Contribution of non-acyclic flat connections in Chern-Simons theory and invariants of 3-manifolds

Pavel Mnev

University of Notre Dame

Fifty years of BRST,
Munich, March 24, 2026

Joint work with Konstantin Wernli,
arXiv:2510.18653, 2512.17638

Plan

1. Reminder: Perturbative Chern-Simons invariants at an acyclic flat connection (Witten-Axelrod-Singer).
2. Non-acyclic flat connection: formal volume form on the moduli space of flat connections.
3. Global volume form on the moduli space. Invariance of total volume under changes of metric.
4. Sketch of proof.

Chern-Simons path integral

Fix:

- M – closed oriented 3-manifold
- G a Lie group (compact, simply-connected, matrix), $\mathfrak{g} = \mathrm{Lie}(G)$.
- Let $\mathcal{P}$ = trivial G-bundle over M.

Chern-Simons action:

$$S_{CS}(A) = \int_M \mathrm{tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A \overset{\wedge}{,} A]\right)$$

Field: $A \in \mathrm{Conn}(\mathcal{P}) = \Omega^1(M, \mathfrak{g})$.

- Critical points: *flat* connections.
- $S_{CS}(A) \bmod 4\pi^2\mathbb{Z}$ is invariant under gauge transformations $A \mapsto g^{-1}Ag + g^{-1}dg$.

Path integral heuristics

Partition function:

$$Z(M)=\int_{\mathrm{Conn}/\mathrm{Gauge}}\mathcal{D}A\; e^{\frac{i}{\hbar}S_{CS}(A)} \underset{\text{stationary phase}}{=} \sum_{[A_0]} Z_{A_0}$$

– putative invariant of a 3-manifold.

Contribution of an *acyclic* flat connection A_0:

$$Z_{A_0} \underset{\mathrm{BV/gauge-fixing}}{=} \text{"} \int_{\mathrm{im}(d^*_{A_0})\subset\Omega^\bullet(M,\mathfrak{g})[1]} \mathcal{D}\mathcal{B}\; e^{\frac{i}{\hbar}S_{CS}(A_0+\mathcal{B})} \text{"}$$

Here we chose an auxiliary *metric* on M.

Note: $S_{CS}(A_0+\mathcal{B})=S_{CS}(A_0)+\int_M \mathrm{tr}\left(\frac{1}{2}\mathcal{B}\wedge d_{A_0}\mathcal{B}+\frac{1}{6}\mathcal{B}\wedge[\mathcal{B}\overset{\wedge}{,}\mathcal{B}]\right)$

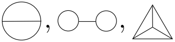
Perturbative (mathematical) formula for Z_{A_0}

Ref: Witten'89, Axelrod-Singer'91,94.

$$Z_{A_0} :\, = e^{\frac{i}{\hbar} S_{CS}(A_0)} \tau_{A_0}^{\frac{1}{2}} e^{\frac{\pi i}{4}\psi_{A_0}} \sum_{\Gamma} \frac{(-i\hbar)^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \Phi_\Gamma \quad \in e^{\frac{i}{\hbar} c} \cdot \mathbb{C}[[\hbar]]$$

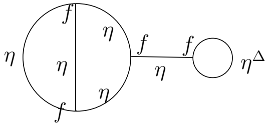
Notations:

- τ_{A_0} is the Ray-Singer torsion.
- ψ_{A_0} is the Atiyah-Patodi-Singer η-invariant of $L_- = d_{A_0} * + * d_{A_0}$ on $\Omega^{\mathrm{odd}}(M, \mathfrak{g})$.
- Γ runs over trivalent graphs:

$\Gamma = \varnothing,$  $, \ldots$

Weights of Feynman graphs

$$\Phi_\Gamma = \int_{C_V(M)} \Big\langle \bigwedge_{\text{edges } (uv)} \pi_{uv}^* \eta \wedge \bigwedge_{\text{short loops } (vv)} \pi_v^* \eta^\Delta, \bigotimes_{\text{vertices } v} f \Big\rangle$$



Here:

