QMA-hardness of Consistency of local density matrices with applications to quantum ZK

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joint work with Anne Broadbent (U. of Ottawa)

arxiv:1911.07782

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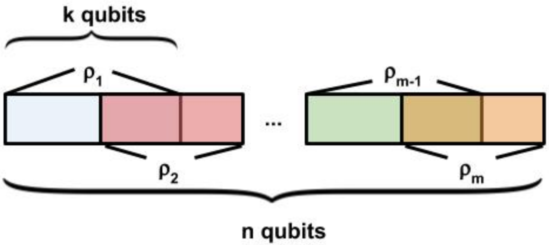
Output: yes: $\exists \psi$ such that $\forall i : \left\| Tr_{\overline{S_i}}(\psi) - \rho_i \right\| \leq \frac{1}{exp(n)}$

no: $\forall \psi, \exists i : \left\| Tr_{\overline{S_i}}(\psi) - \rho_i \right\| \geq \frac{1}{poly(n)}$



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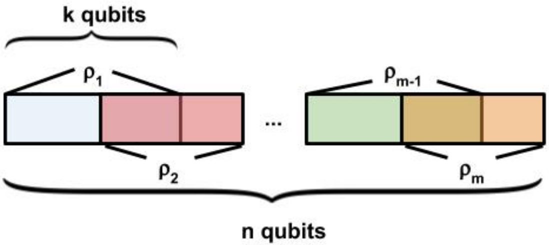

How hard is this problem?

Complexity theory background

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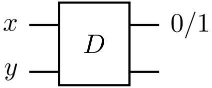
Problem $L \in \text{NP}$



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 $\exists y \; D(x, y) = 1$
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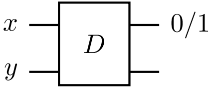

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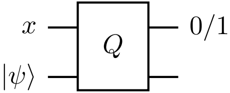

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for $x \in L_{yes}$,
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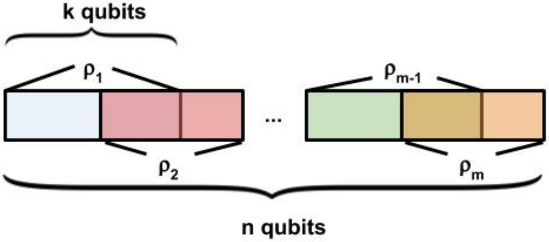

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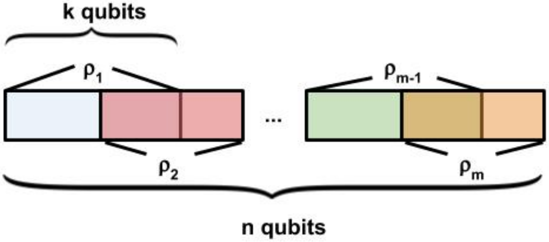

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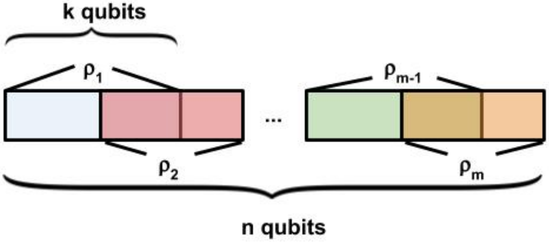

- Liu'06: containment in QMA, and partial result on QMA-hardness
- Our work:
  - QMA-hardness of CLDM
  - Applications to complexity theory and quantum cryptography

Simulatable codes - warm up

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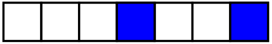

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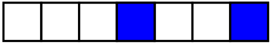

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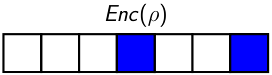

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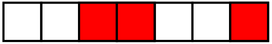

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- Not true anymore for $i, j, k \in [7]$

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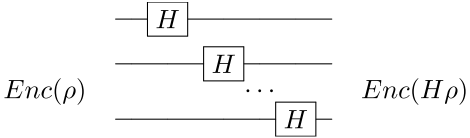

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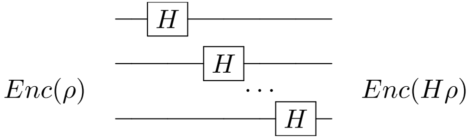

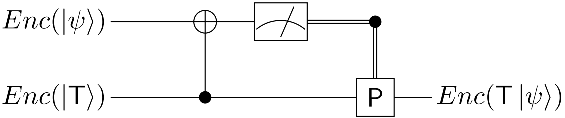

CLDM is QMA-hard

Circuit-to-hamiltonian construction

Given a circuit $V = U_T ... U_1$ and initial state $|\psi_{init}\rangle = |\phi\rangle |0^a\rangle$, there is a reduction to a 5-Local Hamiltonian H_V such that

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$$\frac{1}{\sqrt{T+1}} \sum_{t \in [T+1]} |t\rangle \otimes U_t \dots U_1 |\psi_{init}\rangle$$

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Goal

Tweak the verification algorithm such that we can compute the reduced density matrices of history states.

