Beyond Chern-Simons: The Algebraic Topology of Superconformal Anomalies

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“50 years of BRST”

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Dedicated to the Memory of Carlo Maria Becchi

On September 15, 2025, we lost Carlo Maria Becchi.

BRST symmetry — his most extraordinary discovery, and the very reason we are gathered here — did more than provide the definitive solution to the renormalization of gauge theories. In the decades since, it has become an indispensable and illuminating tool across quantum theory, from gauge theories to string theory and beyond.

I am grateful to the organizers for this opportunity to honor the enduring depth and reach of Carlo's ideas.

Beyond his groundbreaking contributions, Carlo was a master teacher with an inimitable style. His charisma, his penetrating insights, and his insistence on theoretical work that is both rigorous and relevant left a profound mark on generations of students, and on all who had the privilege of engaging with him intellectually.

With his passing, we have lost not only one of the most brilliant physicists of his generation, but also one of the last representatives of a great and noble tradition: that of the universal intellectual — someone capable of a nonspecialist's understanding of the discipline in which he excelled, and gifted with a comprehensive vision of the connections among the highest achievements of thought and inquiry.

We plan to hold a meeting in **Genova in the first half of 2027** to remember Carlo.

You are all warmly invited, and I hope to meet many of you there.

Chern-Simons paradigma for YM anomalies

- Gauge fields 1-forms $A = A^i\, T_i$ are bundled together with the ghost fields 0-forms $c = c^i\, T_i$ into generalized connections of total fermion number $+1$

$$\boldsymbol{A} = \boldsymbol{A}^i\, T_i = A + c$$

where T^i are the generators of the YM gauge algebra.

- BRST operator s and exterior differential d are put together into a generalized BRST operator δ acting on generalized forms

$$\delta = s + d$$

- Generalized curvatures of total fermion number $+2$

$$\boldsymbol{F} = \boldsymbol{F}^i\, T_i = \delta\, \boldsymbol{A} + \boldsymbol{A}^2$$

- $\boldsymbol{F}$ satisfies the Bianchi identity because of δ-nilpotency

$$\delta \boldsymbol{F} = -[\boldsymbol{A}, \boldsymbol{F}]$$

A slight change of perspective

- Consider the ring $P(\boldsymbol{A}, \boldsymbol{F})$ of polynomials freely generated by monomials of $\boldsymbol{A}$ and $\boldsymbol{F}$
- Define the action of δ on $P(\boldsymbol{A}, \boldsymbol{F})$ to be

$$\delta \boldsymbol{A} = \boldsymbol{F} - \boldsymbol{A}^2$$
$$\delta \boldsymbol{F} = -[\boldsymbol{A}, \boldsymbol{F}]$$

- δ defined above is nilpotent independently of the $\delta = s + d$ and $\boldsymbol{A} = A + c$ decompositions.

The topological ring

- Decompose $P(\boldsymbol{A}, \boldsymbol{F})$ into subspace of fixed total fermion number

$$P(\boldsymbol{A}, \boldsymbol{F}) = \bigoplus_{n \geq 0} P_n(\boldsymbol{A}, \boldsymbol{F})$$

- It is elementary to prove that δ-cohomology

$$H_\delta(P_n) = 0 \qquad \forall n > 0$$

- However the functional space relevant for YM anomalies is not the freely generated $P(\boldsymbol{A}, \boldsymbol{F})$.
- This is so because there are vanishing relations — i.e. constraints — among the monomials generators of the polynomial ring $P(\boldsymbol{A}, \boldsymbol{F})$.

The horizontal constraint of Yang-Mills

- For YM all constraints originate from horizontality of the generalized curvature

$$\boldsymbol{F} = F^{(2)}$$

  which encodes the action of the BRST operator s on fields c and A.
- Since ordinary forms of degree $n > d$ vanish, the horizontality of curvatures implies that ideal $\mathrm{W}^{\mathrm{YM}_d}$ of $P(\boldsymbol{A}, \boldsymbol{F})$ generated by homogenous curvature polynomials of degree $\frac{d+2}{2}$ vanishes for YM_d.

The quotient of the polynomial ring

- YM_d vanishing ideal $\mathrm{W}^{\mathrm{YM}_d}$ is δ-invariant, since δ acts linearly on curvatures.
- Hence YM_d anomalies of YM_d are classified by quotient cohomologies

$$H_\delta(P_{d+1}/W_{d+1}^{\mathrm{YM}_d})$$

where $W_{d+1}^{\mathrm{YM}_d}$ is the component of fermion degree $d+1$ of the vanishing ideal.

The cohomology of the quotient

- Since $H_\delta(P_n) = 0$, the δ-cohomology of the quotient is computed by the isomorphism

$$H_\delta(P_{d+1}/W_{d+1}) \cong H_\delta(W_{d+2})$$
$$p_{d+1} \to w_{d+2} = \delta\, p_{d+1}$$

- For YM_d,

$$W_n = 0 \quad \text{for} \;\; n < d+2$$

  since monomials of fermion degree $n < d$ do not have enough curvatures to vanish.

