Multivariate Trace Inequalities, Recovery, and p-Fidelity Beyond Tracial Settings

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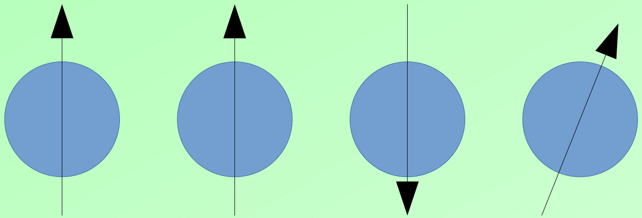
Thanks to the QIP organizers, and the audience

Von Neumann Algebras (type I, finite dimension)

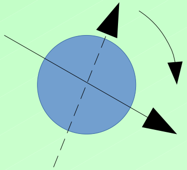
Multi-qubit system



States in Hilbert space: $|\psi\rangle \in \mathcal{H} = \mathcal{H}_2^{\otimes n}$

Inner Product: $\langle \psi | \phi \rangle$

Transformations



Unitary Matrices

$+$ projectors, densities, etc.

Observables

Hermitian Matrices

Bounded Operators

$\mathbb{B}(\mathcal{H})$

Densities ρ, ω, η

Operators X, Y

Von Neumann Algebras (type I)

Type I_d: bounded operators or block diagonal subalgebras on d-dimensional Hilbert space (matrices)

Trace denoted $tr(X)$ = sum of eigenvalues of X

Let $\hat{1}$ denote identity matrix

$tr \sim \hat{1}$ as an element of the dual space

Cyclic property: $tr(XY) = tr(YX)$

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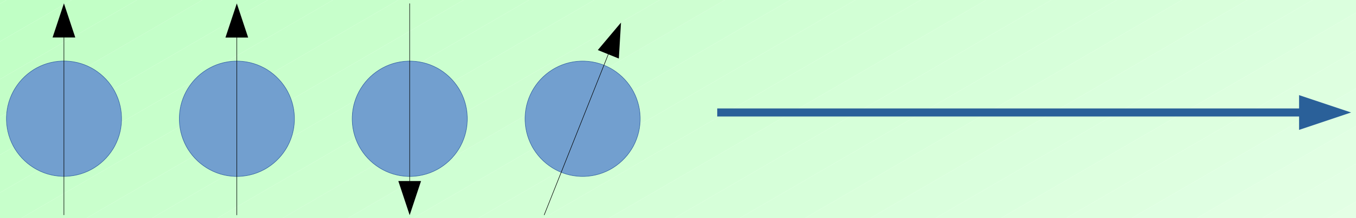
Cyclic property: $tr(XY) = tr(YX)$

Type I_∞: bounded ops on d-dim Hilbert space for $d \to \infty$

$tr(\hat{1}) = \infty$, tr is semifinite (not always infinite)

What happens with infinite qubits?