- $C_V(M)$ FMAS-compactified configuration space of $V = \#\{\text{vertices of } \Gamma\}$ ordered points on M.
- $\pi_{uv} \colon C_V(M) \to C_2(M)$ records the positions of u, v. $\pi_v \colon C_V(M) \to M$.
- Propagator $\eta \in \Omega^2(C_2(M), \mathfrak{g} \otimes \mathfrak{g})$ – integral kernel of $K = d^* G$ – chain contraction of $(\Omega^\bullet(M, \mathfrak{g}), d_{A_0})$. $G = \Delta^{-1}$ Green's function.
- $\eta^\Delta \in \Omega^2(M, \mathfrak{g} \otimes \mathfrak{g})$ regularized evaluation of η on the diagonal $\mathrm{Diag} \subset M \times M$.
- $f \in \mathfrak{g}^{\otimes 3}$ – structure tensor, $\langle , \rangle$ – trace pairing on $\mathfrak{g}$.

Metric-invariance

Theorem (Witten/Axelrod-Singer)

There exists a power series

$$c(\hbar) = i\frac{\dim \mathfrak{g}}{24(2\pi)} + \frac{\langle f, f\rangle}{24(2\pi)^2} i\hbar + \cdots \in \mathbb{C}[[\hbar]]$$

with universal coefficients (independent of M), such that

$$Z^{\text{ren}}_{A_0} := e^{c(\hbar) S_{\text{grav}}(g,\sigma)} Z_{A_0}$$

does not depend on the metric g.

Here:

- $\sigma\colon M \times \mathbb{R}^3 \to TM$ is a framing (trivialization of TM up to homotopy).
- $S_{\text{grav}}(g,\sigma) = S_{CS}(\sigma^* \nabla_{\text{Levi-Civita}})$.

Thus: coefficients a_n in $Z^{\text{ren}}_{A_0} = e^{\frac{i}{\hbar} S_{CS}(A_0)} \sum_{n\geq 0} a_n \hbar^n$ are invariants of (framed 3-manifold, acyclic flat connection).

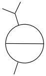
Non-acyclic A_0

Let A_0 – a possibly non-acyclic flat connection.

$$Z_{A_0}(a) = \text{"} \int_{\text{im}(d^*_{A_0})} \mathcal{D}\alpha_{\text{fl}}\; e^{\frac{i}{\hbar} S_{CS}(A_0 + i(a) + \alpha_{\text{fl}})} \text{"}$$

$$:= e^{\frac{i}{\hbar} S_{CS}(A_0)} \tau_{A_0}^{\frac{1}{2}} e^{\frac{\pi i}{4}\psi_{A_0}} \sum_{\Gamma} \frac{(-i\hbar)^{-\chi(\Gamma)}}{|\text{Aut}(\Gamma)|} \Phi_\Gamma(a) \quad \in \text{Dens}^{\frac{1}{2},\text{formal}}(H^\bullet_{A_0}[1])$$

Here:

- $a \in H^\bullet_{A_0}[1]$ "zero-mode".
- $\tau_{A_0} \in \text{Det} H_{A_0}$ Ray-Singer torsion.
- Γ runs over 3-valent graphs with leaves 

Weights of Feynman graphs

$$\Phi_\Gamma(a) = \int_{C_V(M)} \left\langle \bigwedge_{\text{edges }(uv)} \pi^*_{uv}\eta \wedge \bigwedge_{\text{short loops }(vv)} \pi^*_v \eta^\Delta \wedge \bigwedge_{\text{leaves } l} \pi^*_{v(l)} i(a), \bigotimes_{\text{vertices } v} f \right\rangle$$

Propagator: One has Hodge decomposition

$$\Omega^\bullet(M,\mathfrak{g}) = \mathrm{Harm}_{A_0} \oplus \mathrm{im}(d_{A_0}) \underset{\overrightarrow{K}}{\overset{\overleftarrow{d_{A_0}}}{\oplus}} \mathrm{im}(d^*_{A_0})$$

Propagator η = integral kernel of $K = d^*_{A_0} G_{A_0}$, with $G = (\Delta_{A_0} + P_{\mathrm{harm}})^{-1}$, chain homotopy in the Hodge theory SDR

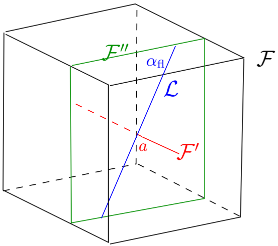
$$K \circlearrowright (\Omega^\bullet(M,\mathfrak{g}), d_{A_0}) \underset{p}{\overset{i}{\leftrightarrows}} H^\bullet_{A_0}$$

with i – inclusion of cohomology as harmonic forms, p – projection to cohomology.