- Therefore, for YM_d

$$H_\delta(W_{d+2}) = \ker_\delta(W_{d+2})/\mathrm{img}_\delta(W_{d+1}) \cong \ker_\delta(W_{d+2})$$

Chern polynomials and anomalies

- Elements of $\ker_\delta(W_{d+2})$ are vanishing polynomials of the curvature which are left invariant by δ. Since δ acts as the adjoint on the curvatures, these are the gauge-invariant polynomials of the curvature of degree $\frac{d+2}{2}$
- This is the celebrated one-to-one correspondence between Yang-Mills anomalies in dimension d and Chern polynomials of degree $d+2$.

Two pictures for anomalies

- The isomorphism

$$H_\delta(P_{d+1}/W_{d+1}) \cong H_\delta(W_{d+2})$$

gives thus two equivalent "pictures" for anomalies:

- the degree $d+1$ "picture", as elements of the quotient P_{d+1}/W_{d+1}: these are the Chern-Simons polynomials.
- the $d+2$ "picture", as elements of fermion degree $d+2$ in the vanishing ideal W_{d+2}: the invariant Chern polynomials.

Richer vanishing ideals

- $d = 4$, $\mathcal{N} = 1$ (super)conformal anomalies fit this very same topological framework, with one major novelty: the constraint ideal for 4D (super)conformal gravity is not just determined by curvature horizontality conditions. It contains additional polynomials mixing curvatures and connections.
- This richer structure naturally explains the coexistence of both Chern-type — which correspond to the so-called a-type (super) conformal anomalies — and non-Chern-type, corresponding to c-type (super)conformal anomalies.

4D bosonic conformal gravity

- Bosonic conformal gravity is the gauge theory of the $d = 4$ $SO(4,2)$ conformal algebra.
- Generalized connections/curvatures are:

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{e}^a\, P_a + \boldsymbol{\omega}^{ab}\, J_{ab} + \boldsymbol{f}^a\, K_a + \boldsymbol{b}\, D \\ \boldsymbol{F} &= \boldsymbol{T}^a\, P_a + \tilde{\boldsymbol{R}}^{ab}\, J_{ab} + \tilde{\boldsymbol{T}}^a\, K_a + \tilde{\boldsymbol{F}}_W\, D \end{aligned}$$

where $\{P_a, J_{ab}, K_a, D\}$ are the generators of translations, Lorentz rotations, special conformal transformation and dilations, respectively.

(Conformal) gravity δ

- The relationship between the generalized δ acting on generalized forms and the BRST operator is modified in (conformal) gravity

$$\delta = \boldsymbol{s} + \mathcal{L}_\xi + \mathrm{d}$$

- This stems from the need to introduce an anticommuting ghost vector field ξ^μ for diffeomorphisms.

$$\begin{aligned} \boldsymbol{s}\,\phi &= -\mathcal{L}_\xi\,\phi + \text{other gauge transformations} \\ \boldsymbol{s}\,\xi^\mu &= -\tfrac{1}{2}\,\mathcal{L}_\xi\,\xi^\mu \end{aligned}$$

- To avoid redundancy, one must not introduce ghosts associated to translations P^a and impose the “horizontality” on the generalized vierbein connection:

$$\boldsymbol{e^a} = e^a$$

Conformal gravity δ

- The conformal δ so defined acts on generalized forms as dictated by the structure constants of $SO(4,2)$ conformal algebra — just as in YM:

$$
\begin{aligned}
\delta \boldsymbol{b} &= \tilde{\boldsymbol{F}}_W - 2\, \boldsymbol{e}^a\, \boldsymbol{f}_a \\
\delta \boldsymbol{e}^a &= \boldsymbol{T}^a - \boldsymbol{b}\, \boldsymbol{e}^a - \omega^{ab}\, \boldsymbol{e}_b \\
\delta \boldsymbol{f}^a &= \tilde{\boldsymbol{T}}^a + \boldsymbol{b}\, \boldsymbol{f}^a - \omega^{ab}\, \boldsymbol{f}_b \\
\delta \omega^{ab} &= \tilde{\boldsymbol{R}}^{ab} - \omega^{ac}\, \omega^{b}{}_{c} - 2\, \boldsymbol{e}^{[a}\, \boldsymbol{f}^{b]} \\
\delta \tilde{\boldsymbol{F}}_W &= -2\, \boldsymbol{e}^a\, \tilde{\boldsymbol{T}}_a + 2\, f^a\, \boldsymbol{T}_a \\
\delta \boldsymbol{T}^a &= \boldsymbol{e}^a\, \tilde{\boldsymbol{F}}_W + \tilde{\boldsymbol{R}}^{ab}\, \boldsymbol{e}_b - \boldsymbol{b}\, \boldsymbol{T}^a - \omega^{ab}\, \boldsymbol{T}_b \\
\delta \tilde{\boldsymbol{T}}^a &= -\boldsymbol{f}^a\, \tilde{\boldsymbol{F}}_W + \tilde{\boldsymbol{R}}^{ab}\, \boldsymbol{f}_b + \boldsymbol{b}\, \tilde{\boldsymbol{T}}^a - \omega^{ab}\, \tilde{\boldsymbol{T}}_b \\
\delta \tilde{\boldsymbol{R}}^{ab} &= \omega^{[ac}\, \tilde{\boldsymbol{R}}^{b]}{}_{c} - 2\, \boldsymbol{e}^{[a}\, \tilde{\boldsymbol{T}}^{b]} - 2\, \boldsymbol{f}^{[a}\, \boldsymbol{T}^{b]}
\end{aligned}
$$