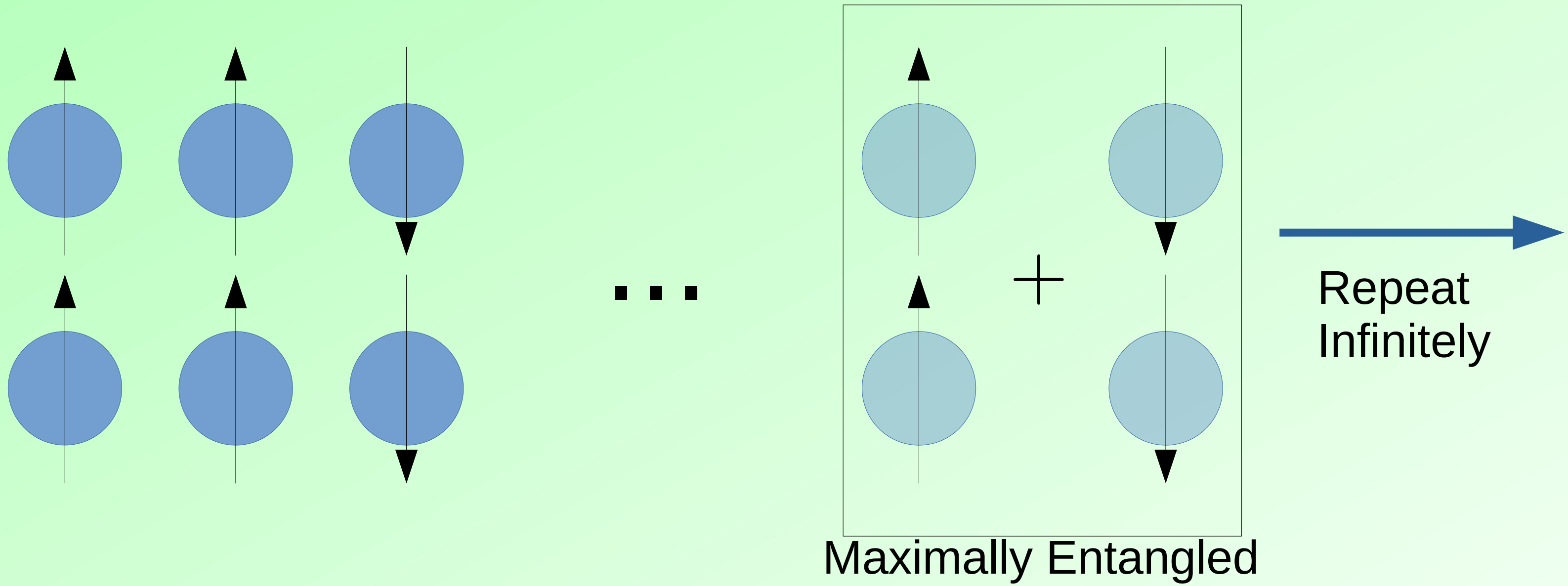
$$\mathcal{H} = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \dots$$

Naively, leads to a troublesomely infinite vector space

Less naively, it leads to higher von Neumann algebra types

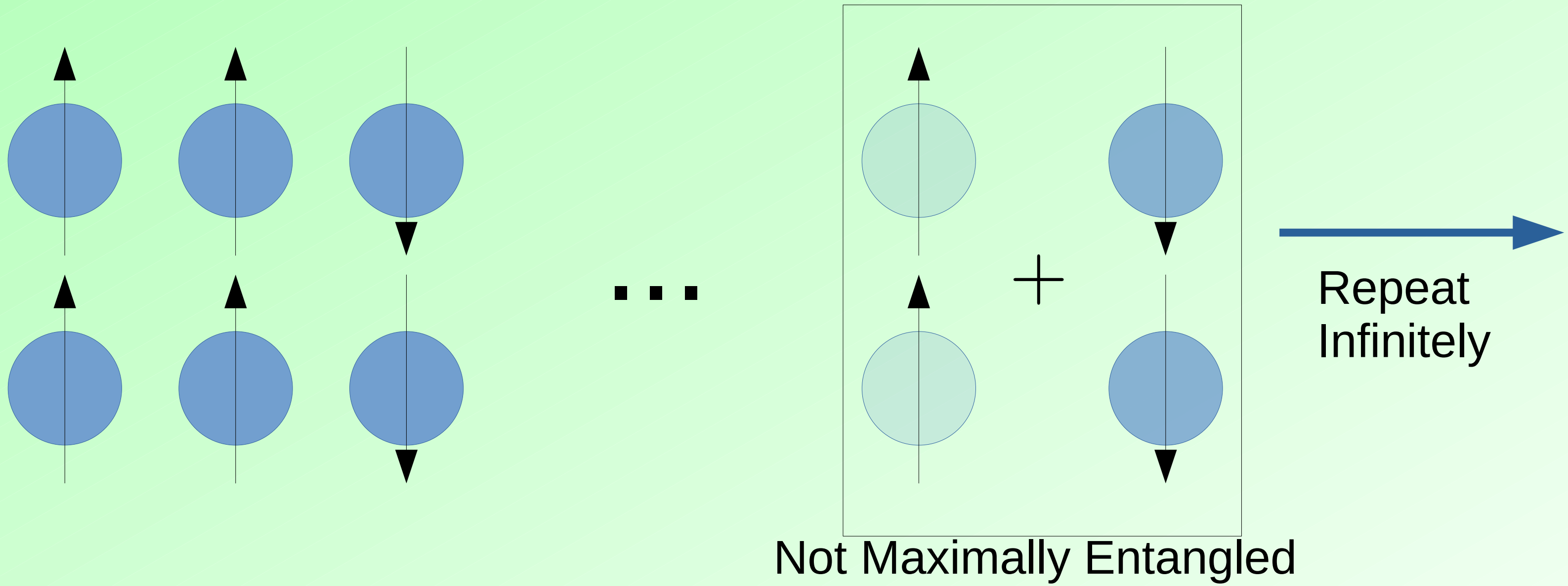
See Witten's 2018 Review Article, Araki-Woods 1968

