Bipartite energy-time uncertainty relation for quantum metrology with noise

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Technical version attached.

- **Introduction.** In quantum metrology, one infers a parameter that is encoded in the quantum state by applying suitable measurements. The fact that one cannot read out perfect time information from a quantum clock, for instance, is a manifestation of an energy-time uncertainty relation [1–4]. Quantum entanglement between multiple probe systems enables a quadratic improvement (the *Heisenberg limit*) over using independent probes [5]. Several innovative ideas allow to counteract the negative effect of the noise on the entanglement of the probes [6–9], although there are general sensitivity limits that apply to metrology with noise [10].

A promising approach is to protect the quantum clock from the noise by using a quantum error correcting code. This is a potentially difficult task, because quantum error correction of a clock requires a code that is time covariant [11] but such codes tend to perform poorly at correcting errors [11–13]. Nevertheless, it was shown that techniques from quantum error correction could be leveraged to recover the Heisenberg scaling in the presence of noise under suitable conditions [14–19]. Our work aims at providing a broad and robust set of tools to determine and optimize the sensitivity of probe states exposed to noise in nonasymptotic regimes.

Our main results are: **(i)** A new type of time-energy uncertainty relation whereby an observer's sensitivity to the time stored in a quantum clock trades off with the environment's ability to sense the energy of the clock; **(ii)** Necessary and sufficient conditions for zero sensitivity loss, which are a weaker form of the Knill-Laflamme quantum error correction conditions [20]; **(iii)** New upper bounds on the Fisher information via our uncertainty relation; and **(iv)** The construction of noise-resilient many-body strongly interacting probe states, whose first-order resilience is verified numerically.

- **Setting.** Alice prepares a noiseless quantum clock in a state $|\psi(t)\rangle$ evolving under the Hamiltonian H. The quantum clock is sent to Bob through a noisy channel $\mathcal{N}_{A\to B}$ (see Fig. 1). Bob's task is to infer the time t from $\rho_B(t)$. In the paradigm of local parameter estimation, we only require measurements to estimate the value of the time parameter in a neighborhood of a given value t. The precision to which one can estimate t of a one-parameter family of states $\rho(t)$ is given by the Cramér-Rao bound $\langle(\delta t)^2\rangle \geqslant [F(t)]^{-1}$, where $\langle(\delta t)^2\rangle$ is the time estimator's average mean squared deviation from the true parameter and $F(t)$ is the *quantum Fisher information*, defined as

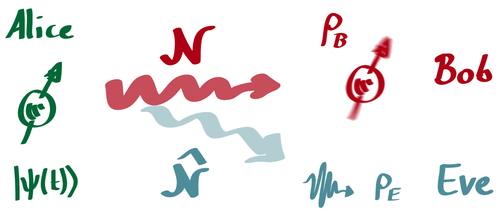

Fig. 1: A noiseless clock possessed by Alice is sent to Bob over a noisy channel $\mathcal{N}$. Bob's sensitivity to time trades off exactly with Eve's sensitivity to a parameter that encodes the energy of $|\psi\rangle$. Eve's state ρ_E is obtained via the complementary channel $\widehat{\mathcal{N}}$.

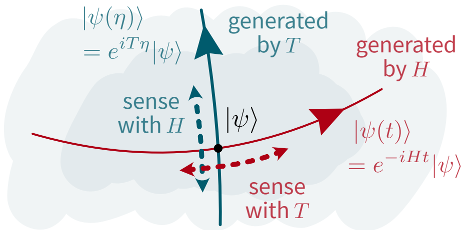

Fig. 2: The "local time observable" T is the observable that optimally distinguishes $|\psi(t)\rangle$ from $|\psi(t+dt)\rangle$ in the neighborhood of t, saturating the Cramér-Rao bound. The complementary parameter η is generated by T, and is the evolution for which the Hamiltonian H saturates the Cramér-Rao bound.

$F(t) = \mathrm{tr}\big(\rho(t)\, L^2\big)$ where L is a solution to $\rho L + L\rho = 2d\rho/dt$. In Alice's case of a pure, noiseless clock evolving according to H, one has $F_{\mathrm{A}}(t) = 4\sigma_H^2$ where $\sigma_H := \langle H^2\rangle_\psi - \langle H\rangle_\psi^2$. A "local time observable" T is one that saturates the Cramér-Rao bound [2], optimally distinguishing $\rho(t)$ from $\rho(t+dt)$. Complementary to the time evolution $|\psi(t)\rangle = e^{-itH}|\psi\rangle$ we define $|\psi(\eta)\rangle = e^{i\eta T}|\psi\rangle$. It turns out that the sensing observable that optimally distinguishes $|\psi(\eta)\rangle$ from $|\psi(\eta+d\eta)\rangle$ is the Hamiltonian H itself, and therefore η represents the energy of the probe (Fig. 2).