Conformal constraints

- Horizontality of the vierbein implies horizontality of conformal generalized curvatures as in the Yang-Mills case.
- Hence the constraint ideal W^{conf} for 4D conformal gravity contains cubic homogeneous curvatures polynomials W^{YM_4}.
- However conformal gravity is characterized by additional constraints.
- One of them is the torsion constraint

$$\boldsymbol{T}^a = 0$$

expressing the spin connection in terms of the vierbein.
- Acting repeatedly with δ on the torsion constraint one obtains more constraints

$$\tilde{\boldsymbol{F}}_W = 0 \qquad \boldsymbol{e}_a \, \tilde{\boldsymbol{R}}^{ab} = 0 \quad \boldsymbol{e}_a \, \tilde{\boldsymbol{T}}^a = 0$$

- To describe the gravitational theory one more constraint is needed — the “Ricci constraints” — which eliminates $\boldsymbol{f}^a$ as an independent fields:

$$\epsilon_{abcd}\, \boldsymbol{e}^b\, \tilde{\boldsymbol{R}}^{cd} = 0$$

- δ-invariance implies more constraints

$$\epsilon_{abcd}\, \boldsymbol{e}^b\, \boldsymbol{e}^c\, \tilde{\boldsymbol{T}}^d = 0$$

- Summarizing, the full δ-invariant ideal conformal ideal W^{conf} is generated by

$$\begin{array}{lllll}
\text{Horizontality}: & \boldsymbol{F}^i \boldsymbol{F}^j \boldsymbol{F}^k & & & \\
\text{Torsionless}: & \boldsymbol{T}^a, & \tilde{\boldsymbol{F}}_W, & \boldsymbol{e}_a \tilde{\boldsymbol{T}}^a, & \boldsymbol{e}_a \tilde{\boldsymbol{R}}^{ab} \\
\text{“Ricci''}: & \epsilon_{abcd}\, \boldsymbol{e}^b \tilde{\boldsymbol{R}}^{cd}, & \epsilon_{abcd}\, \boldsymbol{e}^b \boldsymbol{e}^c \tilde{\boldsymbol{T}}^d & &
\end{array}$$

The conformal cohomology

- $H_\delta(W_6)$ is spanned by two classes having homogenous representatives:

$$W_6^{(A)} = W_{6,3}^{(A)} = -\frac{1}{16\,\pi^2}\,\epsilon_{abcd}\left(\tilde{\boldsymbol{F}}_W\,\tilde{\boldsymbol{R}}^{ab}\,\tilde{\boldsymbol{R}}^{cd} + 8\,\boldsymbol{T}^a\,\tilde{\boldsymbol{T}}^b\,\tilde{\boldsymbol{R}}^{cd}\right)$$

$$W_6^{(B)} = W_{6,4}^{(B)} = \frac{1}{2\,\pi^2}\,\epsilon_{abcd}\left(\boldsymbol{f}^d\,\boldsymbol{b}\,\tilde{\boldsymbol{R}}^{bc}\,\boldsymbol{T}^a + \boldsymbol{e}^d\,\boldsymbol{b}\,\tilde{\boldsymbol{R}}^{bc}\,\tilde{\boldsymbol{T}}^a + \right.$$
$$\left. + \boldsymbol{f}^c\,\boldsymbol{e}^d\,\tilde{\boldsymbol{F}}_W\,\tilde{\boldsymbol{R}}^{ab} + 4\,\boldsymbol{f}^c\,\boldsymbol{e}^d\,\boldsymbol{T}^a\,\tilde{\boldsymbol{T}}^b\right)$$

- $W_{6,3}^{(A)}$ contains only curvatures and thus it is an **invariant polynomial** of the curvatures — i.e. it is the (unique) invariant cubic Chern polynomial of the $SO(4,2)$ algebra.
- $W_{6,3}^{(A)}$ is the so-called a-conformal anomaly

The non-invariant anomaly and holography

- The anomaly described by $W_{6,4}^{(B)}$ does not admit an invariant representative of Chern-type: it is BRST invariant but not gauge-invariant.
- The so-called c-conformal anomaly is related to the A and B classes as follows

$$H_6^{(c)} = c\,(W_{6,4}^{(B)} - W_{6,3}^{(A)})$$

- This relation shows that the homogeneous $W_{6,4}^{(B)}$ describes the anomaly of conformal theories with $a = c$

$$c\,W_{6,4}^{(B)} = (H_6^{(a)} + H_6^{(c)})\big|_{a=c}$$

- It is intriguing to observe that (super)conformal theories with $a = c$ are precisely those which emerge in AdS_5 holography.

