The Quantum Approximate Optimization Algorithm and the Sherrington-Kirkpatrick Model at Infinite Size

Leo Zhou (Harvard University)

with Edward Farhi, Jeffrey Goldstone, and Sam Gutmann

QIP – Feb 1, 2021

Combinatorial Optimization Problems

Cost function

$$C(\boldsymbol{z}) = \sum_{\alpha} C_{\alpha}(\boldsymbol{z})$$

$$\boldsymbol{z} = (z_1, \ldots z_n) \in \{\pm 1\}^n$$

Want $\boldsymbol{z}^*$ so $C(\boldsymbol{z}^*)$ is maximized

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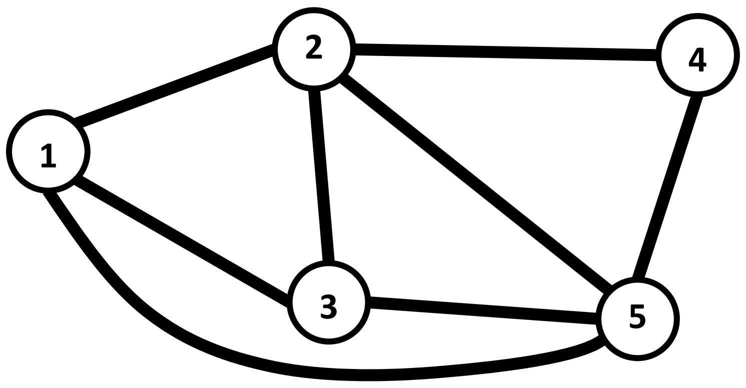

Goal: find a **bipartition** of vertices that cut the maximum # edges

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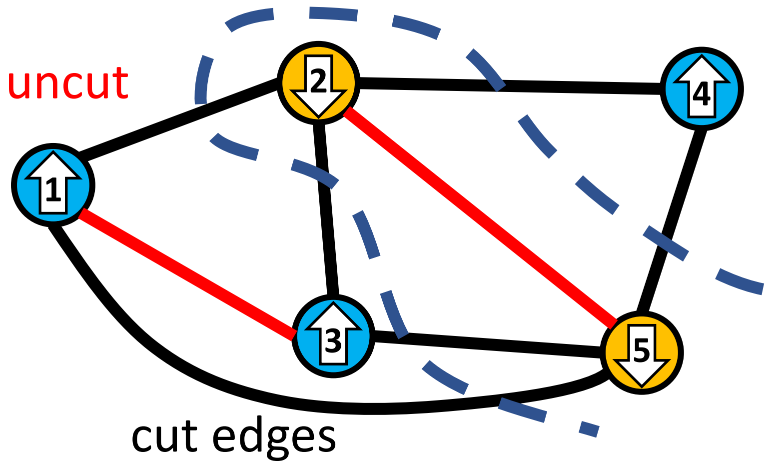

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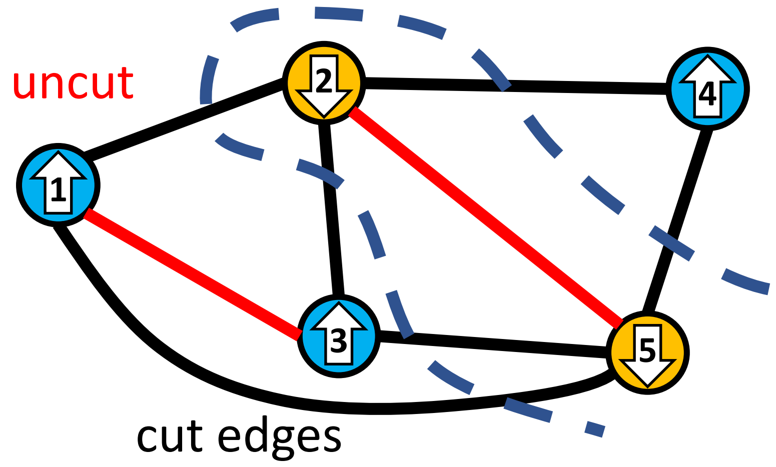

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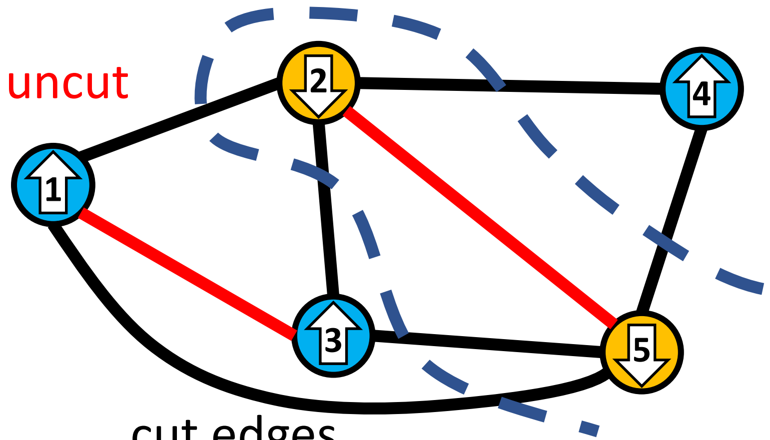

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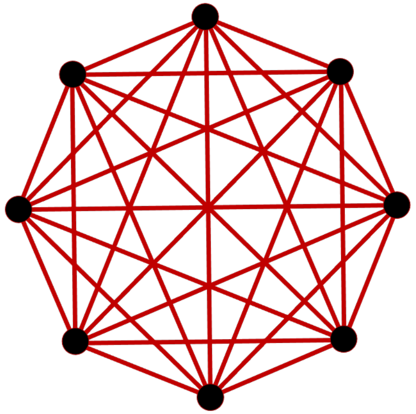

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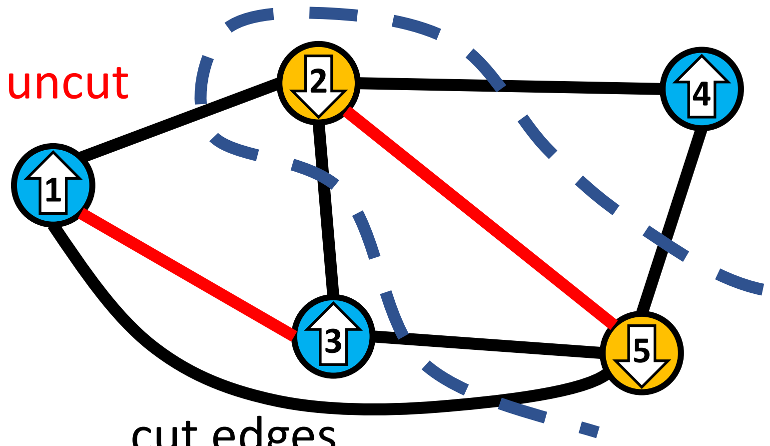

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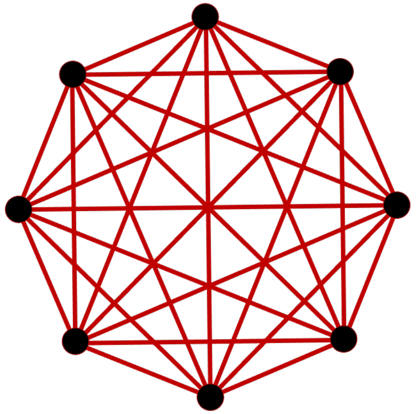

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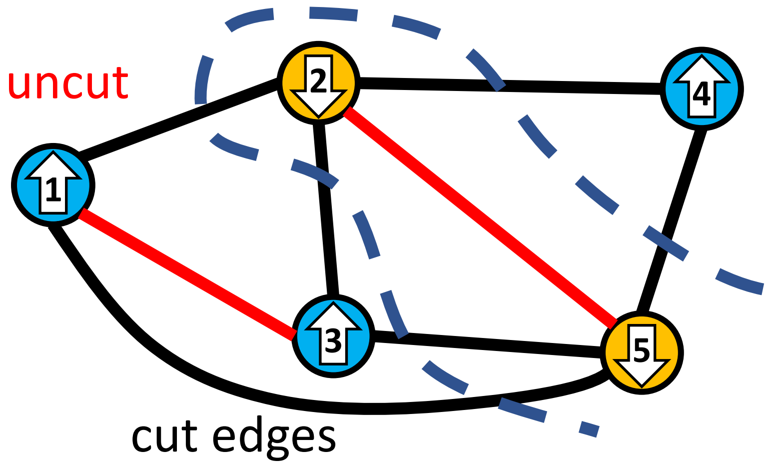