- **Main Results.** We relate Bob's sensitivity to the time t with the sensitivity with which the environment Eve, as described by the output of the complementary channel $\widehat{\mathcal{N}}$, can sense a parameter η that reveals the energy of the noiseless probe state:

Theorem 1 (Bipartite energy-time uncertainty relation): *In the setup of Fig. 1, we have*

$$\frac{F_{\mathrm{Bob}}(t)}{F_{\mathrm{Alice}}(t)} + \frac{F_{\mathrm{Eve}}(\eta)}{F_{\mathrm{Alice}}(\eta)} = 1\ . \tag{1}$$

The proof proceeds by writing $F_{\mathrm{Bob}}(t)$ in terms of the Bures metric, considering the purifying space (Eve), rewriting the resulting expression as a semidefinite program as in Refs. [21, 22], and recasting that program as another semidefinite optimization whose optimal value is $F_{\mathrm{Eve}}(\eta)$.

Theorem 2 (Necessary and sufficient conditions for zero sensitivity loss): *Let $|\psi\rangle$ be the probe state vector and H be the Hamiltonian, and let $|\xi\rangle = P_\psi^\perp H|\psi\rangle$ where $P_\psi^\perp = \mathbb{1} - |\psi\rangle\langle\psi|$. Let $\mathcal{N}(\cdot) = \sum E_k(\cdot)E_k^\dagger$ be the noise channel. Consider a logical qubit spanned by $|+\rangle_L = |\psi\rangle$ and $|-\rangle_L = |\xi\rangle/\||\xi\rangle\|$, along with logical Pauli operators $X_L = |+\rangle\langle+| - |-\rangle\langle-|$, $Z_L = |-\rangle\langle+| + |+\rangle\langle-|$, and let $\Pi = |+\rangle\langle+| + |-\rangle\langle-|$. Then*

$$\begin{aligned} &F_{\mathrm{Eve}}(\eta) = 0 \quad \Leftrightarrow \quad \widehat{\mathcal{N}}(Z_L) = 0 \\ &\quad \Leftrightarrow \quad \langle\psi|\, E_{k'}^\dagger E_k\, |\xi\rangle + \langle\xi|\, E_{k'}^\dagger E_k\, |\psi\rangle = 0 \ \ \forall\, k,k' \quad \Leftrightarrow \quad \mathrm{tr}\big(Z_L \Pi E_{k'}^\dagger E_k \Pi\big) = 0 \ \ \forall\, k,k'\ . \end{aligned} \tag{2}$$

For Π to define a code space, the Knill-Laflamme conditions [20] would require $\Pi E_{k'}^\dagger E_k \Pi \propto \Pi$; the conditions in Eqs. (2) are a weaker version thereof, considering effectively only their projection onto Z_L. Our relation in Eq. (1) can be extended to any two parameters that Bob and Eve might wish to respectively measure. In this case the trade-off is quantified by the commutator of the associated parameter generators.

Theorem 3 (General two-parameter uncertainty relation): *Let H, Z be two Hermitian operators and consider the associated parameters $d\psi/dt = -i[H,\psi]$ and $d\psi/dz = -i[Z,\psi]$. Consider the setup in Fig. 1. Then*

$$\frac{F_{\mathrm{Bob}}(t)}{F_{\mathrm{Alice}}(t)} + \frac{F_{\mathrm{Eve}}(z)}{F_{\mathrm{Alice}}(z)} \leqslant 1 + 2\sqrt{1 - \frac{\big\langle i[H,Z]\big\rangle^2}{4\sigma_H^2\sigma_Z^2}}\ . \tag{3}$$

The proof of Eq. (3) extends to infinite-dimensional spaces and to unbounded operators H, Z such as position and momentum. The right hand side of Eq. (3) reduces to one, thus matching (1), whenever H, Z saturate the Robertson relation $\sigma_H^2 \sigma_Z^2 \geqslant \langle i[H,Z]\rangle^2/4$.