Comment: fiber BV integral

The path integral $Z_{A_0}(a) =$ " $\int_{\mathrm{im}(d^*_{A_0})} \mathcal{D}\alpha_{\mathrm{fl}}\, e^{\frac{i}{\hbar} S_{CS}(A_0 + i(a) + \alpha_{\mathrm{fl}})}$ " is a "fiber BV integral" associated to Hodge decomposition

$$\underbrace{\Omega^\bullet(M,\mathfrak{g})}_{\mathcal{F}} = \underbrace{\mathrm{Harm}_{A_0}}_{\mathcal{F}'} \oplus \underbrace{\mathrm{im}(d_{A_0}) \oplus \overbrace{\mathrm{im}(d^*_{A_0})}^{\mathcal{L}}}_{\mathcal{F}''}$$



Smooth irreducible flat connections

Definition:

1. A flat connection A_0 is *irreducible* if $H^0_{A_0} = 0$.
2. A flat connection A_0 is *smooth*, if the L_∞ algebra on H_{A_0} induced from dgLa $(\Omega^\bullet(M, \mathfrak{g}), d_{A_0}, [-\overset{\wedge}{,} -])$ is has vanishing operations.

Remarks:

- If A_0 is irreducible, smoothness is equivalent to the moduli space of flat connections $\mathcal{M}$ being a smooth manifold around $[A_0]$.
- For A_0 irreducible, Z_{A_0} depends only on $a \in H^1_{A_0} \simeq T_{A_0}\mathcal{M}$ and $Z_{A_0}(a) \in \Omega^{\mathrm{top,\,formal}}(H^1_{A_0})$.

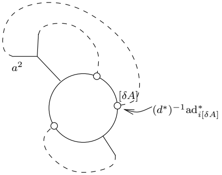
Let $\mathcal{M}' \subset \mathcal{M}$ – moduli space of smooth irreducible flat connections. Want to study Z_{A_0} as a family over A_0 – as a section of

$$\begin{array}{ccc} \mathcal{D} & \longleftarrow & \Omega^{\mathrm{top,\,formal}}(H^1_{A_0}) \\ \downarrow & & \\ \mathcal{M}' & & \end{array}$$

Main result – announcement

We will define $Z^{\text{glob}} \in \Omega^{\text{top}}(\mathcal{M}')$,

$$Z^{\text{glob}}_{A_0} = e^{\frac{i}{\hbar} S_{CS}(A_0)} \tau_{A_0}^{\frac{1}{2}} e^{\frac{\pi i}{4}\psi_{A_0}} \sum$$



$$= Z_{A_0}\Big|_{a=0} \cdot (1 + O(\hbar))$$

such that:

Main Theorem

Variation w.r.t. metric is exact:

$$\delta_g Z^{\text{glob,ren}} = d_{\mathcal{M}'}(\cdots)$$

with $Z^{\text{glob,ren}} = e^{c(\hbar) S_{\text{grav}}(g,\phi)} Z^{\text{glob}}$, $c(\hbar) = i\frac{\dim \mathfrak{g}}{24(2\pi)} + \cdots \in \mathbb{C}[[\hbar]]$, same series as in Witten-Axelrod-Singer.
In particular, for $\mathcal{M}'_\alpha$ a closed smooth connected component of $\mathcal{M}'$, $\int_{\mathcal{M}'_\alpha} Z^{\text{glob,ren}}$ is an invariant of a framed 3-manifold.

Putative relation to RT invariants

Let $RT_{k,N}(M)$ be the Reshetikhin-Turaev invariant of M with $G = SU(N)$ at level k. Let $\{c_j\}$ be the set of values of S_{CS} on $\mathcal{M}$.

Conjecture

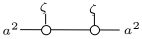
1. (Asymptotic expansion conjecture, Andersen, Reshetikhin)
$$RT_{k,N}(M) \sim_{k\to\infty} \sum_j e^{\frac{i}{\hbar}c_j} f_j(\hbar)$$
with $\hbar = \frac{2\pi}{k+h^\vee}$ and $f_j \in \mathbb{C}[[\hbar]]$ some power series
2. (Refinement) If $\mathcal{M}_j :\ = S_{CS}^{-1}(c_j) \subset \mathcal{M}$ is a closed smooth submanifold contained in $\mathcal{M}'$, then
$$e^{\frac{i}{\hbar}c_j} f_j(\hbar) = \int_{\mathcal{M}_j} Z^{\text{glob,ren}}$$

Here $Z^{\text{glob,ren}}$ is defined using Atiyah's canonical 2-framing.