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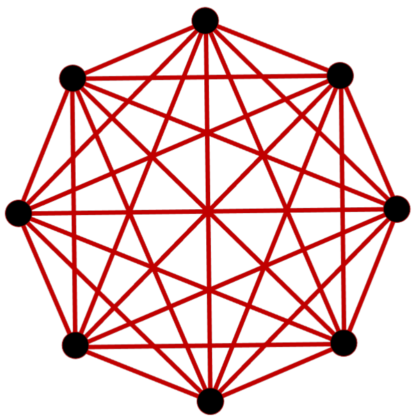

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MaxCut on Erdős-Rényi graphs = SK (average case)

Quantum Approximate Optimization Algorithm (QAOA)

[Farhi Goldstone Gutmann 2014]

Quantum Approximate Optimization Algorithm (QAOA)

$|+\rangle$

$|+\rangle$

$\vdots$

$|+\rangle$

$$|+\rangle^{\otimes n}$$

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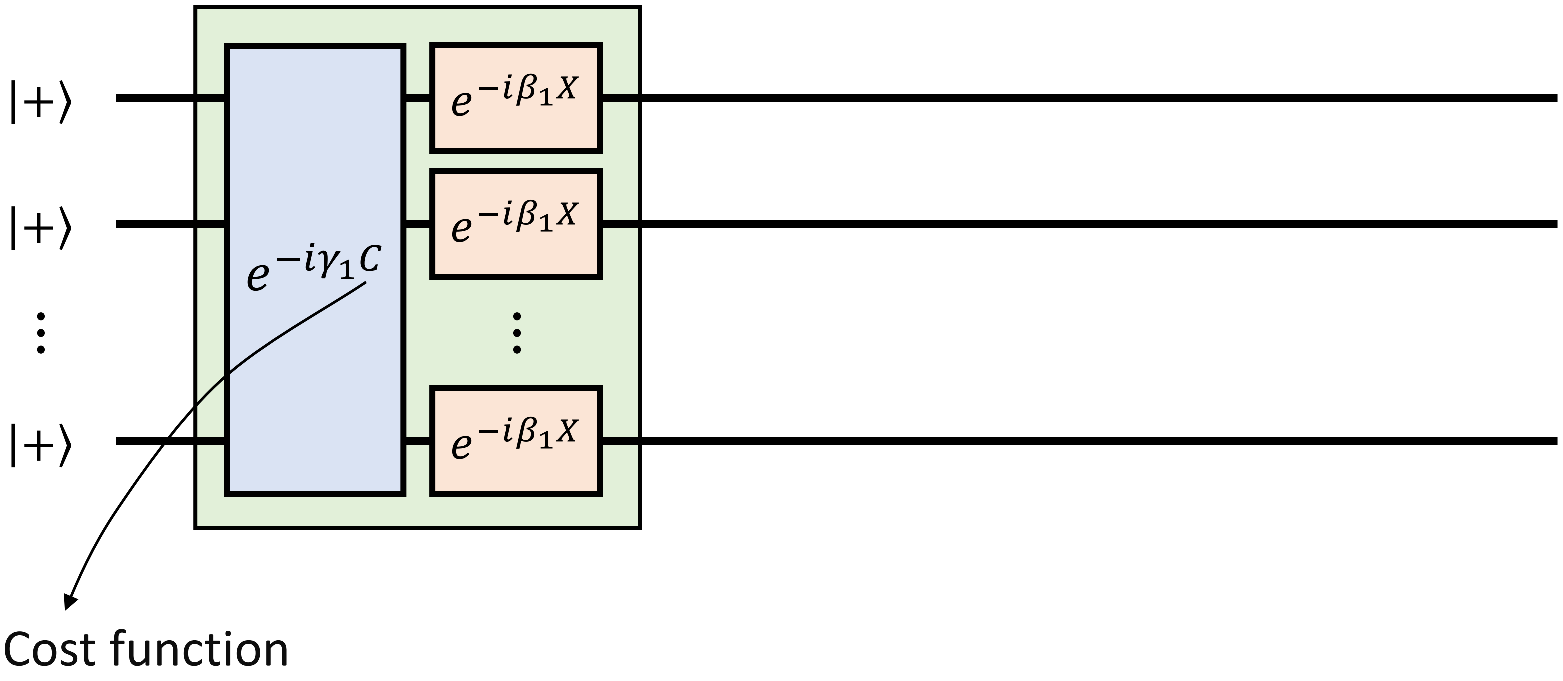

$$e^{-i\beta_1 B} e^{-i\gamma_1 C} \ |+\rangle^{\otimes n}$$

$$B = \sum_{i=1}^{n} X_i$$

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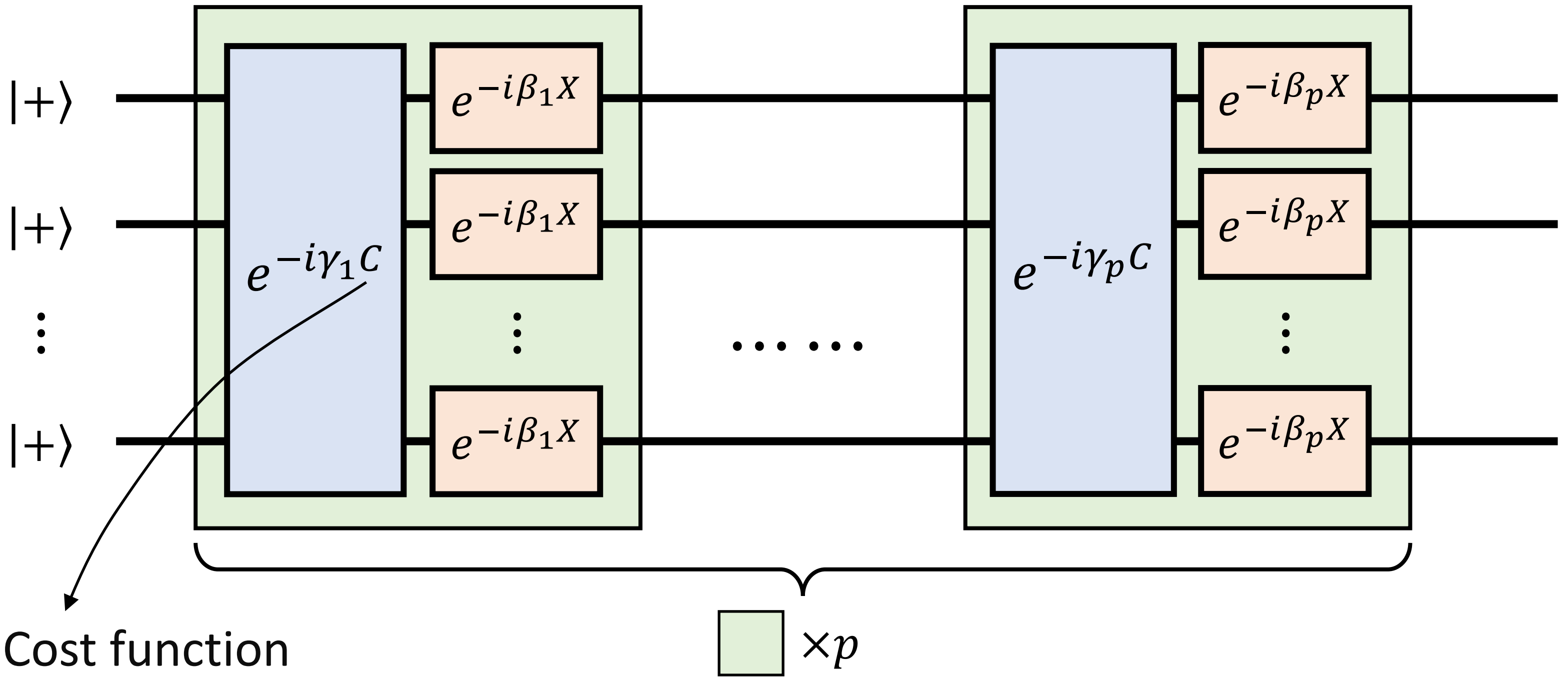

$$|\boldsymbol{\gamma}, \boldsymbol{\beta}\rangle = e^{-i\beta_p B} e^{-i\gamma_p C} \cdots e^{-i\beta_1 B} e^{-i\gamma_1 C} \; |+\rangle^{\otimes n}$$

