Quantum BRST-BV in Topological/Holomorphic Theories

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1. Effective BRST-BV Quantization

We consider two types of QFT. Topological QFT of de Rham type

$$\delta_{\mathsf{BRST}} = d + \cdots$$

Fields are built from de Rham complex $(\Omega^\bullet(X, -), d)$. Examples:

- Chern-Simons theory: $\frac{1}{2}\int \mathcal{A} \wedge d\mathcal{A} + \cdots$
- Topological quantum mechanics
- Topological B-model
- Poisson σ-model
- $\cdots$

The complex analogue: Holomorphic QFT of Dolbeault type

$$\delta_{\mathsf{BRST}} = \bar{\partial} + \cdots$$

Fields are built from Dolbeault complex $(\Omega^{0,\bullet}(X, -), \bar{\partial})$. Examples:

- Holomorphic $\beta - \gamma$ system: $\int \beta \bar{\partial} \gamma$
- Holomorphic Chern-Simons theory: $\frac{1}{2} \int \mathcal{A} \bar{\partial} \mathcal{A} + \cdots$
- Kodaira-Spencer gravity (BCOV theory)
- Holomorphically twisted theory
- $\cdots$

Assume the classical action in the BV form ($Q = d$ or $\bar{\partial}$)

$$S = \frac{1}{2}\int(-, Q-) + I_0 \qquad I_0\text{=classical interaction}$$

The classical BRST transformation is ($\{-,-\}$ = BV anti-bracket)

$$\delta_{\text{BRST}} = \{S, -\} = Q + \{I_0, -\}.$$

$\delta^2_{\text{BRST}} = 0$ is captured by the classical master equation

$$\{S, S\} = 0 \quad \text{or} \quad QI + \frac{1}{2}\{I_0, I_0\} = 0$$

Example: $\delta_{CS}(\mathcal{A}) = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]$. Chevalley-Eilenberg.

At quantum level, BRST-BV quantization asks for

$$I = I_0 + \hbar I_1 + \hbar I_2 + \ldots$$

satisfying quantum master equation

$$\text{``}QI + \frac{1}{2}\{I, I\} + \hbar \Delta I = 0\text{''} \qquad \Delta = \mathsf{BV}$$

Then the quantum BRST transformation would be

$$Q + \hbar \Delta + \{I, -\}.$$

Problem: ΔI is NOT well-defined for local I. Need renormalization.

Choose a metric and let $Q^\dagger$ be the adjoint of Q. The propagator is

$$P = \int_0^\infty Q^\dagger e^{-t[Q,Q^\dagger]} dt = \int_0^\infty G_t dt.$$

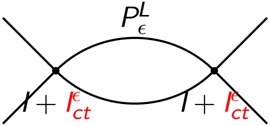
Introduce UV cut-off ϵ and IR cut-off L

$$\text{regularized propagator :} \quad P_\epsilon^L := \int_\epsilon^L G_t dt$$

Costello's effective renormalization theory:

- Construct local counter term I^{ϵ}_{ct} s.t. the effective action $I[L]$

$$I[L] := \lim_{\epsilon \to 0} \sum_{\text{conn}}$$



exist.

- Regularized BV operator Δ_L which is Q-homologous to Δ
- Effective quantum master equation

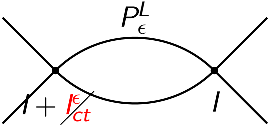
$$QI[L] + \hbar \Delta_L I[L] + \frac{1}{2}\{I[L], I[L]\}_L = 0.$$

is well-defined. Different L lead to equivalent equations.

2. UV (ultra-violet) Finite Theories

There are classes of models where ϵ-dependent counter-terms I^{ϵ}_{ct} are not needed. Such models are UV finite. Start from local I

$$I[L] := \lim_{\epsilon \to 0} \sum_{\text{conn}}$$



exist.

In this case we have $\lim_{L \to 0} I[L] \to I$. Now we expect that in $L \to 0$

$$QI[L] + \hbar \Delta_L I[L] + \frac{1}{2}\{I[L], I[L]\}_L = 0 \quad \stackrel{L \to 0}{\Longrightarrow} \quad QI + \frac{1}{2}\{I, I\} + \cdots = 0$$

will have a local expression that deforms classical master equation.