Hyperfinite II_1



Take operators on respective chains as $\mathcal{M}$ & $\mathcal{M}'$, use truncated inner products

Hyperfinite $\cancel{\mathrm{II}_1}$ III_λ



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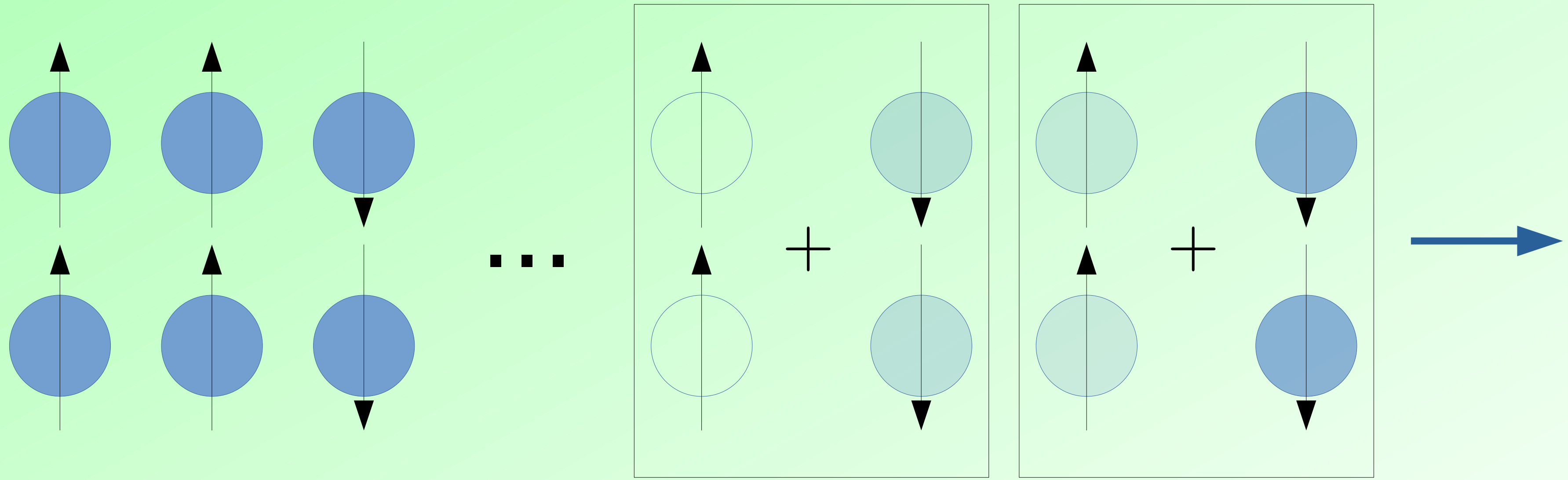
Hyperfinite ~~II_1~~ III_1



"Tail" state cycles between states

Take operators on respective chains as $\mathcal{M}$ & $\mathcal{M}'$, use truncated inner products

Many types of infinite-dimensional vN algebras

Type II_1: finite trace, $tr(\hat{1}) = 1$

Type II_∞: semifinite (not always infinite) trace

Type III_0

Type III_λ for $\lambda \in (0,1)$ has potential thermal analogies

Type III_1 arises in quantum field theory

►No (semifinite) trace

Relative Entropy in General von Neumann Algebras

Let $\mathcal{M}$ be a von Neumann algebra with normal, faithful state ω

$$S_{\rho|\omega} \text{ is given by } S_{\rho|\omega} X \left|\omega\right\rangle = X^\dagger \left|\rho\right\rangle$$

$$S_{\rho|\omega} = J_{\rho|\omega} \Delta_{\rho|\omega}^{1/2} \text{ (polar decomposition)}$$

relative Tomita operator

relative modular operator

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$$\textbf{Relative Entropy: } D(\rho\|\omega) = tr(\rho(\ln\rho - \ln\omega)) \qquad \text{(when there is a trace)}$$

$$= -\left\langle\rho\right| \ln \Delta_{\rho|\omega} \left|\rho\right\rangle$$

$$\textbf{Von Neumann Entropy: } H(\rho) = -D(\rho\|\hat{1})$$

$$\textbf{Subsystem Entropy: } H(A)_\rho = H(\rho^A) \text{ for } \rho^{AB\dots}$$