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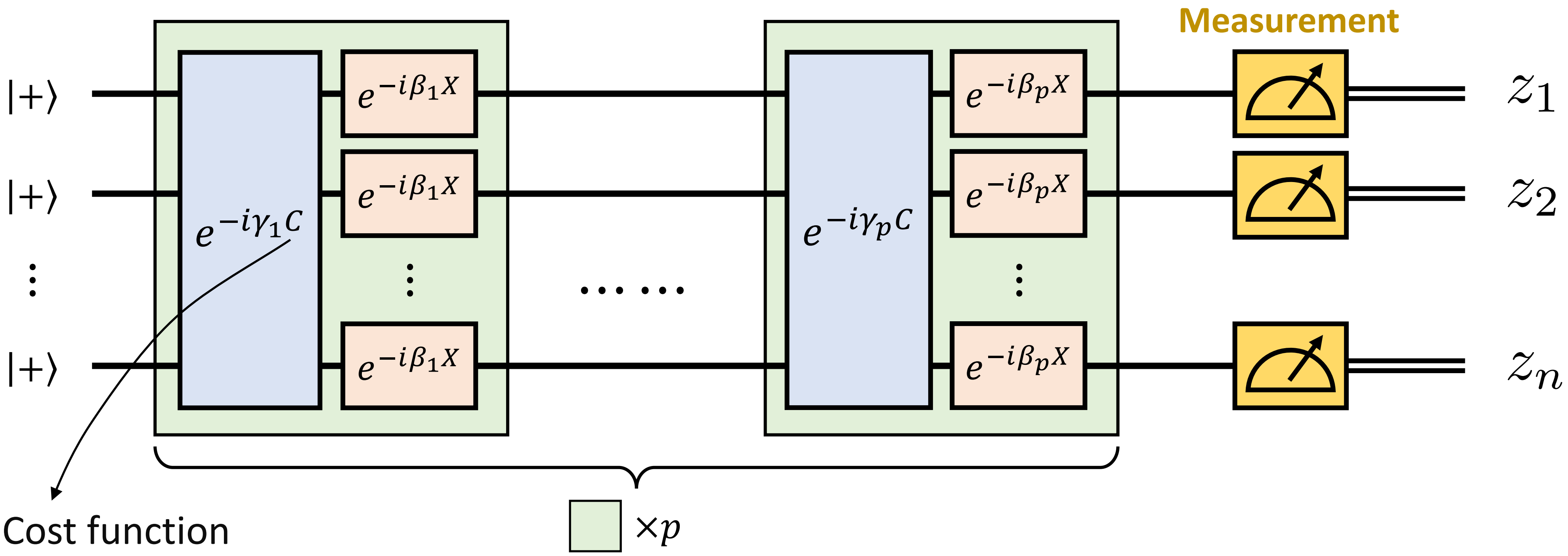

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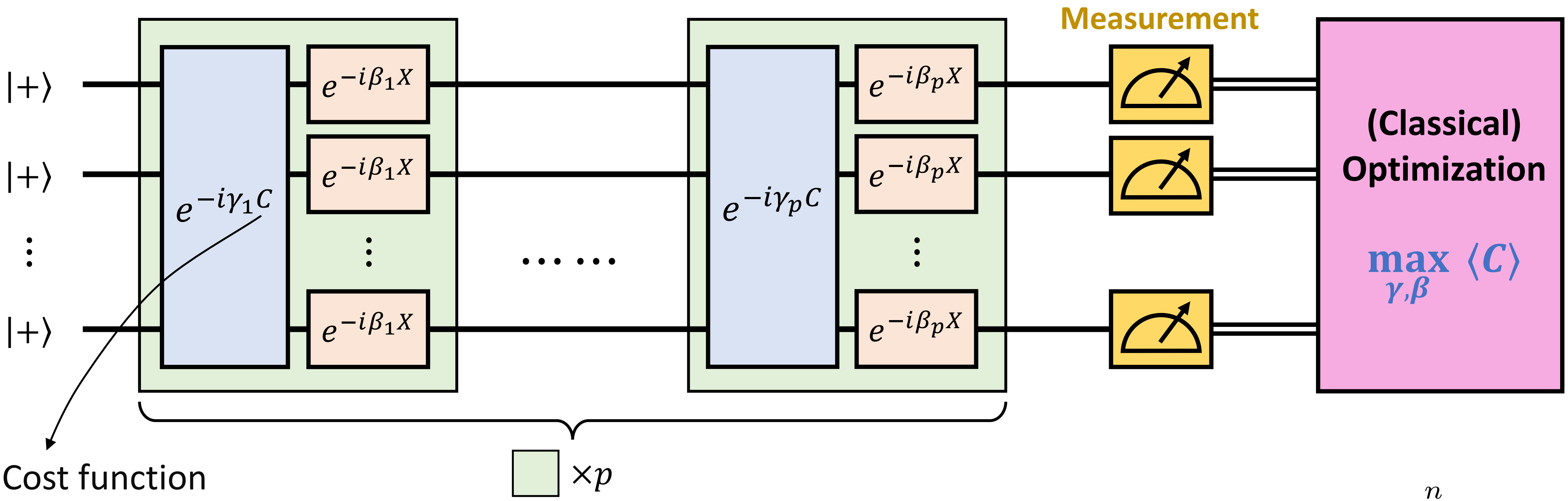

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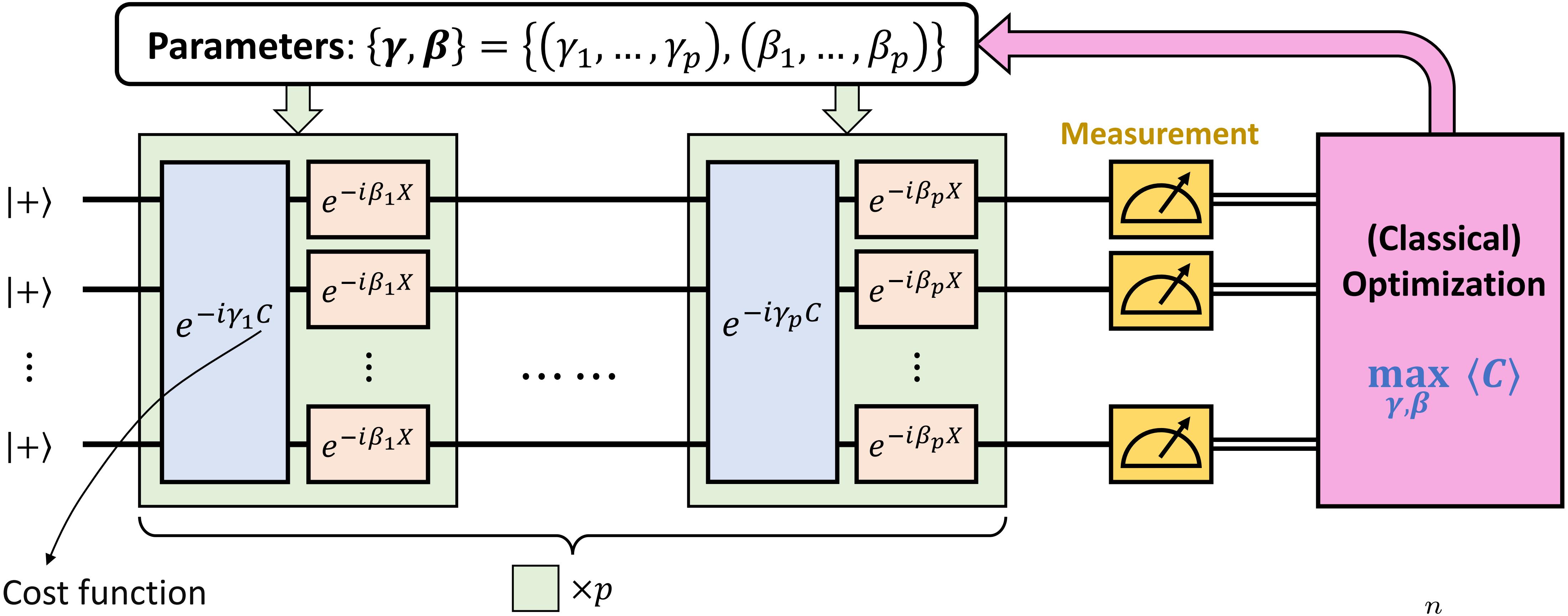