Construction of Z^{glob}, part 1

Let

$$W(a^2,\zeta) = \frac{1}{2}\langle a^2, \Theta a^2\rangle + \sum_\Gamma \frac{(-i\hbar)^{l(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\widehat{\Phi}_\Gamma$$



$$\in \Omega^\bullet(\mathcal{M}', S^\bullet(H^2_A[-1])^*)[[\hbar]] \simeq C^\infty(\mathcal{M}', S^\bullet(H^1_A[1] \oplus H^2_A[-1])^*)[[\hbar]]$$

Here:

- Variables: $a^2 \in H^2_A[-1]$, $\zeta = [\delta A] \in H^1_A[1] \simeq T_A[1]\mathcal{M}'$.
- Γ runs over *connected* 3-valent graphs with leaves.
- $\Theta = p\,\mathrm{ad}^*_{i(\zeta)} G d G \mathrm{ad}^*_{i(\zeta)} i$

Feynman weights $\widehat{\Phi}_\Gamma$

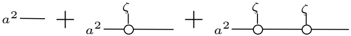
$$\widehat{\Phi}_\Gamma(a^2,\zeta) =$$

$$\int_{C_V(M)} \left\langle \bigwedge_{\text{edges } (uv)} \pi_{uv}^*\widehat{\eta} \wedge \bigwedge_{\text{short loops } (vv)} \pi_v^*\widehat{\eta}^\Delta \wedge \bigwedge_{\text{leaves } l} \pi_{v(l)}^*\widehat{i}(a^2), \bigotimes_{\text{vertices } v} f \right\rangle$$

- $\widehat{\eta} \in \Omega^2(\mathcal{M}' \times C_2(M), \mathfrak{g}\otimes\mathfrak{g})$ – *extended propagator* – integral kernel of

$$\widehat{K} = \sum_{p=0}^{2} K(-\mathrm{ad}^*_{i(\zeta)}G)^p \quad \in \bigoplus_{p=0}^{2} \Omega^2(\mathcal{M}', \mathrm{End}(\Omega^\bullet(M,\mathfrak{g}))_{-1-p})$$

- $\widehat{\eta}^\Delta \in \Omega^2(\mathcal{M}' \times M, \mathfrak{g}\otimes\mathfrak{g})$ – regularized evaluation of $\widehat{\eta}$ on $\mathrm{Diag} \subset M \times M$.
- $\widehat{i} = \sum_{p=0}^{2} (-G\mathrm{ad}^*_{i(\zeta)})^p i \quad \in \bigoplus_{p=0}^{2} \Omega^p(\mathcal{M}', \mathrm{Hom}(H_A^\bullet, \Omega^{\bullet-p}(M,\mathfrak{g})))$



Comment

$\widehat{K}, \widehat{i}, \Theta$ are parts of an SDR – an extension of Hodge retraction

$$K \circlearrowright (\Omega^\bullet(M,\mathfrak{g}), d_A) \underset{p}{\overset{i}{\leftrightarrows}} H^\bullet_A$$

to a family version over $\mathcal{M}'$:

$$\widehat{K} \circlearrowright \left(\Omega^\bullet(\mathcal{M}', \Omega^\bullet(M,\mathfrak{g})), d_A + \delta_{\mathcal{M}'}\right) \underset{\widehat{p}}{\overset{\widehat{i}}{\leftrightarrows}} \left(\Omega^\bullet(\mathcal{M}', H^\bullet_A), \delta_{\mathcal{M}'} + \Theta\right)$$

The extension relies on homological perturbation lemma.