Conjecture: Effective BV quantization (in the sense of Costello) of UV finite theory is equivalent to a local equation

$$\ell_1^{\hbar}(I) + \frac{1}{2}\ell_2^{\hbar}(I, I) + \frac{1}{3!}\ell_3^{\hbar}(I, I, I) + \cdots = 0$$

where $\{\ell_1^{\hbar}, \ell_2^{\hbar}, \cdots\}$ defines a $\hbar$-family of L_∞-operators on local functionals in the model. They can be viewed as traded from BV Δ in terms of the renormalization procedure.

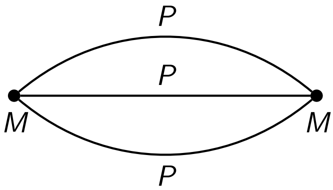
Topological QFT's (e.g Chern-Simons) are UV (ultra-violet) finite.

The UV finite property was established by **Axelrod-Singer** and **Kontsevich** using the compactified configuration space.

$$\int_{\mathrm{Conf}_n(X)} \Phi_\Gamma = \int_{\overline{\mathrm{Conf}_n(X)}} \overline{\Phi}_\Gamma$$

Here

$$\mathrm{Conf}_n(X) = \{(x_1, \cdots, x_n) \in X^n) | x_i \neq x_j, \forall i, j\}$$



$$\Phi_\Gamma = P(x,y)\, P(x,y)\, P(x,y)$$

Remarkably, holomorphic QFT's are also free of UV divergence!

The UV finite property is by completely different analytic reason.

- In $\dim_{\mathbb{C}} = 1$. This is essentially Cauchy principal value. A proof in the sense of effective BV quantization is by **L**.
- In $\dim_{\mathbb{C}} > 1$
  - **Costello-L** and **Williams**: all one-loops are UV finite
  - **Budzik-Gaiotto-Kulp-Wu-Yu**: Laman graphs are UV finite
  - **Minghao Wang**: all graphs in hol QFT on $\mathbb{C}^n$ are UV finite.
  - **Wang-Yan**: hol QFT on Kahler manifolds are UV finite.

The two-point function (propagator) is related to

$$\bar{\partial}^{-1} = \textbf{Bochner–Martinelli kernel}$$

On $\mathbb{C}$,

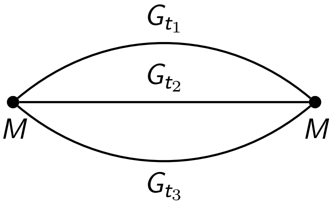
$$\bar{\partial}^{-1} = \frac{1}{z - w}$$

On $\mathbb{C}^n$ $(n > 1)$, the Bochner–Martinelli kernel is

$$\bar{\partial}^{-1} = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\bar{z}_j - \bar{w}_j}{|z-w|^{2n}} \, d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n$$

Wang's argument: Representing the propagator

$$P = \text{"}\bar{\partial}^{-1}\text{"} = \int_0^\infty \bar{\partial}^* e^{-t[\bar{\partial},\bar{\partial}^*]} dt \quad \overset{\text{cut-off}}{\Longrightarrow} P_\epsilon^L = \int_\epsilon^L G_t dt$$



$$\Phi_\Gamma(Q) = G_{t_1}(x,y)\, G_{t_2}(x,y)\, G_{t_3}(x,y)$$

The Feynman graph integral

$$\int_{\mathrm{Conf}_n(X)} \Phi_\Gamma(P_\epsilon^L) = \int_{[\epsilon,L]^E} \int_{\mathrm{Conf}_n(X)} \Phi_\Gamma(G) \overset{\epsilon\to 0}{=} \int_{\overline{(0,L]^E}} \int_{\mathrm{Conf}_n(X)} \Phi_\Gamma(G)$$

$\overline{(0,L]^E}$ is called the compactified Schwinger space.

3. Examples

Topological Quantum Mechanics

Phase space: V graded space with a $\deg = 0$ symplectic pairing ω.

$$\text{fields}: \quad \varphi : S^1_{dR} \to V$$

where $S^1_{dR} = \Omega^\bullet(S^1)$. The topological action is

$$S = \frac{1}{2}\int_{S^1} \omega(\varphi, d\varphi) + \underbrace{\int_{S^1} \Gamma(\varphi)}_{I_\Gamma}, \qquad I \in \mathcal{O}(V)_1$$

where Γ is a $\deg = 1$ function on V. The CME is equivalent to

$$\{S, S\}_{\mathsf{BV}} = 0 \qquad \Longleftrightarrow \qquad \{\Gamma, \Gamma\}_{\omega^{-1}} = 0.$$

This model is UV finite.