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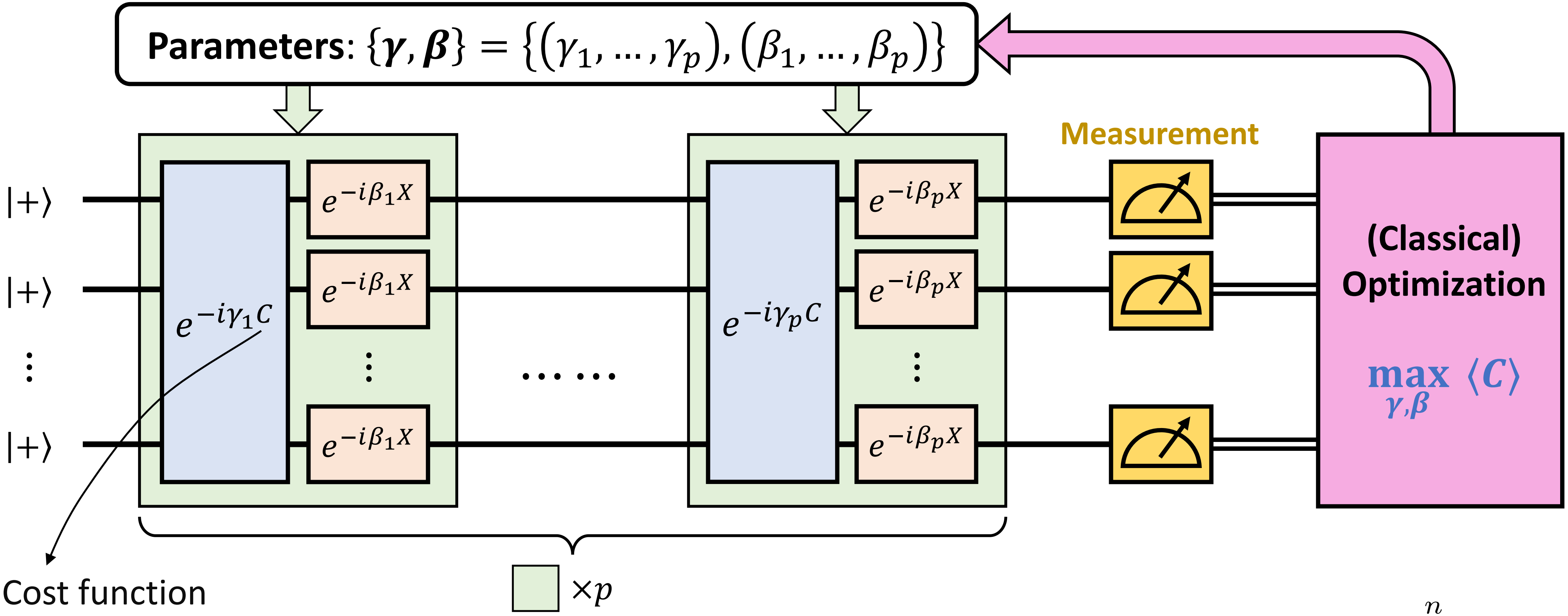

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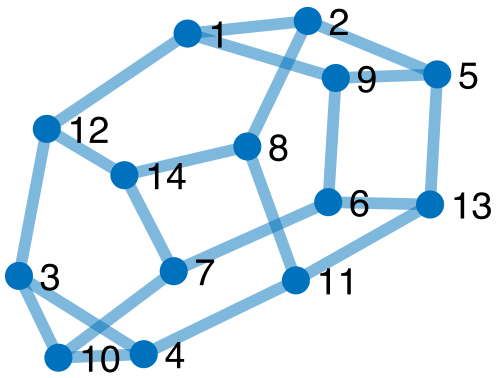
As $p \to \infty$ QAOA can get the global optimum

[Farhi Goldstone Gutmann 2014]

Previous Results on the QAOA

- Analyze performance via “subgraphs”

e.g. MaxCut on 3-regular graphs



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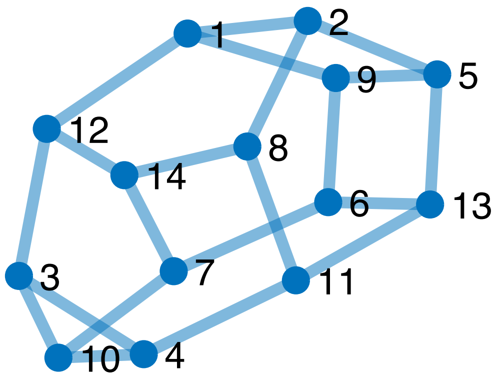

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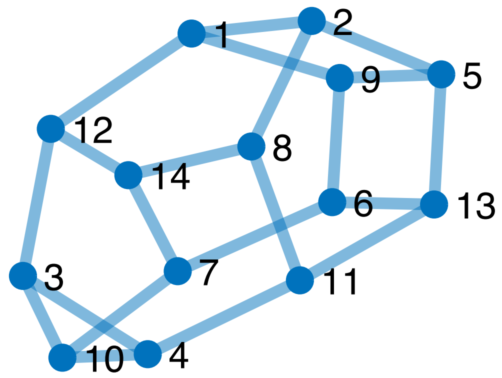

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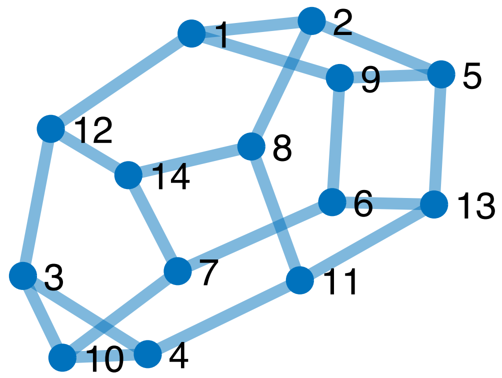

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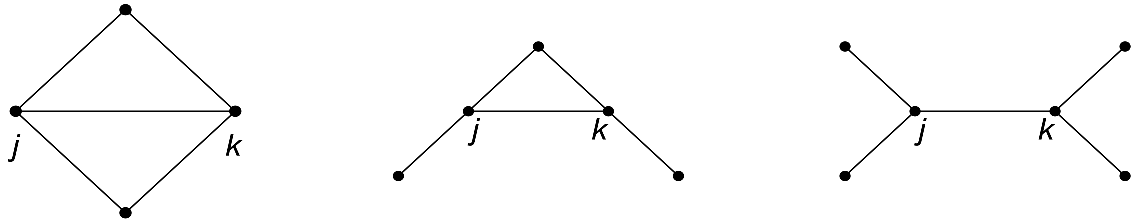

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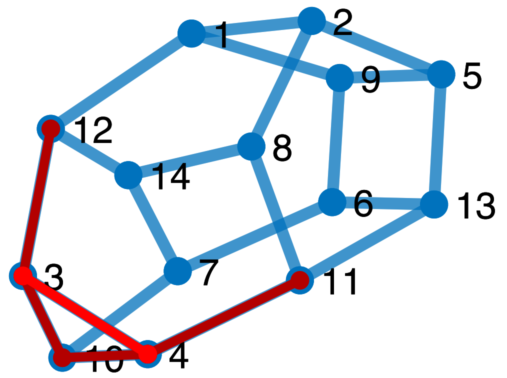

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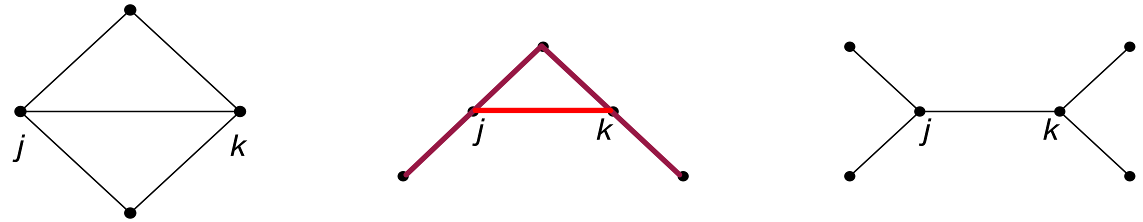

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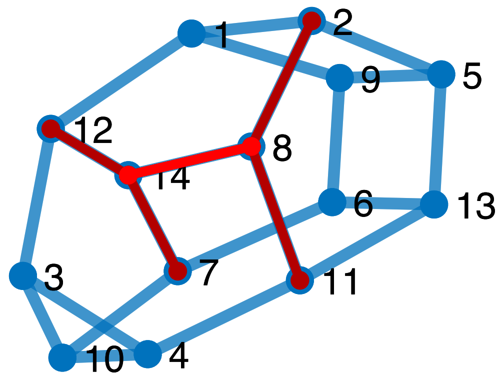