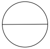
Definition of Z^{glob}

Definition of Z^{glob}

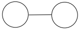
$$Z_A^{\text{glob}} := e^{\frac{i}{\hbar} S_{CS}(A)} e^{\frac{\pi i}{4}\psi_A} \tau_A^{\frac{1}{2}} \underbrace{\int_{H_A^2[-1]\oplus H_A^1[1]} \mathcal{D}a^2\, \mathcal{D}\zeta\; e^{\frac{i}{\hbar}(W(a^2,\zeta)+\langle a^2,\zeta\rangle)}}_{Z^{\text{glob,hl}}} \in \Omega^{\text{top}}(\mathcal{M}')$$

Low orders:

$Z^{\text{glob,hl}} = 1 +$

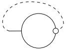


$\frac{i\hbar}{12}\int_{C_2}\langle \eta\wedge\eta\wedge\eta, f\otimes f\rangle$ $+$

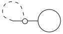


$\frac{i\hbar}{8}\int_{C_2}\langle \pi_1^*\eta^\Delta\wedge\eta\wedge\pi_2^*\eta^\Delta, f\otimes f\rangle$

$+$

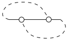


$\frac{i\hbar}{2}\int_M\langle (K\text{ad}^*_{\chi_i}G)^\Delta\wedge\chi^i, f\rangle$ $+$

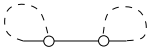


$\frac{i\hbar}{2}\int_M\langle \eta^\Delta\wedge(G\text{ad}^*_{\chi_i}\chi^i), f\rangle$

$+$



$\frac{i\hbar}{2}\int_M\langle \chi^i \overset{\wedge}{,} \text{ad}^*_{\chi_j} G dG \text{ad}^*_{\chi_i}\chi^j\rangle$ $+$



$\frac{i\hbar}{2}\int_M\langle \chi^i \overset{\wedge}{,} \text{ad}^*_{\chi_i} G dG \text{ad}^*_{\chi_j}\chi^j\rangle$ $+ O(\hbar^2)$

Then Main Theorem holds: $\delta_g Z^{\text{glob,ren}} = d_{\mathcal{M}'}(\cdots)$.

Sketch of proof. Step 1: Desynchronized Hodge decomposition

Let FC' be the space of smooth irreducible flat connections.

Lemma

For $(A, A') \in \mathcal{U}$ – an open neighborhood of $\mathrm{Diag} \subset \mathrm{FC}' \times \mathrm{FC}'$, one has

$$\Omega^\bullet(M, \mathfrak{g}) = \mathrm{Harm}_{A,A'} \oplus \mathrm{im}\, d_A \oplus \mathrm{im}\, d^*_{A'}$$

where $\mathrm{Harm}_{A,A'} = \mathrm{ker} d_A \cap \mathrm{ker} d^*_{A'} = \mathrm{ker}\, \underbrace{[d_A, d^*_{A'}]}_{\Delta_{A,A'}} \simeq H^\bullet_A$

One has SDR

$$\underbrace{K_{A,A'}}_{d^*_{A'}(\Delta_{A,A'}+P_{A,A'})^{-1}} \quad \circlearrowright \; (\Omega^\bullet(M,\mathfrak{g}), d_A) \underset{p_{A,A'}}{\overset{i_{A,A'}}{\leftrightarrows}} H^\bullet_A$$

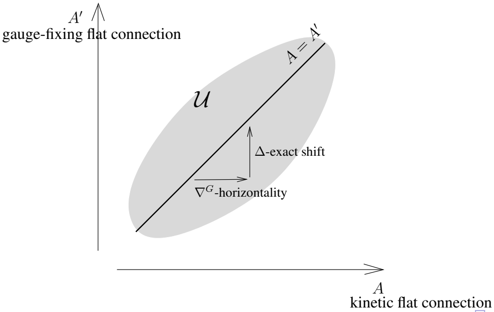
Desynchronized Chern-Simons partition function

Consider "desynchronized" Chern-Simons path integral expanded around A, with gauge-fixing $\mathcal{L} = \mathrm{im}\, d^*_{A'}$:

$$Z_{A,A'}(a) = \text{"} \int_{\mathrm{im}\, d^*_{A'}} \mathcal{D}\alpha_{\mathrm{fl}}\, e^{\frac{i}{\hbar} S_{CS}(A + i_{A,A'}(a) + \alpha_{\mathrm{fl}})} \text{"}$$

$= $ same formula as for $Z_A(a)$, replacing $K, i \to K_{A,A'}, i_{A,A'}$

$$\in \mathrm{Dens}^{\frac{1}{2},\mathrm{formal}}(H_A[1])$$



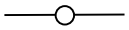
Step 2: Extended partition function: ∇^{Hodge}