Theorem[**Grady-L-Li**, 2017] The effective quantum master equation is equivalent to its $L \to 0$ limit as

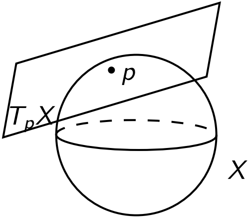
$$dI_\Gamma[L] + \hbar \Delta_L I_\Gamma[L] + \frac{1}{2}\{I_\Gamma[L], I_\Gamma[L]\}_L = 0$$

$$\Updownarrow$$

$$\frac{1}{\hbar}[\Gamma, \Gamma]_\star = 0$$

where $\star$ is the Moyal product.

Such construction can be glued on a symplectic manifold (X, ω).



We have a bundle of fiberwise Weyl algebras

$$\mathcal{W} := \prod_{k=0}^{\infty} \mathrm{Sym}^k(T^*X)[[\hbar]].$$

QME is geometrically equivalent to a $\Gamma \in \Omega^1(X, \mathcal{W})$ such that

$$\nabla + \frac{1}{\hbar}[\Gamma, -]_\star \quad \text{is flat} \quad \text{(Fedosov's connection)}$$

2d Chiral QFT

Consider an elliptic curve E and a graded space V with $\deg = 0$ symplectic pairing ω. Fields in the BV formalism is

$$\Omega^{0,\bullet}(E) \otimes V \quad \ni \quad \varphi = \{\beta_i, \gamma^i, b_j, c^j\}$$

It defines a free chiral CFT

$$\frac{1}{2}\int_E dz\, \omega(\varphi, \bar{\partial}\varphi) = \int_E dz\, \beta_i \bar{\partial}\gamma^i + \int_E dz\, b_j \bar{\partial} c^j$$

We are interested in chiral interactions of the form

$$I_{\mathcal{L}} = \int_E dz\, \mathcal{L}(\varphi, \partial_z \varphi, \cdots)$$

Let us denote

$$\mathcal{V} := \mathrm{Sym}(\mathbb{C}[\partial_z] \otimes V^\vee) = \mathbb{C}[\varphi, \partial_z \varphi, \cdots]$$

which carries a structure of chiral VOA. In a symplectic basis

$$\beta_i(z)\gamma^i(w) \sim \frac{\hbar}{z-w}, \qquad b_j(z)c^j(w) \sim \frac{\hbar}{z-w}.$$

The chiral interaction $\mathcal{L}$ defines an element of modes

$$\oint \mathcal{V} = \mathrm{Span}\left\{ \oint dz z^k A(z) \right\}_{A \in \mathcal{V}, k \in \mathbb{Z}}$$

which is a graded Lie algebra by Borcherds commutator formula.

Theorem [**L**,2023]: 2d chiral QFT is UV finite. The effective quantum master equation is equivalent to its $L \to 0$ limit as

$$\bar{\partial} I[L] + \hbar \Delta_L I[L] + \frac{1}{2}\{I[L], I[L]\}_L = 0$$

$$\Updownarrow$$

$$\frac{1}{\hbar}\left[\oint dz \mathcal{L}, \oint dz \mathcal{L}\right] = 0$$

where $[-,-]$ is the Lie bracket in $\oint \mathcal{V}$. The quantum BRST

$$\delta_{\mathrm{BRST}} = \frac{1}{\hbar}\left[\oint dz \mathcal{L}, -\right].$$

The above discussion can be glued for a chiral σ-model

$$\varphi : E \to X$$

which produces a bundle $\mathcal{V}(X) \to X$ of chiral vertex algebras on X.

Then effective QME asks for a flat connection on $\mathcal{V}(X)$ of the form

$$D = \nabla + \frac{1}{\hbar}[\oint \gamma, -], \text{ such that } D^2 = 0.$$

Here $\gamma \in \Omega^1(X, \mathcal{V}(X))$ and $\oint \gamma$ is fiberwise chiral mode operator.

This can be viewed as the chiral analogue of Fedosov connection.

Remark: For holomorphic theories in $\dim_{\mathbb{C}} > 1$, we do not know precise form of local QME at $L = 0$ yet. A closely related result is

- [**Gui-Wang-Williams**, 2025]: Jouanolou models and higher chiral operations

where a system of higher chiral OPE $\{\ell_2^\hbar, \ell_3^\hbar, \cdots\}$ are explicitly constructed via combinatorics modeled on the propagator $\bar{\partial}^{-1}$.