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Encoded circuit

Instead of $V = U_T ... U_1$ and proof $|\phi\rangle$, we use the following circuit V':

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Theorem

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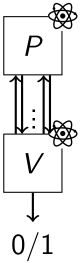
3. There is a polynomial-time algorithm that computes the density matrices of the history state
   - Most of clock qubits are traced-out, so the remaining state is a mixture of intervals

Quantum Zero-knowledge for L



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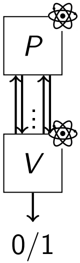

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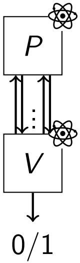

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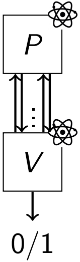

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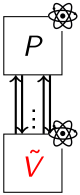

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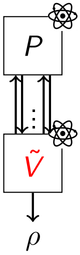

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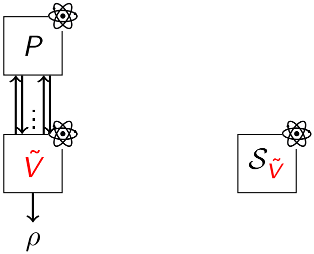

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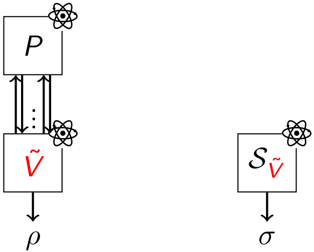

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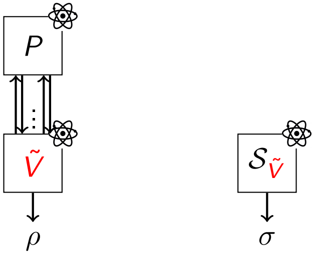

Quantum computational zero-knowledge

ρ and σ cannot be **efficiently** distinguished:

$$\forall \text{ quantum poly-time } \mathcal{A} : |Pr[\mathcal{A}(\rho) = 1] - Pr[\mathcal{A}(\sigma) = 1]| \leq negl(n)$$

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- Assuming qOWF: QMA $\subseteq$ QZK since PSPACE $=$ CZK $\subseteq$ QZK
  - Need to go through QMA $\subseteq$ PP
  - Desired: Efficient prover with QMA witness
- BJSW'16: QMA $\subseteq$ QZK with efficient prover
  - Multiple rounds of communication
  - Somewhat complicated

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Input: $\rho_1, \ldots, \rho_m$

1. Prover sends $\psi^{\otimes \ell}$, where ψ is consistent with all ρ_i
2. Verifier picks i and random k-qubit Pauli P
3. Verifier measures the qubits corresponding to ρ_i on each (supposed) copy of ψ
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- Soundness
  - Prover sends state σ
  - Let ψ_j be the register that should be the j-th copy of ψ
  - Let $\tilde{\psi} = \frac{1}{\ell} \sum_j \psi_j$
  - Expected value of the outcomes is $Tr(P\tilde{\psi})$ + Hoeffding's inequality

Very simple ZK proof for QMA

P V

$\rho_1, \ldots, \rho_m$

Very simple ZK proof for QMA

P	V
$\psi^{\otimes \ell}$	$\rho_1, \ldots, \rho_m$

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P	V
$X^a Z^b \psi^{\otimes \ell} Z^b X^a$	$\rho_1, \ldots, \rho_m$
a_1, b_1	
a_2, b_2	
$\ldots$	
a_{n-1}, b_{n-1}	
a_n, b_n	

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Very simple ZK proof for QMA

P

$a_1, b_1 \to 564651$
$a_2, b_2 \to 984565$
$\dots$
$a_n, b_n \to 894102$

V

$\rho_1, \dots, \rho_m$

$X^a Z^b \psi^{\otimes \ell} X^a Z^b$

...

Very simple ZK proof for QMA

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$\xleftarrow{\quad i \quad}$

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$\longrightarrow$ 984565, 894102
keys to open otp of copies of ρ_i

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...

a_n, b_n

Completeness ✓ Soundness ✓ ZK ✓

Open questions

- Complexity of CLDM with density matrices of size $\{2, 3, 4\}$
- Complexity of approximation of CLDM
- QNIZK protocol for QMA in the CRS model
- More efficient Proof of Quantum knowledge protocols

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Thank you for your attention!