4D $\mathcal{N} = 1$ superconformal gravity

- $\mathcal{N} = 1$ 4D superconformal gravity is the gauge theory of the superconformal algebra $\mathfrak{su}(2,2|1)$, whose generalized connections and curvatures are

$$\boldsymbol{A}^i = \{\boldsymbol{e}^a,\ \boldsymbol{\omega}^{ab},\ \boldsymbol{f}^a,\ \boldsymbol{b},\ \boldsymbol{a},\ \psi^\alpha,\ \tilde{\psi}^\alpha\}$$
$$\boldsymbol{F}^i = \{\boldsymbol{T}^a,\ \tilde{\boldsymbol{R}}^{ab},\ \tilde{\boldsymbol{T}}^a,\ \tilde{\boldsymbol{F}}_W,\ \tilde{\boldsymbol{F}}_R,\ \boldsymbol{\rho}^\alpha,\ \tilde{\boldsymbol{\rho}}^\alpha\},$$

  corresponding to the (graded) generators $T_i = \{P_a,\ J_{ab}, K_a,\ D,\ R,\ Q_\alpha,\ S_\alpha\}$, relative to translations, Lorentz rotations, special conformal transformations, dilations, R-symmetry, supersymmetry and special conformal supersymmetry.

Supergravity δ

- The relationship between the generalized δ acting on generalized forms and the BRST operator is modified in supergravity:

$$\delta = \boldsymbol{s} + \mathcal{L}_\xi + \mathrm{d} - \boldsymbol{i}_\gamma$$

  where i_γ is the contraction of a form with the ghost number 2 commuting vector field γ^μ

$$\gamma^\mu \equiv \bar{\zeta}\, \Gamma^\mu\, \zeta$$

  and ζ is the commuting spinorial ghost associated to the supersymmetry generator Q_α.
- As in the bosonic case — to avoid redundancy — the generalized vierbein connection must be horizontal

$$\boldsymbol{e^a} = e^a$$

The superconformal δ acts on generalized forms as dictated by the structure constants of the superconformal Lie algebra — just as in YM:

$$
\begin{aligned}
\delta \boldsymbol{b} &= \tilde{\boldsymbol{F}}_W - 2\, \boldsymbol{e}^a\, \boldsymbol{f}_a + 2\, i\, \bar{\psi}\, \tilde{\psi} \\
\delta \boldsymbol{a} &= \tilde{\boldsymbol{F}}_R + 2\, \bar{\psi}\, \Gamma_5\, \tilde{\psi} \\
\delta \boldsymbol{e}^a &= \boldsymbol{T}^a - \boldsymbol{b}\, \boldsymbol{e}^a - \omega^{ab}\, \boldsymbol{e}_b - \bar{\psi}\, \Gamma^a\, \psi \\
\delta \boldsymbol{f}^a &= \tilde{\boldsymbol{T}}^a + \boldsymbol{b}\, \boldsymbol{f}^a - \omega^{ab}\, \boldsymbol{f}_b + \bar{\tilde{\psi}}\, \Gamma^a\, \tilde{\psi} \\
\delta \omega^{ab} &= \tilde{\boldsymbol{R}}^{ab} - \omega^{ac}\, \omega_c{}^b - 2 \boldsymbol{e}^{[a}\, \boldsymbol{f}^{b]} + 2\, i\, \bar{\psi}\, \Gamma^{ab}\, \tilde{\psi} \\
\delta \psi &= \rho - \frac{1}{2}\, \boldsymbol{b}\, \psi - i\, \boldsymbol{e}^a\, \Gamma_a\, \tilde{\psi} + \frac{3}{2} i\, \boldsymbol{a}\, \Gamma_5\, \psi - \frac{1}{4}\, \omega_{ab}\, \Gamma^{ab}\, \psi \\
\delta \tilde{\psi} &= \tilde{\rho} + \frac{1}{2}\, \boldsymbol{b}\, \tilde{\psi} + i\, \boldsymbol{f}^a\, \Gamma_a\, \psi - \frac{3}{2}\, i\, \boldsymbol{a}\, \Gamma_5\, \tilde{\psi} - \frac{1}{4}\, \omega_{ab}\, \Gamma^{ab}\, \tilde{\psi}
\end{aligned}
$$

The action of the superconformal δ

$$\delta \tilde{\boldsymbol{F}}_W = -2\, \boldsymbol{e}^a\, \tilde{\boldsymbol{T}}_a + 2\, \boldsymbol{f}^a\, \boldsymbol{T}_a - 2\, i\, \bar{\rho}\, \tilde{\psi} + 2\, i\, \bar{\tilde{\rho}}\, \psi$$

$$\delta \tilde{\boldsymbol{F}}_{\boldsymbol{R}} = -2\, \bar{\rho}\, \Gamma_5\, \tilde{\psi} + 2\, \bar{\tilde{\rho}}\, \Gamma_5\, \psi$$