Let $\underline{\mathcal{U}} \subset \mathrm{FC}' \times \mathrm{FC}' \times \mathrm{Met} = \{(A, A', g)\}$ a neighborhood of $\mathrm{Diag} \times \mathrm{Met}$. On the trivial bundle

$$\begin{array}{ccc} E & \longleftarrow & \Omega^\bullet(M, \mathfrak{g}) \\ \downarrow & & \\ \underline{\mathcal{U}} & & \end{array}$$

one has a connection $\nabla^{\mathrm{Hodge}} = \delta + H$ preserving Hodge decomposition in fibers. Connection 1-form:
$H = H_{\delta A} + H_{\delta A'} + H_{\delta g} \in \Omega^1(\underline{\mathcal{U}}, \mathrm{End}(\Omega^\bullet(M, \mathfrak{g})))$,

$$\begin{aligned} H_{\delta A} &= -(K\mathrm{ad}_{\delta A} dK + K\mathrm{ad}_{\delta A} P + P\mathrm{ad}_{\delta A} K) \\ H_{\delta A'} &= -(dG\mathrm{ad}^*_{\delta A'} Kd + dG\mathrm{ad}^*_{\delta A'} P + P\mathrm{ad}^*_{\delta A'} Gd) \\ H_{\delta g} &= dK\lambda_{\delta g} Kd + dK\lambda_{\delta g} P + P\lambda_{\delta g} Kd \end{aligned}$$



where $\lambda_{\delta g} = *^{-1}\delta_g *$.

Extended partition function

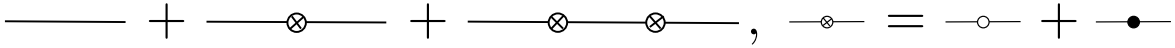
Set

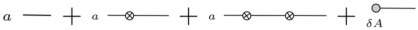
$$\check{Z}(a) := \text{“} \int_{\mathrm{im}\, d^*_{A'}} \mathcal{D}\alpha_{\mathrm{fl}}\, e^{\frac{i}{\hbar}\left(S_{CS}(A+\mathcal{B}) - \int_M \langle \delta A, \mathcal{B}\rangle - \int_M \frac{1}{2}\langle \mathcal{B}, H(\mathcal{B})\rangle\right)}\Big|_{\mathcal{B}=i(a)+\alpha_{\mathrm{fl}}} \text{”}$$

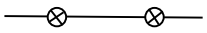
$$= e^{\frac{i}{\hbar}S_{CS}(A)} \tau_A^{\frac{1}{2}} e^{\frac{\pi i}{4}\psi_A} e^{\frac{i}{\hbar}\left(-\langle [\delta A], a\rangle + \frac{1}{2}\langle a, \check{\Theta} a\rangle\right)} \sum_\Gamma \frac{(-i\hbar)^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \check{\Phi}_\Gamma(a)$$

$$\in \Omega^\bullet(\underline{\mathcal{U}}, \mathrm{Dens}^{\frac{1}{2},\mathrm{formal}}(H_A[1]))$$

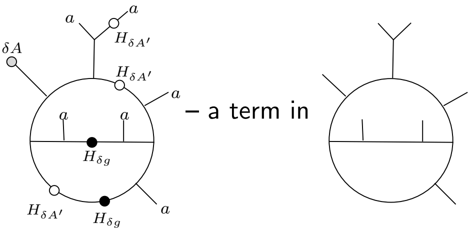
Here $\check{\Phi}_\Gamma$ is defined using $\check{K}$ on edges, $\check{i}(a)$ on leaves with

$$\check{K} = \sum_{p=0}^{2} K(HK)^p =$$ 

$$\check{i}(a) = \sum_{p=0}^{2} (KH)^p i(a) + K\delta A =$$ 

$$\check{\Theta} = -pHKHi =$$ 

A typical graph in $\check{Z}$



Horizontality of $\check{Z}$ (differential quantum master equation)

Theorem

The extended partition function satisfies the dQME:

$$\underbrace{\left(\nabla^{\mathcal{D}} - i\hbar\Delta_a - \frac{i}{\hbar}\frac{1}{2}\langle a, F_{\nabla^{\mathbb{H}}} a\rangle\right)}_{\text{flat ``Gauss--Manin'' superconnection}} \check{Z}^{\mathrm{ren}} = 0$$