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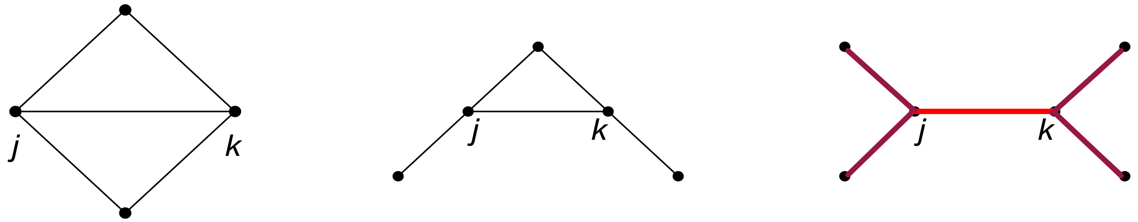

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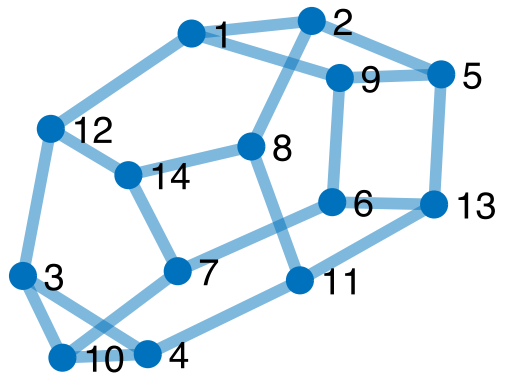

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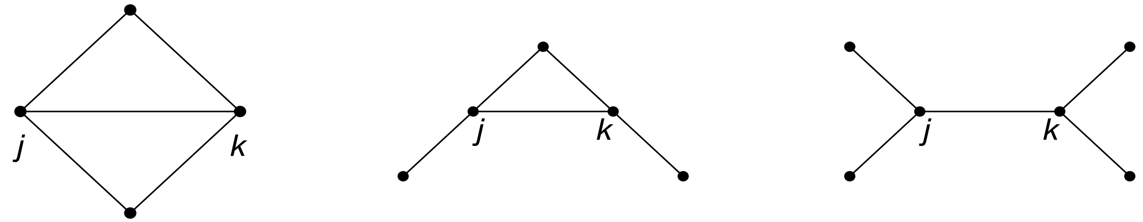

Worst case guarantee:

$$\langle C\rangle / C_{\max} \geq 0.6924 \text{ @ } p = 1$$

[Farhi Goldstone Gutmann 2014]

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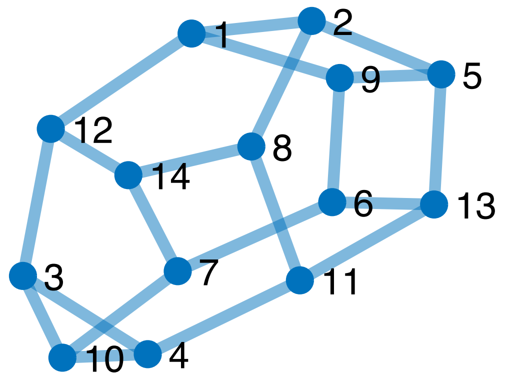

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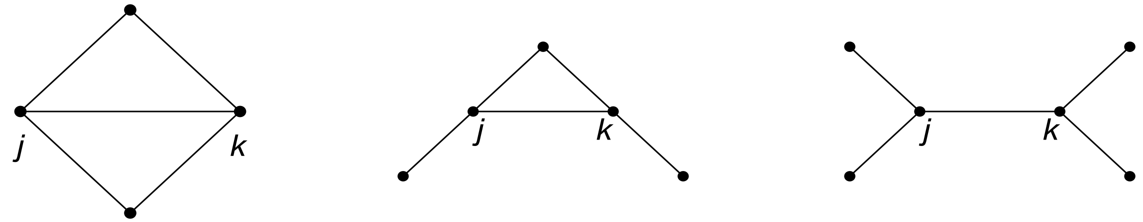

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Difficult for higher p as the complexity of classical simulation grow as $O\left(2^{2^p}\right)$!

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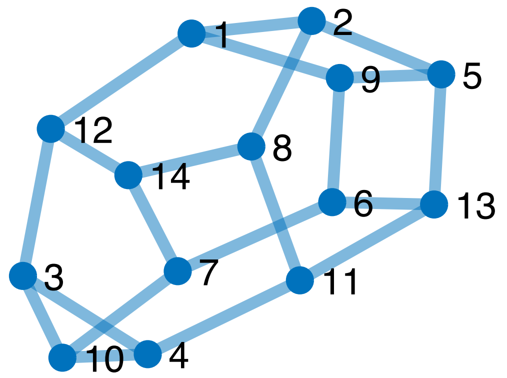

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[**LZ** *et al.* 2018]
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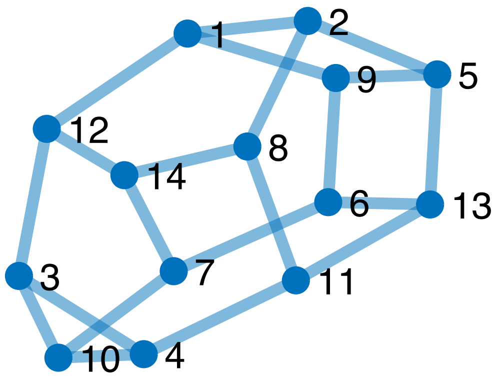

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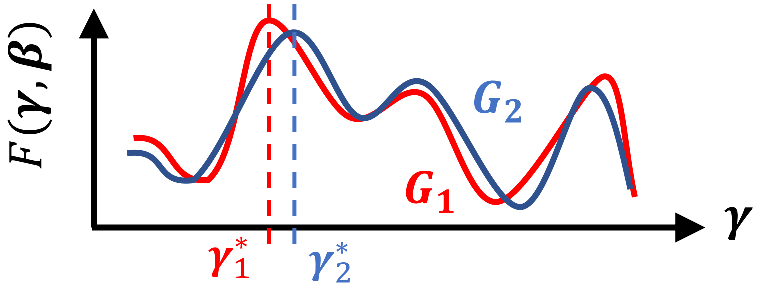

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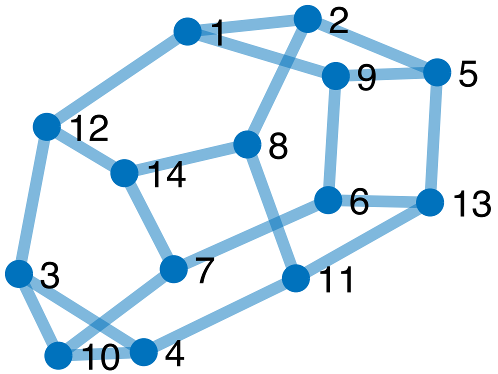

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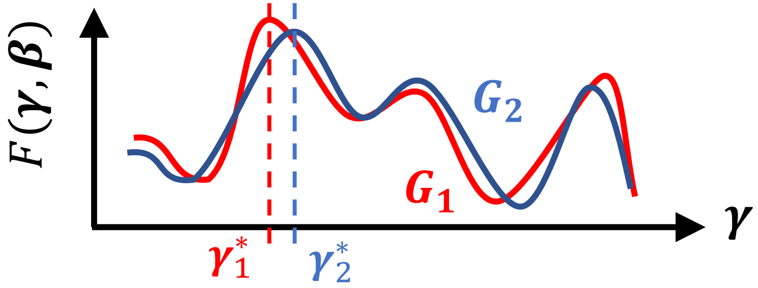

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[Bravyi Kliesch Koenig Tang 2019]
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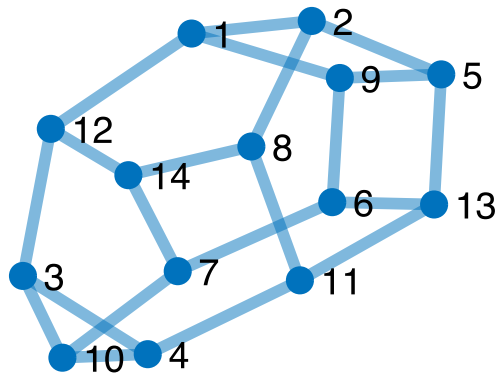

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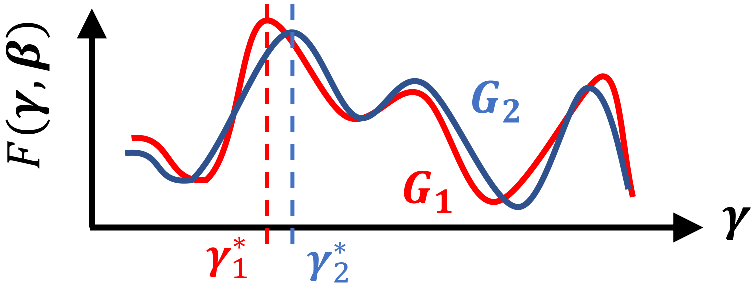