$$\delta \boldsymbol{T}^a = \boldsymbol{e}^a\, \tilde{\boldsymbol{F}}_W + \tilde{\boldsymbol{R}}^{ab}\, \boldsymbol{e}_b - \boldsymbol{b}\, \boldsymbol{T}^a - \omega^{ab}\, \boldsymbol{T}_b + 2\, \bar{\rho}\, \Gamma^a\, \psi$$

$$\delta \tilde{\boldsymbol{T}}^a = -\boldsymbol{f}^a\, \tilde{\boldsymbol{F}}_W + \tilde{\boldsymbol{R}}^{ab}\, \boldsymbol{f}_b + \boldsymbol{b}\, \tilde{\boldsymbol{T}}^a - \omega^{ab}\, \tilde{\boldsymbol{T}}_b - 2\, \bar{\tilde{\rho}}\, \Gamma^a\, \tilde{\psi}$$

$$\delta \tilde{\boldsymbol{R}}^{ab} = -\omega^{[ac}\, \tilde{\boldsymbol{R}}_c{}^{b]} - 2\, \boldsymbol{e}^{[a}\, \tilde{\boldsymbol{T}}^{b]} - 2 \boldsymbol{f}^{[a}\, \boldsymbol{T}^{b]} - 2\, i\, \bar{\rho}\, \Gamma^{ab}\, \tilde{\psi} - 2\, i\, \bar{\tilde{\rho}}\, \Gamma^{ab}\, \tilde{\psi}$$

$$\delta \rho = -\frac{1}{2}\, \boldsymbol{b}\, \rho - i \boldsymbol{e}_a\, \Gamma^a\, \tilde{\rho} - \frac{1}{4}\, \omega_{ab}\, \Gamma^{ab}\, \rho + \frac{3}{2}\, i\, \boldsymbol{a}\, \Gamma_5\, \rho - \frac{3}{2}\, i\, \Gamma_5\, \psi\, \tilde{\boldsymbol{F}}_{\boldsymbol{R}} + \frac{1}{2}\, \psi\, \tilde{\boldsymbol{F}}$$
$$+ \frac{1}{4}\, \Gamma^{ab}\, \psi\, \tilde{\boldsymbol{R}}_{ab} + i\, \Gamma^a\, \tilde{\psi}\, \boldsymbol{T}_a$$

$$\delta \tilde{\rho} = \frac{1}{2}\, \boldsymbol{b}\, \tilde{\rho} + i\, \boldsymbol{f}_a\, \Gamma^a\, \rho - \frac{1}{4}\, \omega_{ab}\, \Gamma^{ab}\, \tilde{\rho} - \frac{3}{2}\, i\, \boldsymbol{a}\, \Gamma_5\, \tilde{\rho} + \frac{3}{2}\, i\, \Gamma_5\, \tilde{\psi}\, \tilde{\boldsymbol{F}}_{\boldsymbol{R}} - \frac{1}{2}\, \tilde{\psi}\, \tilde{\boldsymbol{F}}_W$$
$$+ \frac{1}{4}\, \Gamma^{ab}\, \tilde{\psi}\, \tilde{\boldsymbol{R}}_{ab} - i\, \Gamma^a\, \psi\, \tilde{\boldsymbol{T}}_a$$

The superconformal ideal

- Constraints of superconformal gravity are described in the literature as relations among *ordinary* curvatures of superconformal gravity. They express the connections $\omega^{ab}, f^a, \tilde{\psi}$ in terms of the "physical" connections e^a, b, ψ.
- Unlike the bosonic case, the superconformal generalized curvatures associated to composite fields

$$\tilde{\boldsymbol{R}}^{ab} \qquad \tilde{\boldsymbol{T}}^a \qquad \tilde{\boldsymbol{\rho}}$$

are not horizontal.

- "Miraculously" both the supergravity constraints and the non-horizontal components of the generalized curvatures are encoded in the vanishing of polynomials involving generalized curvatures and connections, generating the constraint ideal $W^{\text{super-conf}}$

$$\{\boldsymbol{T}^a,\ \boldsymbol{e}^a\,\Gamma_a\,\boldsymbol{\rho},\ \epsilon_{abcd}\,\boldsymbol{e}^b\,\tilde{\boldsymbol{R}}^{cd} + 2\,\boldsymbol{e}_a\,\tilde{\boldsymbol{F}}_R - 4\,i\,\bar{\boldsymbol{\psi}}\,\Gamma_a\,\Gamma_5\,\boldsymbol{\rho}\}$$

Superconformal cohomology

- $H_\delta(W^{\text{super-conf}})$ admits a 2-dimensional basis made of homogeneous representatives which are characterized analogously to the bosonic case.
- Despite non-horizontality of all curvatures, the unique $\mathfrak{su}(2,2|1)$ invariant cubic curvature polynomial sits in $W^{\text{super-conf}}$. It corresponds to the a-anomaly.
- A second cohomology class — the c-anomaly — emerges from a non-gauge-invariant but δ-closed polynomial.