Here:

- $\Delta_a = \frac{\partial^2}{\partial a^1 \partial a^2}$ the BV Laplacian on $\mathrm{Dens}^{\frac{1}{2},\mathrm{formal}}(H_A[1])$.
- ∇^{Hodge} restricted to harmonic forms yields a connection $\nabla^{\mathbb{H}}$ (with curvature $F_{\nabla^{\mathbb{H}}}$) on the cohomology bundle $\begin{array}{ccc} \mathbb{H} & \leftarrow & H_A \\ \downarrow & & \\ \mathcal{U} & & \end{array}$

  and in turn connection $\nabla^{\mathcal{D}}$ on $\begin{array}{ccc} \mathcal{D} & \leftarrow & \mathrm{Dens}^{\frac{1}{2},\mathrm{formal}}(H_A[1]) \\ \downarrow & & \\ \mathcal{U} & & \end{array}$

Remark: low degrees along $\underline{\mathcal{U}}$

- $\check{Z}|_{\Omega^0(\underline{\mathcal{U}})} = Z$ the usual (non-extended, desynchronized) partition function.
- Restriction of dQME to $\Omega^1(\underline{\mathcal{U}})$ yields infinitesimal variation statements:
  - $\delta_{A'} Z = \Delta_a(\cdots)$
  - $\delta_g Z^{\mathrm{ren}} = \Delta_a(\cdots)$
  - $\delta_A Z = \langle [\delta A], \frac{\delta}{\delta a} \rangle Z$ for δA harmonic.

Proof of dQME is based on Stokes' theorem on configuration spaces.

Step 3: Reduction of $\check{Z}$ to the moduli space

1. Restrict $\check{Z}$ to $A' = A, \quad \delta A' = \delta A$ harmonic;
2. Pass to gauge classes of connections;
3. Remove trees which are 1-forms along $\underline{\mathcal{U}}$.

Result: $\overline{Z} \in \Omega^\bullet(\mathcal{M}' \times \mathrm{Met}, \mathcal{D})$.

$$\mathrm{dQME} \Rightarrow \qquad (\nabla^G_{\mathcal{M}'\times\mathrm{Met}} - i\hbar\Delta_a)\overline{Z}^{\mathrm{ren}} = 0$$

Here:

- $\nabla^G_{\mathcal{M}'} = \nabla^{\mathcal{D}}_{\mathcal{M}'} + \{\xi, -\} + \Delta_a \xi$ the flat Grothendieck connection on $\mathcal{D} \to \mathcal{M}'$,

  $$\xi = -\langle[\delta A, a]\rangle + \sum \text{trees with one white vertex} \in \Omega^1(\mathcal{M}', S(H_A[1])^*)$$

- $\nabla^G_{\mathcal{M}'\times\mathrm{Met}} = \nabla^G_{\mathcal{M}'} + \delta_g + \mu + \Delta_a \mu$ the flat Grothendieck connection on $\mathcal{D} \to \mathcal{M}' \times \mathrm{Met}$,

  $$\mu = \sum \text{trees with one black vertex} \in \Omega^{0,1}(\mathcal{M}' \times \mathrm{Met}, S(H_A[1])^*)$$

Step 4: Constructing a global object on the moduli space

One has SDR

$$\cdots \circlearrowright (\Omega^\bullet(\mathcal{M}',\mathcal{D}),\nabla^G) \underset{\cdots|_{a^1=0,[\delta A=0]}}{\overset{\cdots}{\leftrightarrows}} (\mathrm{Dens}^{\frac{1}{2}}(T^*[-1]\mathcal{M}'),0)$$

Considering it in a family over Met and perturbing the differential, one gets the SDR

$$K \circlearrowright \Big(\underbrace{\Omega^\bullet(\mathcal{M}'\times\mathrm{Met},\mathcal{D}),\nabla^G_{\mathcal{M}'\times\mathrm{Met}}-i\hbar\Delta_a}_{(I)}\Big) \underset{p}{\overset{i}{\leftrightarrows}} \Big(\underbrace{\mathrm{Dens}^{\frac{1}{2}}(T^*[-1]\mathcal{M}')\otimes\Omega^\bullet(\mathrm{Met}),\delta_g-i\hbar\Delta_{\mathcal{M}'}}_{(II)}\Big)$$