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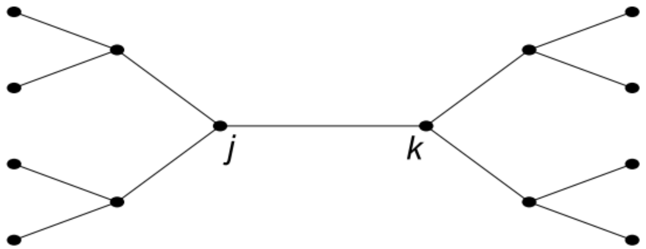

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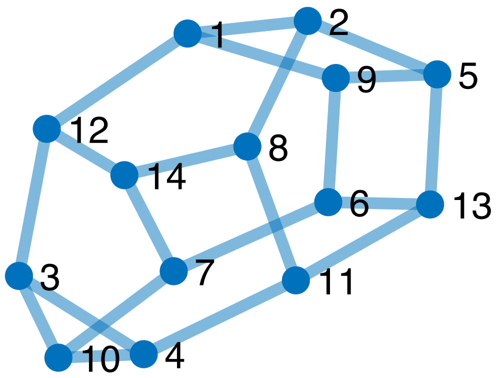

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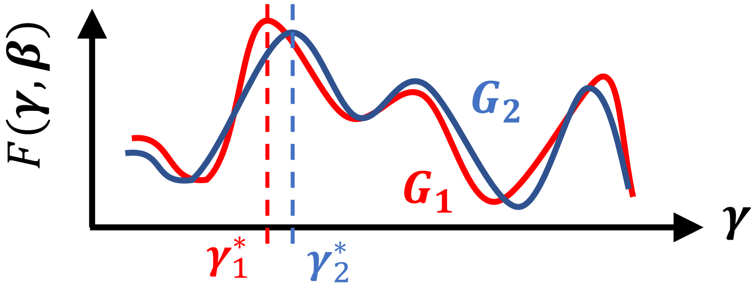

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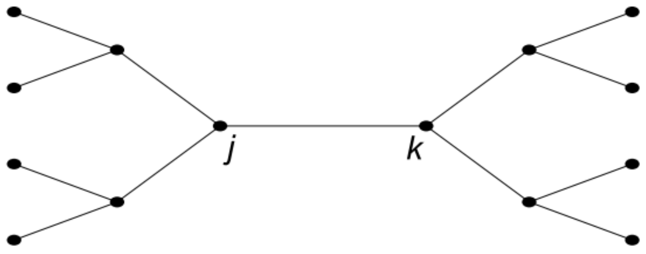

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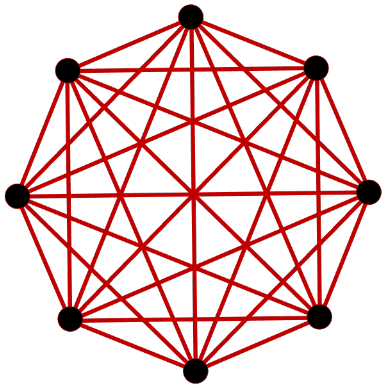
Cannot distinguish bipartite vs. typical (frustrated) graphs

The Sherrington-Kirkpatrick model



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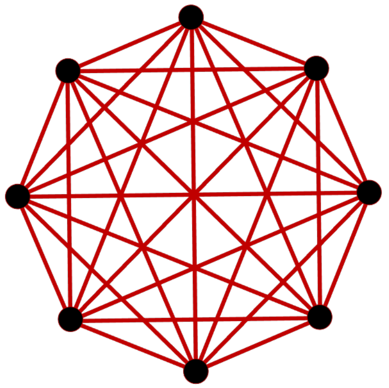

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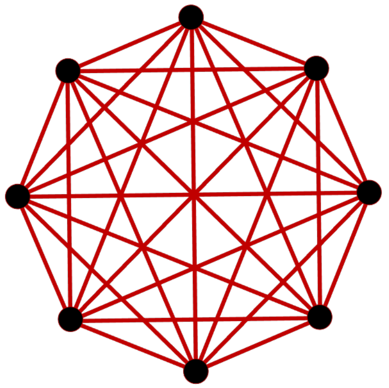

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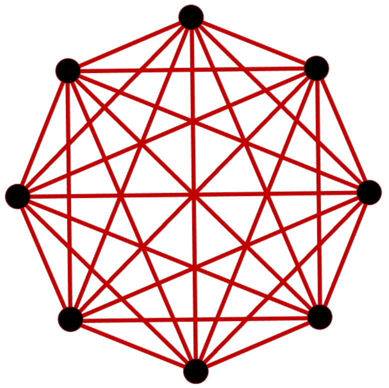
- **Worst case**: NP-hard to approximate within $O(1/\log^{c}(n))$ factor [Arora *et al.* 2005]

The Sherrington-Kirkpatrick model



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- **Typical case**: Famously, Parisi (1979) predicted and Talagrand (2006) proved that

$$\lim_{n\to\infty} \frac{1}{n} \max_{\boldsymbol{z}} C_J(\boldsymbol{z}) = \Pi_* = 0.763166\ldots$$

Complexity of solving a **typical** SK instance?

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- Known results of typical-case complexity:

1. Simulated Annealing is believed to fail for this problem [Parisi]
2. Semi-Definite Programming obtains $C/n = 2/\pi \approx 0.6366$ [Montanari Sen 2016]
3. Assuming the conjecture that the SK model has no "***overlap gap property***" (OGP), Andrea Montanari's algorithm (2018) outputs $\hat{\boldsymbol{z}}$ with

$$C/n \geq (1-\epsilon)\Pi_* \quad \text{in time} \quad O(n^2/\epsilon^k)$$

Main Result 1:
Performance of the QAOA applied to the SK model

We give an $O(16^p)$-time method to evaluate

$$V_p(\boldsymbol{\gamma},\boldsymbol{\beta}) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_J[\langle \boldsymbol{\gamma},\boldsymbol{\beta}|C|\boldsymbol{\gamma},\boldsymbol{\beta}\rangle]$$

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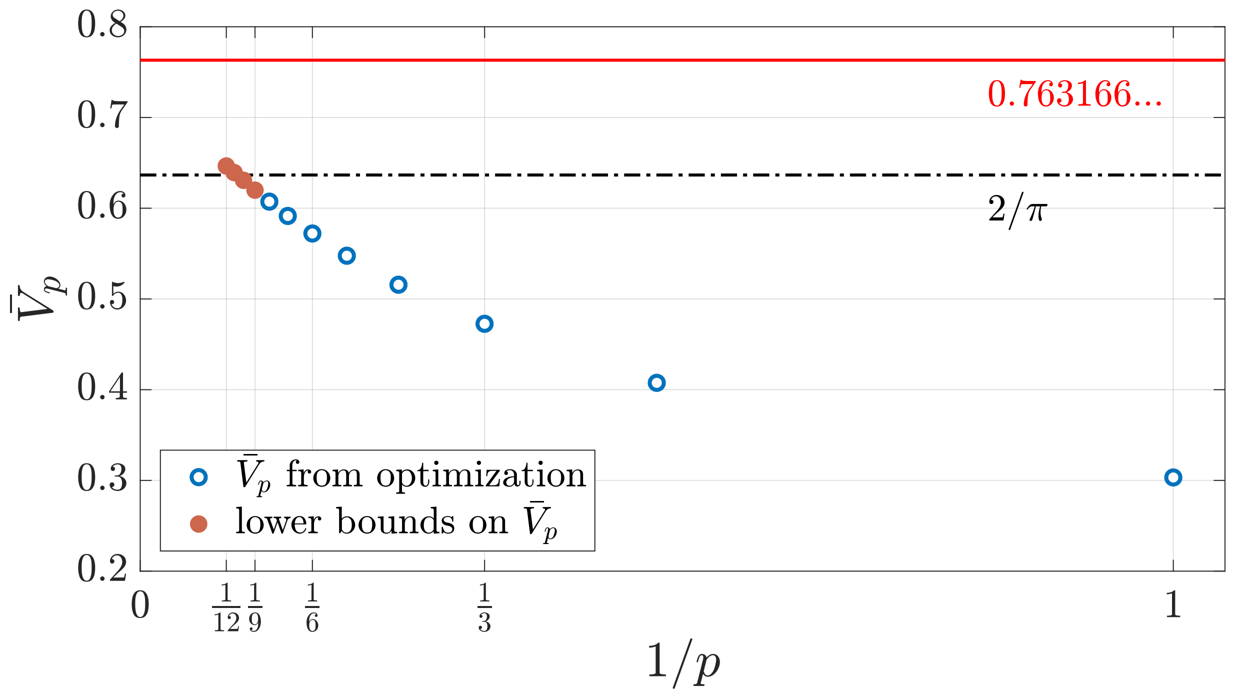