Strong Subadditivity (Lieb & Ruskai 1973)

$$H(AC)_\rho + H(BC)_\rho - H(ABC)_\rho - H(C)_\rho \geq 0$$

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Holographic Version from Headrick & Takayanagi, 2007

Minimal Surfaces for AC & BC are... ...non-minimal for ABC, C.

High energy use of entropy inequalities motivate recovery maps in type III

Strong Subadditivity (Lieb & Ruskai 1973)

$$H(AC)_\rho + H(BC)_\rho - H(ABC)_\rho - H(C)_\rho \geq 0$$

$$\leftrightarrow \text{ Data Processing: } D(\rho \| \eta) - D(\Phi(\rho) \| \Phi(\eta)) \geq 0$$

for densities ρ, η, quantum channel (open physical process) Φ

Recall Data Processing, Recovery

(Lindblad 1975), (Petz 1986 & 1988), (Fawzi & Renner 2015), (Wilde 2015), (Junge et al. 2018), (Carlen & Vershynina 2020)...

$$\text{Petz Map: } R_{\omega,\Phi}(\eta) = \omega^{1/2}\Phi^\dagger(\Phi(\omega)^{-1/2}\eta\Phi(\omega)^{-1/2})\omega^{1/2}$$

$$\boxed{D(\rho\|\omega) = D(\Phi(\rho)\|\Phi(\omega)) \implies \rho = R_{\omega,\Phi}\circ\Phi(\rho)}$$

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$$\text{Rotated: } R^{it}_{\omega,\Phi}(\eta) = \omega^{-it}R_{\omega,\Phi}(\Phi(\omega)^{it}\eta\Phi(\omega)^{-it})\omega^{it}$$

$$\text{Fidelity: } f_1(\rho,\eta) = \|\sqrt{\rho}\sqrt{\eta}\|_1$$

$$\boxed{D(\rho\|\omega) - D(\Phi(\rho)\|\Phi(\omega)) \geq -2\ln\left(\sup_{t\in\mathbb{R}} f_1(\rho, R^{it}_{\omega,\Phi}\circ\Phi(\rho))\right)}$$

Generalized to von Neumann algebras in 2020 (Gao & Wilde)

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$$\text{Universal: } \tilde{R}_{\omega,\Phi}(\eta) = \int_{\mathbb{R}} \frac{\pi R^{it/2}_{\omega,\Phi}(\eta)dt}{2(\cosh(\pi t)+1)}$$

$$\boxed{D(\rho\|\omega) - D(\Phi(\rho)\|\Phi(\omega)) \geq -2\ln f_1(\rho, \tilde{R}_{\omega,\Phi}\circ\Phi(\rho))}$$

Generalized to von Neumann algebras in 2020 (Faulkner, Hollands, Swingle & Wang; Junge & L)

Trace inequalities often become quantum (noncommutative) replacements for functional calculus of (classical) info theory.

For vectors $\vec{x}, \vec{y}$, element-wise multiplication: $\exp(\vec{x} + \vec{y}) = \exp(\vec{x}) \exp(\vec{y})$

For Hermitian matrices X, Y, $tr(\exp(X + Y)) \leq tr(\exp(X) \exp(Y))$

(Golden-Thompson inequality)

$$\text{Recall } \|X\|_p = tr(|X|^p)^{1/p}$$

$$\text{For Hermitian matrices } \{X_k\}_{k=1}^n, p \geq 1,\ \beta_0(t) = \frac{\pi}{2}\Big(\cosh(\pi t) + 1\Big)^{-1},$$

$$\ln \Big\| \exp \sum_{k=1}^n X_k \Big\|_p \leq \int_{\mathbb{R}} dt \beta_0(t) \ln \Big\| \prod_{k=1}^n \exp((1+it)X_k) \Big\|_p$$

(Generalized Golden-Thompson inequality from Sutter, Berta, & Tomamichel 2017)