If Eve further processes her output into a system Eve′, the data processing inequality for the Fisher information [23] along with Eq. (1) ensures that $F_{\mathrm{Bob}}(t)/F_{\mathrm{Alice}}(t) \leqslant 1 - F_{\mathrm{Eve}'}(\eta)/F_{\mathrm{Alice}}(\eta)$. We obtain new general upper bounds to the Fisher information, of which relatively few are known [10, 21, 22]. E.g., for an independent and identically distributed (i.i.d.) noise channel, we obtain:

Theorem 4 (Upper bound on the Fisher information): *Consider i.i.d. noise with two Kraus operators E_0, E_1. For any k,*

$$F_{\mathrm{Bob}}(t) \;\leqslant\; 4\,\sigma_H^2 \;-\; 4 \sum_{\boldsymbol{x}:\ |\boldsymbol{x}|\leqslant k} \frac{\left[2\,\mathrm{Re}\langle\psi|\bar{H} E_{\boldsymbol{x}}^\dagger E_{\boldsymbol{x}}|\psi\rangle\right]^2}{\langle\psi|E_{\boldsymbol{x}}^\dagger E_{\boldsymbol{x}}|\psi\rangle}\;; \quad \text{with } E_{\boldsymbol{x}} = E_{x_1}\otimes\cdots\otimes E_{x_n}\;. \tag{4}$$

The sum in Eq. (4) ranges over bit-strings $\boldsymbol{x}$ of maximal Hamming weight $|\boldsymbol{x}| \leqslant k$. There are a polynomial number of terms and the individual terms can be computed efficiently given an efficient description of $|\psi\rangle$. Eq. (4) can be generalized to multiple Kraus operators.

Starting from any time-covariant quantum error-correcting code, one can construct a probe state in the logical subspace that evolves nontrivially in time and loses no sensitivity under the action of the noise. Exploiting this fact leads to metrological schemes for metrology with many-body interacting Hamiltonians:

Theorem 5 (Code state for many-body metrology): *Consider any graph with vertices representing qubits, and let the Hamiltonian H contain an Ising or a Heisenberg interaction for each graph edge. Let c^n be a assignment of bits for each vertex that violates c of the Hamiltonian ZZ terms, and which is such that the bit-strings $0\,0\,\ldots\,0$, $1\,1\,\ldots\,1$, c^n all differ on at least four sites. Let $|\psi\rangle = \frac{1}{2}\left[|0\,0\,\ldots\,0\rangle + |1\,1\,\ldots\,1\rangle + |c^n\rangle + |\overline{c^n}\rangle\right]$. Then the state vector $|\psi\rangle$ loses no sensitivity under any single localized error, and $F_{\mathrm{Bob}}(t) = 4\sigma_H^2 \geqslant \Omega(c^2)$.*

Finally, our results can be applied to strongly interacting probe states subject to amplitude damping noise. We compute the time sensitivity of a strongly interacting 1D spin chain with Ising interactions for different probe states (Fig. 3).

- **Discussion.** Our relation Eq. (1) offers a new paradigm of uncertainty relations for the Fisher information, where the two physical quantities in question are accessed by complementary observers. This setting mirrors analogous results for entropic uncertainty relations [4, 24, 25]. Our results furthermore complement existing studies of quantum error correction in metrology [17–19, 26, 27] by offering quantitative expressions for the sensitivity in nonasymptotic regimes for realistic noise models and strongly interacting probes, and by accounting for imperfect error correction.

Ultra-precise quantum atomic clocks [28] has been one of the major successes of precise coherent control of isolated quantum systems. As quantum clocks and metrology probes evolve into the interacting many-body regime [29–31], we anticipate exciting possibilities for the use of our results and other quantum error-correction inspired schemes for metrology applications.

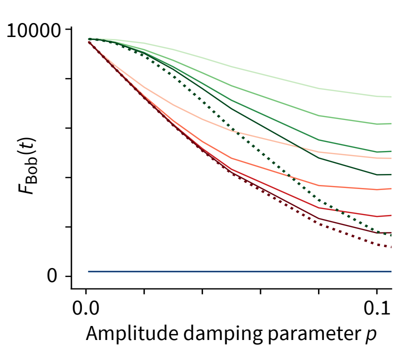

Fig. 3: Numerical calculation of the sensitivity of a 1D chain of $n = 50$ Ising spins as a function of amplitude-damping parameter p. Depicted in red is the state vector $|\psi\rangle = \left(|0\,0\,0\,0\ldots\rangle + |0\,1\,0\,1\ldots\rangle\right)/\sqrt{2}$: solid lines are Eq. (4) for varying $k = 1,\ldots,4$ and the dotted line is a lower bound. The same calculation is depicted in green for the code state of Theorem 5. This state does not appear to lose sensitivity to first order in p. The blue line is the sensitivity achieved by a (nonentangled) spin-coherent state.

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