mu\int_{\partial\mathcal{M}} d^3x\sqrt{h}$$



$$\simeq -\frac{1}{8\pi G}\, n^\mu\partial_\mu\int_{r\simeq r_H+\varepsilon} d^3x\sqrt{h}$$

$$\int d^3x = \int d\tau\, d^2\Omega$$

$$ds^2 \approx \kappa^2x^2d\tau^2 + dx^2 + r_H^2\,d\Omega_2 \qquad r-r_H \propto x^2$$

$\sqrt{h} = \kappa x\, r_H^2\,\omega_2$ $\quad$ $\omega_2 = \sin\theta$: volume element of S^2 (it could be any transverse space)

Euclidean regularity requires $\kappa = \frac{2\pi}{\beta}$

$$n^x = 1 \qquad n^\mu\partial_\mu = \partial_x$$

$$A_H = 4\pi r_H^2$$

$$I_E^{(\varepsilon)} = -\frac{1}{8\pi G}\,\underbrace{\beta}_{\int d\tau}\,\underbrace{4\pi}_{\int d^2\Omega}\,\kappa r_H^2\,\partial_x x = -\frac{1}{4G}\,4\pi r_H^2 = -\frac{1}{4G}A_H$$

$$\Rightarrow \quad S = (\beta\partial_\beta - 1)I_E = (\beta\partial_\beta - 1)I_E^{(\varepsilon)}$$

$$\Rightarrow \qquad S = (\beta\,\partial_\beta - 1)\, I_E = (\beta\,\partial_\beta - 1)\, I_E^{(c)}$$

$$\Rightarrow \quad \boxed{S = \frac{A_H}{4G}} \qquad \text{The correct Bekenstein–Hawking entropy}$$

There's a general formula for the entropy of higher-curvature gravities due to Wald, which generalizes Bekenstein-Hawking.

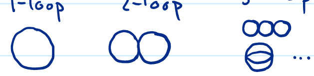
This entropy is due to the non-trivial topology of the Euclidean black hole solution, specifically to the fixed point set of the isometry $\partial/\partial\tau$.

- This is a classical contribution to entropy, which is unusual.

In QFT $\quad \sum_i \int d^4p \; \bigcirc \quad$: sum over states running in the circle

The action receives contributions in a loop expansion

$$I = \underset{\substack{\uparrow \\ \text{1-loop}}}{\hbar} + \underset{\substack{\uparrow \\ \text{2-loop}}}{\hbar^2} + \underset{\substack{\uparrow \\ \text{3-loop}}}{\hbar^3} + \dots$$



In QFT at finite β, we have $F, S \sim \dfrac{I}{\hbar}$

so $\quad S \propto \hbar^0 + \hbar + \hbar^2 + \dots$

But here we have found

$$S_{BH} = \frac{A_H}{4G\hbar} \quad : \text{0-loop calculation of entropy?!}$$

what microscopic states are being counted?

How does the gravitational path integral give us the correct result for the entropy, while not performing any sum over microscopic states?

In string theory, it has been possible to perform countings over microstates of a fundamental theory that correctly reproduce the

In string theory, it has been possible to perform countings over microstates of a fundamental theory that correctly reproduce the BH entropy:

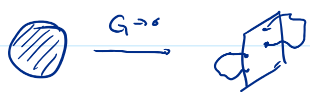
$S_{micro} = S_{BH}$ $\qquad$ S_{micro}: stat-mech counting of entropy

Build a susy bh w/ certain charges, mass, and area $S_{BH} = \frac{A_H}{4G}$

It's very similar to the extremal Reissner-Nordstrom solution, for which $S_{BH} = \pi Q^2$

Now consider the gravitational coupling G to be variable.

$G \to 0$: The size of horizon decreases until its small enough that it doesn't look like a black hole, but a configuration of strings (on D-branes)



As we change G : mass changes (renormalization of mass)

Keep charges fixed

susy $\Rightarrow$ entropy as a function of charges doesn't change

In the string side we can do a stat-mech calculation of S_{micro}.

Strominger+Vafa (1996) found $S_{micro} = S_{BH}$

The result has been generalized and extended for large classes of black holes, in different dimensions, with different kinds of charges, different horizon topologies, and sometimes with higher-curvature corrections.