Recall Data Processing, Recovery

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$$\text{Universal: } \tilde{R}_{\omega,\Phi}(\eta) = \int_{\mathbb{R}} \frac{\pi R^{it/2}_{\omega,\Phi}(\eta)dt}{2(\cosh(\pi t)+1)}$$

Measured relative entropy: $D_M(\rho\|\omega) := \sup_{\text{measurements } M} D(M(\rho)\|M(\omega))$

$$\boxed{D(\rho\|\omega) - D(\Phi(\rho)\|\Phi(\omega)) \geq D_M(\rho\|\tilde{R}_{\omega,\Phi}\circ\Phi(\rho))}$$

(via the generalized Golden-Thompson inequality of Sutter, Berta & Tomamichel)

What happens without a (semifinite) trace?

What happens without a (semifinite) trace?

It will still work.

(but we need more even to make sense of it)

Haagerup L_p Spaces (Haagerup, Junge, Xu 2008)

Given algebra $\mathcal{M}$ with normal, faithful state ω...

1. Construct the *crossed product* $\mathcal{M} \rtimes \mathbb{R}$, of type II_∞
2. Construct the Haagerup Spaces $L_p(\mathcal{M})$ for $0 < p \leq \infty$

Then we have...

- $L_p(\mathcal{M})$ is a linear subspace, $\mathcal{M}$-bimodule
- $L_1(\mathcal{M})$ has a "trace" Tr, densities $d_\eta \leftarrow \eta$ for $\eta \in \mathcal{M}_*^+$
- $\{L_p(\mathcal{M})\}$ have Hölder conjugates & inequality
- $L_p(\mathcal{M})$ supports polar decompositions, $\|X\|_{L_p(\mathcal{M})} = Tr(|X|^p)^{\frac{1}{p}}$

Kosaki L_p Spaces

(Kosaki 1984)

$$\|X\|_{L_p^w(M),\rho,\eta} = \|d_\rho^{(1-w)/p} X d_\eta^{w/p}\|_{L_p(\mathcal{M})}$$

form an interpolation family for $1 \leq p \leq \infty$, where $L_p(\mathcal{M})$ is the Haagerup L_p space for $X \in \mathcal{M}$, ρ, η states $\in \mathcal{M}_*$, $w \in [0,1]$

$\alpha - z$ Rényi relative entropy $\leftrightarrow$ Kosaki L_p^w norm

$$D_{\alpha,z}(\rho\|\eta) = \frac{z}{\alpha - 1} \ln \|d_\rho^{1/z'} d_\eta^{-1/z'}\|_{L_z^{z-\alpha}(M,\rho,\eta)}$$

See (Audenaert & Datta 2015)
Subsumes sandwiched, Petz-Rényi forms

Intuitive Connection...

Haagerup L_p Spaces $\cong$ Kosaki L_p Spaces

For $X \in L_p(\mathcal{M})$

$Tr(|X|^p)$ makes sense

$\|X\|_{L_p(\mathcal{M})}$ makes sense

$L_\infty(\mathcal{M}) \cong \mathcal{M}$

For $X \in \mathcal{M}$

and states $\rho, \eta \in \mathcal{M}_*$

$d_\rho^{(1-w)/p} X d_\eta^{w/p} \in L_p(\mathcal{M})$

$\mathcal{M}$ finite $\rightarrow \| \cdot \|_{L_p(\mathcal{M}),\hat{1},\hat{1}} = \| \cdot \|_p$

Can yield usual p-norms if original algebra had finite trace

(Haagerup, Junge, Xu 2008)

Haagerup Reduction

Given algebra $\mathcal{M}$ with normal, faithful state η, construct the *crossed product* $\mathcal{M} \rtimes G$, where $G = \cup_n 2^{-n}\mathbb{Z} \subset \mathbb{R}$. Then...

- $\exists$ a norm-preserving, faithful conditional expectation $\mathcal{E} : \mathcal{M} \rtimes G \to \mathcal{M}$
- There exists an increasing family of subalgebras $\tilde{\mathcal{M}}_k$ and normal conditional expectation $F_k : \mathcal{M} \rtimes G \to \tilde{\mathcal{M}}_k$ such that $(\eta \circ \mathcal{E})F_k = (\eta \circ \mathcal{E})$;
- $\lim_k \|F_k(\psi) - \psi\|_{(\mathcal{M} \rtimes G)_*} = 0$ for every normal state $\psi \in (\mathcal{M} \rtimes G)_*$;
- $\tilde{\mathcal{M}}_k$ has a finite trace (type II_1) for all k.