- The **unique** gauge-invariant Chern anomaly polynomial of the $\mathfrak{su}(2,2|1)$ super Lie algebra

$$W^{(A)}_{6,3} = \frac{1}{\pi^2}\Bigg(\frac{5}{4}(\tilde{\boldsymbol{F}}_{\boldsymbol{R}})^3 + \frac{1}{4}(\tilde{\boldsymbol{F}}_{\boldsymbol{W}})^2\,\tilde{\boldsymbol{F}}_R - \frac{1}{16}\,\epsilon_{abcd}\,\tilde{\boldsymbol{R}}^{ab}\,\tilde{\boldsymbol{R}}^{cd}\,\tilde{\boldsymbol{F}}_{\boldsymbol{W}} +$$
$$-\frac{1}{8}\,\tilde{\boldsymbol{F}}_{\boldsymbol{R}}\,\mathrm{Tr}\tilde{\boldsymbol{R}}^2 - \frac{1}{2}\,\bar{\rho}\,\Gamma_{ab}\,\Gamma_5\,\tilde{\rho}\,\tilde{\boldsymbol{R}}^{ab} + 5\,i\,\bar{\rho}\,\tilde{\rho}\,\tilde{\boldsymbol{F}}_R + \bar{\rho}\,\Gamma_5\,\tilde{\rho}\,\tilde{\boldsymbol{F}}_{\boldsymbol{W}} +$$
$$+i\,\bar{\rho}\,\Gamma_a\,\Gamma_5\,\rho\,\tilde{\boldsymbol{T}}^{\boldsymbol{a}} + \tilde{\boldsymbol{F}}_R\,\boldsymbol{T}_a\,\tilde{\boldsymbol{T}}^a - \frac{1}{2}\,\epsilon_{abcd}\,\tilde{\boldsymbol{R}}^{cd}\,\boldsymbol{T}^a\,\tilde{\boldsymbol{T}}^b +$$
$$-i\,\bar{\tilde{\rho}}\,\Gamma^a\,\Gamma_5\,\tilde{\rho}\,\boldsymbol{T}_a\Bigg)$$

Superconformal non-Chern-like anomaly

- This is the non-Chern-like anomaly polynomial:

$$W_{6,4}^{(B)} = \frac{1}{\pi^2}\Big(-4\,\bar{\tilde{\rho}}\,\Gamma_{ab}\,\Gamma_5\,\rho\,\boldsymbol{f}^a\boldsymbol{e}^b + 2\,i\,\bar{\rho}\,\Gamma_a\,\Gamma_5\,\rho\,\boldsymbol{f}^a\,\boldsymbol{b} - 2\,i\,\bar{\tilde{\rho}}\,\Gamma_a\,\Gamma_5$$

$$-2\,i\,\bar{\tilde{\psi}}\,\rho\,\boldsymbol{b}\,\tilde{\boldsymbol{F}}_R + 2\,\bar{\tilde{\psi}}\,\Gamma_a\,\tilde{\rho}\,\boldsymbol{e}^a\,\tilde{\boldsymbol{F}}_R - 2\,\bar{\tilde{\psi}}\,\Gamma_5\,\rho\,\boldsymbol{a}\,\tilde{\boldsymbol{F}}_R +$$

$$-2\,i\,\bar{\tilde{\rho}}\,\psi\,\boldsymbol{b}\,\tilde{\boldsymbol{F}}_R - 4\,\bar{\rho}\,\Gamma_a\,\psi\,\boldsymbol{f}^a\,\tilde{\boldsymbol{F}}_R - 2\,\bar{\tilde{\rho}}\,\Gamma_5\,\psi\,\boldsymbol{a}\,\tilde{\boldsymbol{F}}_R +$$

$$-2\,\bar{\tilde{\psi}}\,\Gamma_5\,\psi\,\tilde{\boldsymbol{F}}_R^2 - 2\,i\,\bar{\tilde{\psi}}\,\rho\,\boldsymbol{a}\,\tilde{\boldsymbol{F}}_W + 2\,\bar{\tilde{\psi}}\,\Gamma_5\,\rho\,\boldsymbol{b}\,\tilde{\boldsymbol{F}}_W +$$

$$-2\,i\,\bar{\tilde{\psi}}\,\Gamma_a\,\Gamma_5\,\tilde{\rho}\,\boldsymbol{e}^a\,\tilde{\boldsymbol{F}}_W - 2\,i\,\bar{\tilde{\rho}}\,\psi\,\boldsymbol{a}\,\tilde{\boldsymbol{F}}_W + 2\,\bar{\tilde{\rho}}\,\Gamma_5\,\psi\,\boldsymbol{b}\,\tilde{\boldsymbol{F}}_W +$$

$$-2\,i\,\bar{\rho}\,\Gamma_a\,\Gamma_5\,\psi\,\boldsymbol{f}^a\,\tilde{\boldsymbol{F}}_W + \boldsymbol{f}^a\,\boldsymbol{e}_a\,\tilde{\boldsymbol{F}}_R\,\tilde{\boldsymbol{F}}_W - 4\,i\,\bar{\tilde{\psi}}\,\psi\,\tilde{\boldsymbol{F}}_R\,\tilde{\boldsymbol{F}}_W +$$