UV properties and Computations

We are interested in chiral correlations on Riemann surface Σ

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\cdots\mathcal{O}_n(z_n)\rangle_\Sigma$$

- holomorphic on $\mathrm{Conf}_n(\Sigma) = \{(z_1, \cdots, z_n) \in \Sigma^n | z_i \neq z_j, \forall i, j\}$
- singular with meromorphic poles when $z_i \to z_j$

$$\mathcal{O}_i(z_i)\mathcal{O}_j(z_j) \sim \frac{*}{(z_i - z_j)^{2026}} + \cdots$$

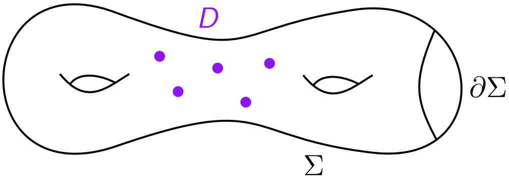
We are led to consider

$$\int_{\Sigma^n} \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\cdots\mathcal{O}_n(z_n)\rangle_\Sigma \, dVol$$

Unlike the situation in topological field theory, $\langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle$ is very singular along diagonals and there is no way to extend it to certain compactification of $\mathrm{Conf}_n(\Sigma)$.

Regularized integral [**L-Zhou**, 2021]

Let us first consider the integral of a 2-form ω on Σ with meromorphic poles of arbitrary orders along a finite subset $D \subset \Sigma$.



Locally we can write $\omega = \frac{\eta}{z^n}$ where η is smooth 2-form and $n \in \mathbb{Z}$.

We can decompose ω into

$$\omega = \alpha + \partial\beta$$

- α is a 2-form with at most logarithmic pole along D
- β is a $(0,1)$-form with arbitrary order of poles along D
- $\partial = dz\frac{\partial}{\partial z}$ is the holomorphic de Rham

We define the regularized integral

$$\boxed{\fint_{\Sigma} \omega := \int_{\Sigma} \alpha + \int_{\partial\Sigma} \beta}$$

This does not depend on the choice of the decomposition and is equvalent to the Cauchy principal value.

The regularized integral can be viewed as a "homological integration" by the holomorphic de Rham ∂

$$\fint_{\Sigma} \partial(-) = \int_{\partial \Sigma} (-).$$

The $\bar{\partial}$-operator intertwines the residue

$$\fint_{\Sigma} \bar{\partial}(-) = \mathrm{Res}(-).$$

Here

$$\mathrm{Res}_0 \frac{\rho(z,\bar{z})}{z^n} dz = \lim_{\epsilon \to 0} \oint_{|z|=\epsilon} \frac{\rho(z,\bar{z})}{z^n} dz$$

Example:

$$\int_{\mathbb{C}} \frac{d^2 z}{(z-a)(z-b)(z-c)}$$

$$= \frac{\bar{a}}{(a-b)(a-c)} + \frac{\bar{b}}{(b-a)(b-c)} + \frac{\bar{c}}{(c-a)(c-b)}$$

In general, we can define

$$\fint_{\Sigma^n}(-) := \fint_{\Sigma}\fint_{\Sigma}\cdots\fint_{\Sigma}(-)\,.$$

This does not depend on the choice of the ordering (**Fubini** type theorem holds). This gives an intrinsic definition of

$$\fint_{\Sigma^n}\langle\mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle\, dVol$$

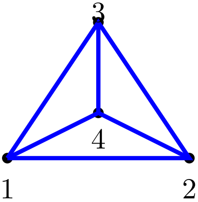
Example: Consider chiral boson on elliptic curve E_τ.

$$\langle \partial\phi(z_1)\partial\phi(z_2)\rangle_{E_\tau} = \widehat{P}(z_1, z_2; \tau, \bar{\tau})$$

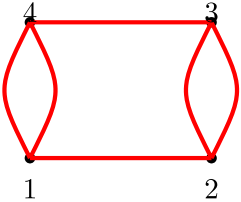
Here

$$\widehat{P}(z_1, z_2; \tau, \bar{\tau}) = \wp(z_1 - z_2; \tau) + \frac{\pi^2}{3}\widehat{E}_2(\tau, \bar{\tau})$$
$$\widehat{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi}\frac{1}{\operatorname{im}\tau}$$

$\wp$ is the Weierstrass $\wp$-function, E_2 is the 2nd Eisenstein series.