Theorems

Let ρ, η be states on $\mathcal{M}$, $p \geq 1$, $r \in (0,1]$, $n \in \mathbb{N}$, $w \in [0,1]$, and consider a collection $\{Y_k\}_{k=1}^n \subset \mathcal{M}$ of positive semidefinite, bounded operators. Then (generalized Araki-Lieb-Thirring, see Sutter, Berta & Tomamichel 2017)

$$\ln \left\| \prod_{k=1}^{n} Y_k^r \right\|_{L^w_{p/r}(\mathcal{M},\rho,\eta)} \leq r \int_{-\infty}^{\infty} dt \beta_r(t) \ln \left\| \prod_{k=1}^{n} Y_k^{1+it} \right\|_{L^w_p(\mathcal{M},\rho,\eta)} .$$

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Assume $d_\rho = \exp(X_0)$ has full support, and consider a collection $\{X_k\}_{k=1}^n \subseteq \mathcal{M}$ of bounded, Hermitian operators. Then (generalized Golden-Thompson)

$$\ln \left\| \exp\left(\frac{X_0}{p} + \sum_{k=1}^n X_k \right) \right\|_{L_p(M)} \leq \int_{\mathbb{R}} dt \beta_0(t) \ln \left\| \prod_{k=1}^n \exp((1+it)X_k) \right\|_{L^1_p(M,\rho)} .$$

Theorem: Recovery & p-Fidelity

Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a normal, completely positive map from von Neumann algebra $\mathcal{M}$ to algebra $\mathcal{N}$. Let ρ, ω be states on $\mathcal{M}$. Then

$$\boxed{D(\rho\|\omega) \geq D(\Phi(\rho)\|\Phi(\omega)) - 2p \int_{\mathbb{R}} dt \beta_0(t) \ln f_p(\rho^{1/p}, R_{\omega,\Phi}^{(1+it)/p}(\Phi(\rho)^{1/p}))}$$

for $p \geq 1$, $\beta_0(t) = \frac{\pi}{2}\big(\cosh(\pi t) + 1\big)^{-1}$, $f_p(\rho, \omega) = \|\sqrt{\rho}\sqrt{\omega}\|_p$. f_1 is usual fidelity up to square. See (Liang et al. 2018) for p-fidelities.

Subsumes recovery from (Junge et al. 2018)

Also yields forms of "nonlinear" recovery for p ≠ 1

Recovery Via Trace Inequalities

$$D(\rho\|\omega) - D(\Phi(\rho)\|\Phi(\omega)) \geq D_M(\rho\|\tilde{R}_{\omega,\Phi} \circ \Phi(\rho))$$

in general von Neumann algebras

- In principle, follows from generalized Golden-Thompson
- In practice, many subtle continuity issues
- Easier: Golden-Thompson in type II_1 + Haagerup reduction

Theorem

Let $\exists \lambda > 0 : \rho \leq \lambda\omega$, & $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$. The following are equivalent

1. $D(\Phi(\rho)\|\Phi(\omega)) = D(\rho\|\omega)$;
2. There exists a ω-conditioned subalgebra $\mathcal{M}_0 \subset \mathcal{M}$ and a completely positive L_1-isometry u such that

$$u(\omega) = \Phi(\omega)\ ,\ u(\rho) = \Phi(\rho).$$

Conclusions

Multivariate trace inequalities extend to Kosaki or Haagerup norm inequalities

$$D(\rho\|\omega) \geq D(\Phi(\rho)\|\Phi(\omega)) + \max\left\{ -2p \int_{\mathbb{R}} dt \beta_0(t) \ln f_p(\rho^{1/p}, R_{\omega,\Phi}^{(1+it)/p}(\Phi(\rho)^{1/p})), D_M(\rho\|\tilde{R}_{\omega,\Phi} \circ \Phi(\rho)) \right\}$$

in general von Neumann algebras

$$D(\rho\|\omega) = D(\Phi(\rho)\|\Phi(\omega)) \iff L_1\text{-isometry}$$