$$
\begin{aligned}
&+2\,\bar{\tilde{\psi}}\,\Gamma_5\,\psi\,\tilde{\boldsymbol{F}}_W^2+\frac{1}{2}\,\epsilon_{abcd}\,\boldsymbol{f}^c\boldsymbol{e}^d\,\tilde{\boldsymbol{F}}_W\,\tilde{\boldsymbol{R}}^{ab}+2\,i\,\bar{\tilde{\psi}}\,\Gamma_a\,\Gamma_5\,\tilde{\rho}\,\boldsymbol{b}\,\boldsymbol{T}^a+ \qquad (3)\\
&+\bar{\tilde{\psi}}\,\Gamma_a\,\tilde{\psi}\,\tilde{\boldsymbol{F}}_R\,\boldsymbol{T}^a+2\,\bar{\tilde{\psi}}\,\Gamma_{ab}\,\Gamma_5\,\rho\,\boldsymbol{f}^b\,\boldsymbol{T}^a-2\,\bar{\tilde{\rho}}\,\Gamma_{ab}\,\Gamma_5\,\psi\,\boldsymbol{f}^b\,\boldsymbol{T}^a+\\
&+\boldsymbol{f}_a\,\boldsymbol{b}\,\tilde{\boldsymbol{F}}_R\,\boldsymbol{T}^a-2\,\boldsymbol{f}_a\,\boldsymbol{a}\,\tilde{\boldsymbol{F}}_W\,\boldsymbol{T}^a+\frac{1}{2}\,\epsilon_{abcd}\,\boldsymbol{f}^d\,\boldsymbol{b}\,\tilde{\boldsymbol{R}}^{bc}\,\boldsymbol{T}^a+\\
&+2\,i\,\bar{\rho}\,\Gamma_a\,\Gamma_5\,\psi\,\boldsymbol{b}\,\tilde{\boldsymbol{T}}^a-2\,\bar{\psi}\,\Gamma_a\,\psi\,\tilde{\boldsymbol{F}}_R\,\tilde{\boldsymbol{T}}^a+2\,\bar{\tilde{\psi}}\,\Gamma_5\,\psi\,\boldsymbol{T}^a\,\tilde{\boldsymbol{T}}_a+\\
&+2\,\bar{\tilde{\psi}}\,\Gamma_5\,\rho\,\boldsymbol{e}_a\,\tilde{\boldsymbol{T}}^a+2\,\bar{\tilde{\psi}}\,\Gamma_{ab}\,\Gamma_5\,\rho\,\boldsymbol{e}^b\,\tilde{\boldsymbol{T}}^a+2\,\bar{\tilde{\rho}}\,\Gamma_5\,\psi\,\boldsymbol{e}_a\,\tilde{\boldsymbol{T}}^a+\\
&-2\,\bar{\tilde{\rho}}\,\Gamma_{ab}\,\Gamma_5\,\psi\,\boldsymbol{e}^b\,\tilde{\boldsymbol{T}}^a-\boldsymbol{e}_a\,\boldsymbol{b}\,\tilde{\boldsymbol{F}}_R\,\tilde{\boldsymbol{T}}^a-2\,\boldsymbol{a}\,\boldsymbol{e}_a\,\tilde{\boldsymbol{F}}_W\,\tilde{\boldsymbol{T}}^a+\\
&\left.+\frac{1}{2}\,\epsilon_{abcd}\,\boldsymbol{e}^d\,\boldsymbol{b}\,\tilde{\boldsymbol{R}}^{bc}\,\tilde{\boldsymbol{T}}^a+2\,\epsilon_{abcd}\,\boldsymbol{f}^c\,\boldsymbol{e}^d\,\boldsymbol{T}^a\,\tilde{\boldsymbol{T}}^b\right)
\end{aligned}
$$

- $W^{(B)}_{6,4}$ describes the anomaly of superconformal — “holographic” — theories with $a = c$

$$c\, W^{(B)}_{6,4} = (H^{(a)}_6 + H^{(c)}_6)\big|_{a=c}$$

- Just like in the bosonic case the superconformal so-called c-anomaly is a linear combination of the homogenous classes $W^{(A)}_{6,3}$ and $W^{(B)}_{6,4}$

$$W^{(c)}_6 = c\,(W^{(B)}_{6,4} - W^{(A)}_{6,3})$$

Anomaly densities

$$\sigma \mathcal{A}_W = -\frac{a}{16\pi^2}\,\sigma\, E_4 + \frac{c}{16\pi^2}\,\sigma\, W^{\alpha\beta\gamma\delta}\, W_{\alpha\beta\gamma\delta} - \frac{c}{6\pi^2}\,\sigma\, F_R^{\alpha\beta} F_{R\,\alpha\beta} + O(\psi^2$$

$$\alpha \mathcal{A}_R = \frac{(5a-3c)}{54\pi^2}\,\alpha\, \epsilon_{\alpha\beta\gamma\delta}\, F_R^{\alpha\beta} F_R^{\gamma\delta} - \frac{(a-c)}{48\pi^2}\,\alpha\, \epsilon_{\alpha\beta\gamma\delta}\, W^{\alpha\beta\lambda\mu}\, W^{\gamma\delta}{}_{\lambda\mu}$$