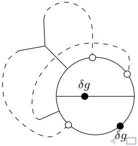
with

$$p(\cdots) = \Big(\int_{H^2_A[-1]\oplus H^1_A[1]} \mathcal{D}a^2\mathcal{D}\underbrace{[\delta A]}_{\zeta}\, e^{\frac{i}{\hbar}\langle a^2,\zeta\rangle}\cdots\Big)\Big|_{a^1=0}$$

We have

$\overline{Z}^{\mathrm{ren}}$ is a cocycle of (I) $\Rightarrow$

$$\check{Z}^{\mathrm{glob,ren}} := p(\overline{Z}^{\mathrm{ren}}) = \cdots\sum$$



is a cocycle of (II)

Cocycle property of $\check{Z}^{\mathrm{glob,ren}}$ in lowest degree along Met

Let $\check{Z}^{\mathrm{glob,ren}(k)}$ be the k-form component of $\check{Z}^{\mathrm{glob,ren}}$ along Met.

$$(\delta_g - i\hbar\Delta_{\mathcal{M}'})\check{Z}^{\mathrm{glob,ren}} = 0$$

$$\underset{\cdots|_{\Omega^1(\mathrm{Met})}}{\Rightarrow} \qquad \delta_g \check{Z}^{\mathrm{glob,ren}(0)} = i\hbar\Delta_{\mathcal{M}'}\check{Z}^{\mathrm{glob,ren}(1)} \qquad (1)$$

Remark

One has isomorphisms of chain complexes

$$(\mathrm{Dens}^{\frac{1}{2}}(T^*[-1]\mathcal{M}'),\Delta_{\mathcal{M}'}) \cong (\Omega^{\mathrm{top}}(\mathcal{M}')\otimes\mathcal{V}^\bullet(\mathcal{M}'),\mathrm{div}) \cong (\Omega^\bullet(\mathcal{M}'),d_{\mathcal{M}'})$$

Under this iso, $Z^{\mathrm{glob,ren}} = \check{Z}^{\mathrm{glob,ren}(0)}$ is a top form, $\check{Z}^{\mathrm{glob,ren}(1)}$ is a (top-1)-form. (1) becomes

$$\delta_g Z^{\mathrm{glob,ren}} = d_{\mathcal{M}'}(\cdots)$$

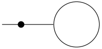
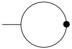
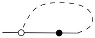
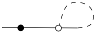
This proves the Main Theorem.

Q.E.D.

Remark

$d_{\mathcal{M}'}$-primitive for the metric variation of $Z^{\text{glob,ren}}$ is:

$$\check{Z}^{\text{glob,ren}(1)} = e^{\frac{i}{\hbar}S_{CS}(A)} e^{\frac{\pi i}{4}\psi_A} \tau_A^{\frac{1}{2}} \cdot$$

$\cdot\Big($  $+$  $+$  $+$  $+ O(\hbar)\Big)$

Open questions/wish list

1. Explicit computations. E.g., 2-loop invariant for components of $\mathcal{M}'$ for Seifert homology 3-spheres.
2. Extension from $\mathcal{M}'$ to $\mathcal{M}$ – incorporating singular and reducible connections.
3. Extension to manifolds with boundary/corners.
4. Comparison with RT.
5. Other models where perturbation theory is in a family over a parameter space X, with gauge-fixing depending on X?

References

- E. Witten, "Quantum field theory and the Jones polynomial." Commun. Math. Phys. 121.3 (1989): 351–399.
- S. Axelrod, I. M. Singer, "Chern-Simons perturbation theory." In: Differential geometric methods in theoretical physics, Proceedings, New York. Vol. 1. (1991): 3–45.
- S. Axelrod, I. M. Singer, "Chern-Simons perturbation theory." II. In: J. Diff. Geom. 39.1 (1994): 173–213.
- P. Mnev, K. Wernli, "Globalization of perturbative Chern-Simons theory on the moduli space of flat connections in the BV formalism." arXiv:2510.18653.
- P. Mnev, K. Wernli, "Perturbative Chern-Simons invariants from non-acyclic flat connections." arXiv:2512.17638.