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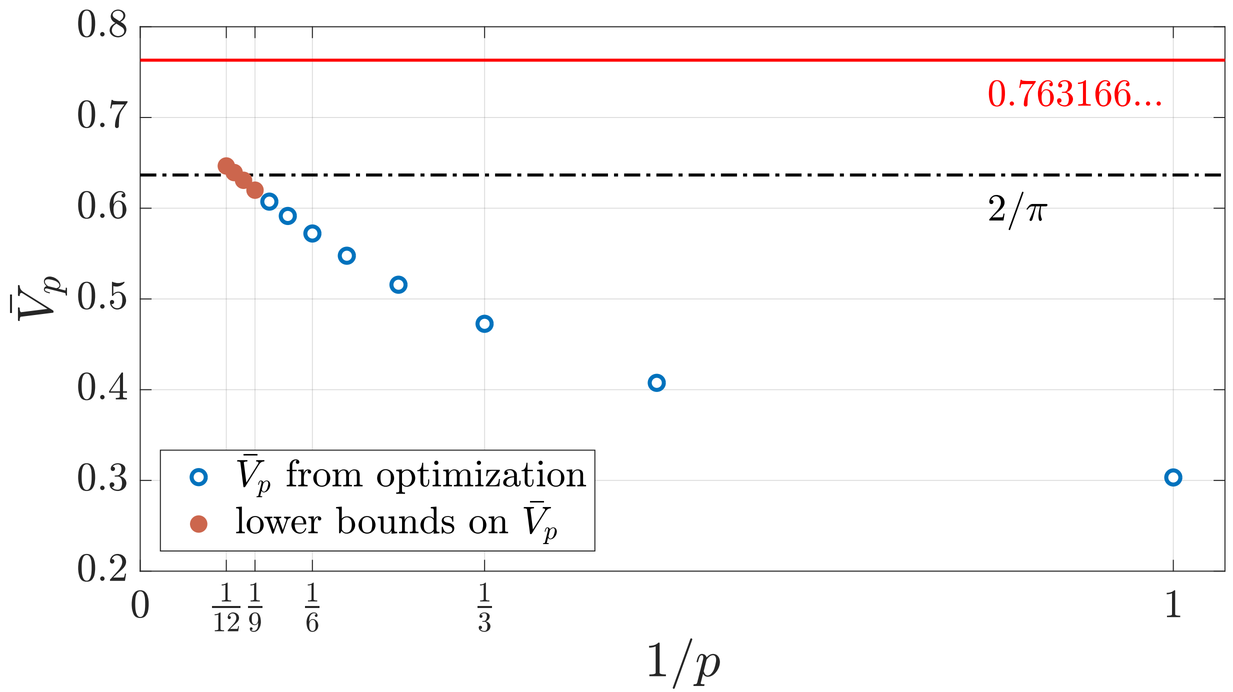

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QAOA beats SDP @ *p*=11

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Concentration of QAOA on the SK model

- We also prove, for any fixed depth p:

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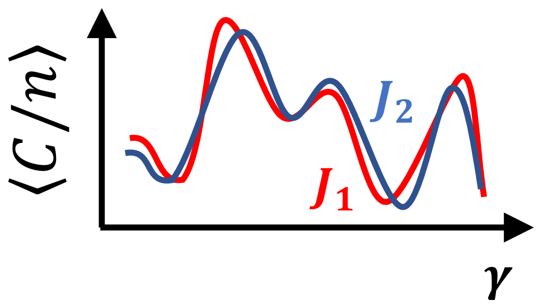

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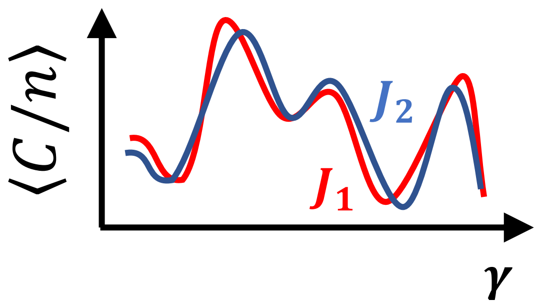

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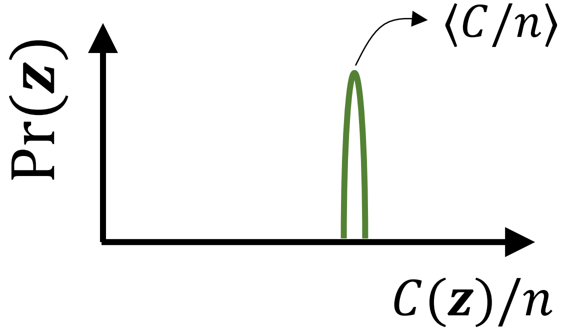

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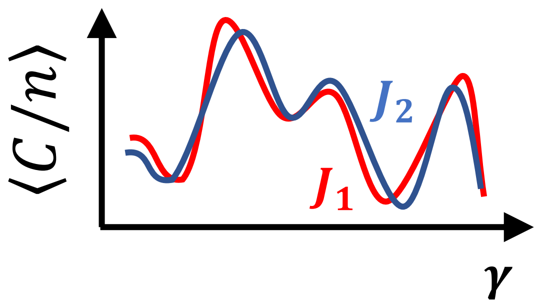

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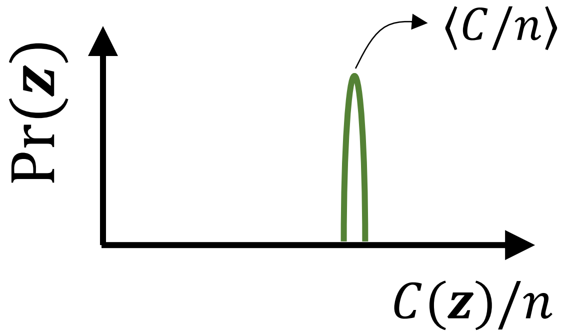

- With probability $\to 1$ as $n \to \infty$, applying QAOA and measuring will give us a bit string $\boldsymbol{z}$ which has

$$C(\boldsymbol{z})/n \approx \langle C/n \rangle \approx V_p$$

Key Idea: Average over instances

- Parisi's formalism requires delicate tricks
  - A replica-symmetry-breaking ansatz for the free energy:

$$\mathbb{E}_J[\log Z_J]$$

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For ϕ small, use $\mathbb{E}_J[e^{iJ\phi}] = 1 - \frac{1}{2}\phi^2 + \cdots \qquad \mathbb{E}_J[Je^{iJ\phi}] = i\phi + \cdots$

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n-bit strings

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For general p, there are 2^{2p} configurations

$\exp(O(p))$ complexity

Performance of the QAOA on the SK model

- Turn the crank, we get at $p = 1$

$$V_1 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_J[\langle C\rangle] = \gamma e^{-2\gamma^2} \sin 4\beta$$

Optimum @ $\beta = \frac{\pi}{8}, \gamma = \frac{1}{2}$

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Generic QAOA state has $\langle C\rangle = e^{-O(n)}$!!