$$\bar\eta \mathcal{A}_S = -\frac{i(5a-3c)}{24\pi^2}\,\bar\eta\, \epsilon_{\alpha\beta\gamma\delta}\, D^{[\alpha}\,\psi^{\beta]}\, F_R^{\gamma\delta} + \frac{c}{12\pi^2}\,\bar\eta\, \Gamma_5\, D_{[\alpha}\,\psi_{\beta]}\,(\delta_\mu{}^\alpha \delta_\nu{}^\beta +$$

$$- \frac{ia}{4\pi^2}\,\bar\eta\, \Gamma_{\alpha\gamma}\, D^{[\beta}\,\psi^{\gamma]}\, S^\alpha{}_\beta + \frac{ia}{48\pi^2}\,\bar\eta\, \Gamma_{\alpha\beta}\, D^{[\alpha}\,\psi^{\beta]}\, R + \frac{5a-3c}{18\pi^2}\,\bar\eta\, \epsilon_{\alpha\beta\gamma}$$

$$+ \frac{i\,(a-c)}{24\pi^2}\,\bar\eta\, \epsilon_{\alpha\beta\lambda\mu}\, \Gamma^{\gamma\delta}\,\psi^\beta\, a^\alpha\, R^{\lambda\mu}{}_{\gamma\delta} + \frac{i(a-c)}{16\pi^2}\,\bar\eta\, \Gamma^{\gamma\delta}\, D^{[\alpha}\,\psi^{\beta]}\, W_{\gamma\delta\alpha\beta}$$

$$\bar\zeta \mathcal{A}_Q = -\frac{(5a-3c)}{108\pi^2}\,\bar\zeta\, \Gamma_\alpha\, D_{[\beta}\,\psi_{\gamma]}\, a^\alpha F_R^{\beta\gamma} + \frac{(5a-3c)}{54\pi^2}\,\bar\zeta\, \Gamma_\beta\, D_{[\alpha}\,\psi_{\gamma]}\, a^\alpha F_R^{\beta\gamma}$$

$$+ \frac{i(5a-3c)}{108\pi^2}\,\bar\zeta\, \epsilon_{\beta\gamma\delta\lambda}\, \Gamma_\alpha\, \Gamma_5\, D^{[\beta}\,\psi^{\gamma]}\, a^\alpha F_R^{\delta\lambda} - \frac{i(5a-3c)}{54\pi^2}\,\bar\zeta\, \epsilon_{\alpha\gamma\delta\lambda}\, \Gamma_\beta\, \Gamma_5$$

$$- \frac{(a-c)}{12\pi^2}\,\bar\zeta\, \epsilon_{\alpha\gamma\delta\lambda}\, \Gamma^\beta\, D^{[\delta}\,\psi^{\lambda]}\, a^\alpha\, S_\beta{}^\gamma + \frac{(a-c)}{24\pi^2}\,\bar\zeta\, \epsilon_{\gamma\delta\lambda\mu}\, \Gamma_\beta\, D^{[\gamma}\,\psi^{\delta]}\, a_\alpha$$

Summary

- The Chern-Simons paradigm can be enlarged to describe all 4D $\mathcal{N} = 1$ (super)conformal anomalies, encompassing both a-type and c-type anomalies.
- All 4D anomalies — Yang–Mills as well as (super)conformal — are captured by the cohomology $H_\delta(W_6)$ of the vanishing ideal sitting inside the topological ring $P(\boldsymbol{A}, \boldsymbol{F})$ of the gauge (super)algebra.

- In 4D Yang–Mills theories, the constraint ideal is generated by cubic curvature monomials; as a result, $H_\delta(W_6)$ reduces to the classical invariant Chern polynomials.
- In (super)conformal gravity, the enlarged constraint ideal includes polynomials in both curvatures and connections.
- The a-anomaly arises from the unique invariant Chern polynomial of the (super)conformal algebra, while the c-anomaly originates from BRST-closed, non-gauge-invariant polynomials mixing curvatures and connections.

Outlook

- In Yang–Mills theory, the description of anomalies in terms of degree- $d+1$ Chern-Simons polynomials fits naturally into the holographic picture, where generalized connections uplift to ordinary connections in $d+1$ dimensions.
- By contrast, the current holographic understanding of Weyl and supersymmetric anomalies relies on structurally different — regularization dependent and non-topological — mechanisms.
- Our description of 4D superconformal anomalies via generalized Chern-Simons polynomials suggests that a unified, topological and regularization independent holographic framework for all kind of anomalies might exist.

Outlook (continued)

- Moreover, in our framework, the $\mathrm{AdS}_5/\mathrm{CFT}_4$ relation $a = c$ is naturally linked to the special homogeneous polynomial $W_{6,4}^{(B)}$. This raises the possibility that the holographic $a = c$ relation may have a purely BRST cohomological explanation.
- From a mathematical viewpoint, it is natural to ask about the algebraic-topological meaning of the generalized Chern classes introduced here.