Γ_1



Γ_2

$$\Phi_{\Gamma_1}(z_1, z_2, z_3, z_4; \tau) = \widehat{P}(z_1 - z_2)\widehat{P}(z_2 - z_3)\widehat{P}(z_3 - z_1)\widehat{P}(z_1 - z_4)\widehat{P}(z_2 - z_4)\widehat{P}(z_3 - z_4)$$

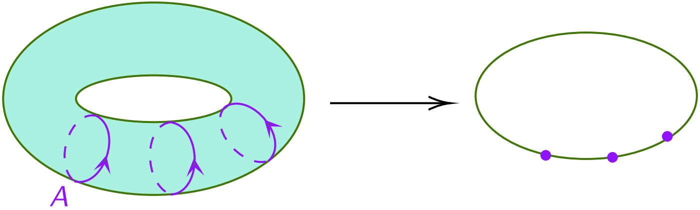
$$\fint_{E_\tau^4} \left(\prod_{i=1}^{4} \frac{d^2 z_i}{\operatorname{im} \tau} \right) \Phi_{\Gamma_1} = \frac{(2\pi i)^{12}}{2^{11} \cdot 3^5} (-\widehat{E}_2^6 + 3\widehat{E}_2^4 E_4 - 3\widehat{E}_2^2 E_4^2 + E_4^3)$$

$$\Gamma_2 = \frac{(2\pi i)^{12}}{2^{10} \cdot 3^7} (-3\widehat{E}_2^6 + 6\widehat{E}_2^4 E_4 + 4\widehat{E}_2^3 E_6 - 3\widehat{E}_2^2 E_4^2 - 12\widehat{E}_2 E_4 E_6 + 4E_4^3 + 4E_6^2)$$

They are almost holomorphic modular forms.

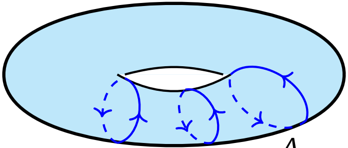
2d Chiral $\Longrightarrow$ 1d Top

In physics, the partition functions on elliptic curves are described by reducing to a quantum mechanical system on S^1.



$$\langle - \rangle_{2d} \longrightarrow \mathrm{Tr}\ \langle \cdots \rangle$$

In 2d we have the *regularized integral* $⨍_E$. In 1d, operators are described by A-cycle $\oint_A$. These two integrals are not exactly the same, but related to each other by ***holomorphic limit***.



Theorem (L-Zhou,2021)

$$\fint_{E_\tau^n} \left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{im}\tau}\right) \langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_E \quad \textit{lies in} \quad \mathcal{O}_{\mathbf{H}}[\frac{1}{\operatorname{im}\tau}]$$

Let $\lim_{\bar\tau\to\infty} : \mathcal{O}_{\mathbf{H}}[\frac{1}{\operatorname{im}\tau}] \to \mathcal{O}_{\mathbf{H}}$ *which sends* $\frac{1}{\operatorname{im}\tau} \to 0$. *Then*

$$\lim_{\bar\tau\to\infty} \fint_{E_\tau^n} \left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{im}\tau}\right) \langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_E$$

$$= \frac{1}{n!}\sum_{\sigma\in S_n} \int_{A_{\sigma(1)}} dz_1 \cdots \int_{A_{\sigma(n)}} dz_n \, \langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_E$$

Here $A_1, \cdots, A_n$ be n disjoint A-cycles on E_τ.

This theorem gives a precise relation on reduction of torus to circle

$$\fint_{E_\tau^n} \xrightarrow{\lim_{\bar{\tau}\to\infty}} \text{Weyl ordered } \oint_A$$

Theorem [**Hou-L-Zhu**,2026] The following Elliptic trace formula

$$\lim_{\bar{\tau}\to\infty} \left\langle e^{\frac{1}{\hbar}\fint_{E_\tau}\mathcal{L}} \right\rangle_{E_\tau} = \frac{\mathrm{Tr}\ q^{L_0-\frac{c}{24}} e^{\frac{1}{\hbar}\oint \mathcal{L}}}{\mathrm{Tr}\ q^{L_0-\frac{c}{24}}}$$

Q: Chiral index in higher $\dim_{\mathbb{C}}$?

Thanks!