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2	0.407545	
…	…	
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$11^\dagger$	0.639311	**beats SDP!**
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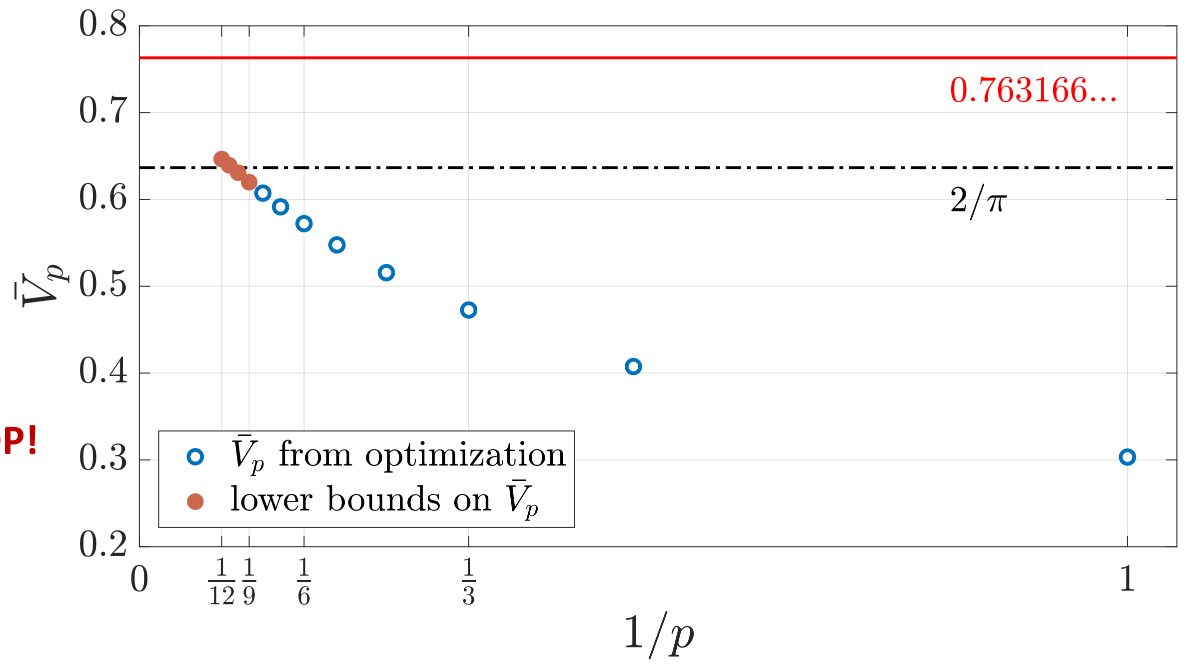

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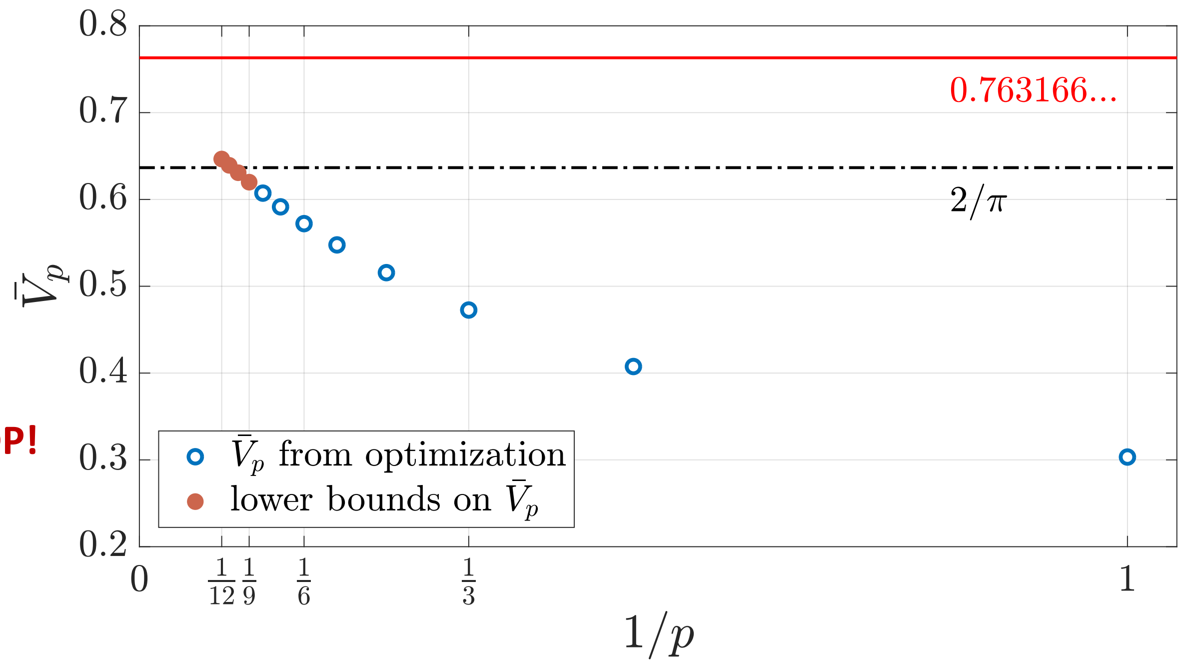

$$\lim_{p\to\infty}\lim_{n\to\infty} \stackrel{?}{=} \lim_{n\to\infty}\lim_{p\to\infty}$$

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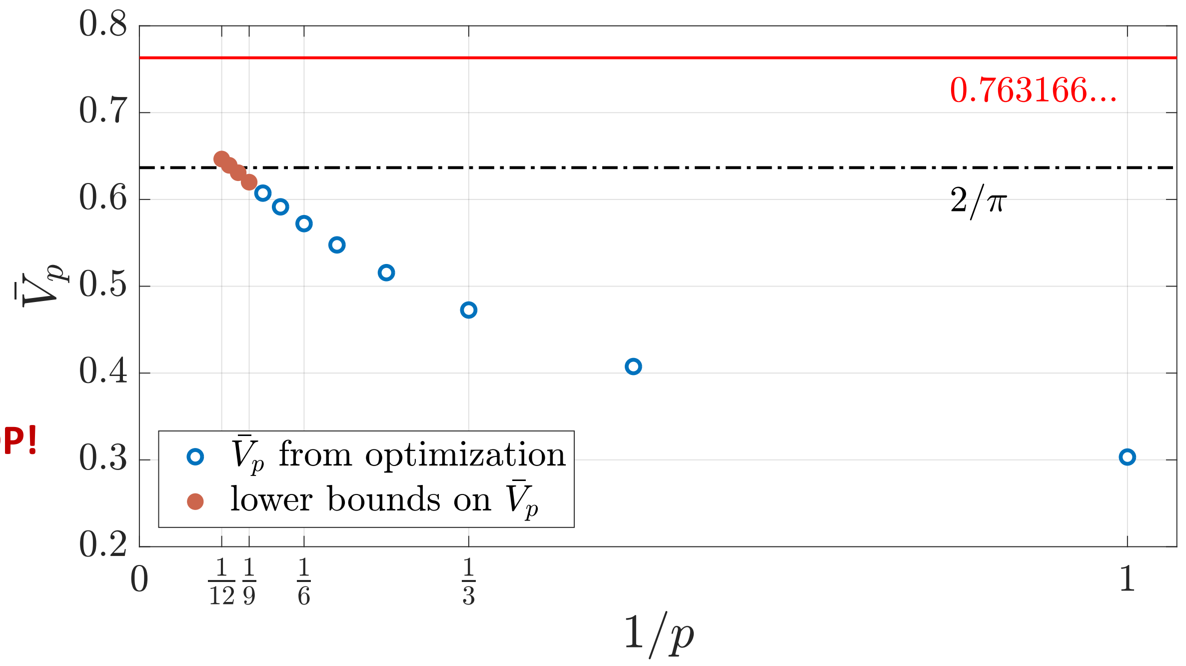
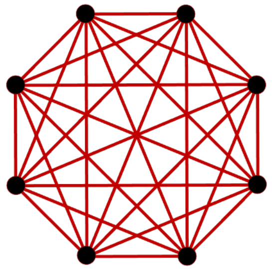


If $\lim_{p\to\infty} \bar{V}_p = \Pi_*$, then a power law fit of optimized V_p yields

$$\bar{V}_p \approx \Pi_* - \frac{1.2}{(p+2)^{0.9}}$$

Summary



- We *analytically* obtain a formula for **typical case performance** of the QAOA on the SK model at high p
  - Evaluation takes $O(16^p)$ currently but may be improvable

- QAOA **beats** Semi-Definite Programming at $p = 11$

- **Concentration** over instances and measurements

[Farhi Goldstone Gutmann **LZ**, arXiv:1910.08187]

https://github.com/leologist/QAOA-SK

Outlook

- Show **convergence** of QAOA as $p \to \infty$? $\lim_{p\to\infty} \lim_{n\to\infty} \stackrel{?}{=} \lim_{n\to\infty} \lim_{p\to\infty}$

Outlook

- Show **convergence** of QAOA as $p \to \infty$? $\lim_{p\to\infty} \lim_{n\to\infty} \stackrel{?}{=} \lim_{n\to\infty} \lim_{p\to\infty}$

- **Average over instances** for harder problems for provable speedup?

q-spin model

$$C = \sum_{i_1 < \cdots < i_q} J_{i_1 \ldots i_q} Z_{i_1} \cdots Z_{i_q}$$

Provably hard for classical algorithms due to their "**Overlap Gap Property**"

[Gamarnik Jagannath 2019]
[Gamarnik Jagannath Wein 2020]

Montanari's algorithm *stuck at 98.4% approximation ratio for q=3*
[Alaoui Montanari 2020]

QAOA @ p=1
gets 33% for q=3