Random quantum circuits anti-concentrate in log depth

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Introduction

Random quantum circuits (RQCs) are a crucial model for understanding a diverse set of phenomena in both quantum information and quantum many-body physics. They have been used to study the onset of quantum chaos and dynamical spread of entanglement in strongly interacting quantum systems, including information processing in black holes. They are also optimal decouplers, good error-correcting codes, rapid scramblers of quantum information, and efficient generators of quantum pseudorandomness. Moreover, recent benchmarks for demonstrating exponential quantum computational advantage entail sampling the output distribution of random quantum circuits.

In this submission we rigorously study another important phenomenon exhibited by RQC evolution: *anti-concentration*. Measuring the evolved state in the computational basis, anti-concentration refers to the distribution over measurement outcomes being sufficiently "spread out." Understanding the circuit depth at which RQCs anti-concentrate is particularly important for knowing when RQCs are hard to simulate. Anti-concentration is both an essential ingredient in formal arguments that RQC simulation is hard [1–8] as well as a necessary condition for certain classical algorithms for noisy circuit simulation [9, 10] or pseudo-simulation [11] to be efficient.

In much of the previous work that relies on anti-concentration, it is often asserted as an implication of the 2-design property, where convergence to an approximate 2-design implies anti-concentration. For instance, it is known that both n-qubit RQCs on a fully connected architecture and geometrically local RQCs on a 1D chain form approximate 2-designs after $O(n)$ depth [12, 13] (i.e. circuit size $O(n^2)$). This was later improved to $O(n^{1/D})$ in D spatial dimensions [14]. Such results then imply anti-concentration at that circuit depth. As we show, the 2-design property is far stronger than what is required for anti-concentration. *We prove, for a number of circuit architectures, that anti-concentration occurs in RQCs at exponentially shorter circuit depths.*

More precisely, we focus on the *collision probability*, the probability that two measurement outcomes from two different circuit realizations are the same. Given some instance of a quantum circuit U, denote the classical probability distribution over its measurement outcomes as $p_U(x) = |\langle x|U|1^n\rangle|^2$, i.e. for the circuit acting on an initial all ones state. For an ensemble of RQCs, we say that the ensemble is anti-concentrated if the collision probability Z obeys

$$Z := \mathop{\mathbb{E}}_{U}\left[\sum_x p_U(x)^2\right] \le \frac{1}{\alpha}\frac{1}{q^n} \tag{1}$$

for some $0 < \alpha \le 1$ independent of n. That is, the RQCs are anti-concentrated if Z is at most a constant factor larger than its minimal value.

We show upper and lower bounds on the collision probability Z for random quantum circuits of different architectures, as a function of the circuit size s (typically a factor of n larger than the circuit depth) and the qudit local dimension q. These bounds are used to form upper and lower bounds on the anti-concentration size s_{AC}, defined as the minimum circuit size required for the anti-concentration condition to be met. We show general bounds on s_{AC} that apply regardless of how the gate are arranged, and then give more specific results for a local 1D architecture and also a "complete-graph" architecture with all-to-all couplings.

Results

For random quantum circuits on n qudits of local dimension q, and for varying circuit architectures, the *circuit size* at which the output probabilities anti-concentrate, denoted s_{AC}, obeys:

Architecture	s_{AC} upper bound	s_{AC} lower bound
general	$O(n^2)$	$\Omega(n\log(n))$
1D	$c_{1\mathrm{D}}\, n\log(n) + O(n)$	$c_{1\mathrm{D}}\, n\log(n) - O(n)$
complete-graph	$c_{\mathrm{cg}}\, n\log(n) + O(n)$	$c_{\mathrm{cg}}\, n\log(n) - O(n)$

where we exactly compute the constants which appear in both the upper and lower bounds as a function of the local dimension q, with $c_{1\mathrm{D}} := \left(2\log\left(\frac{q^2+1}{2q}\right)\right)^{-1}$ for 1D RQCs and $c_{\mathrm{cg}} := \frac{q^2+1}{2(q^2-1)}$ for the non-local RQCs defined on a complete graph.

The upper and lower bounds together allow us to conclude that $s_{AC} = \Theta(n\log(n))$ for both the 1D architecture and the complete-graph architecture. (In fact, we have matching upper and lower bounds on the constant prefactor of the $n\log(n)$ in both cases.) Since 1D and complete graph have the same $\Theta(n\log(n))$ scaling despite being opposite extremes of geometric locality, we conjecture that the $\Theta(n\log(n))$ scaling holds more generally assuming only that a natural connectivity condition is satisfied, but manage only to prove a weaker $O(n^2)$ upper bound.

Significance and implications

We emphasize here a few ways in which the above results are impactful. First of all, they are rigorous bounds that show anti-concentration is achieved much faster than the 2-design property. The dependence of the anti-concentration size on architecture appears to be confined only to the value of the constant pre-factors and not the form of the asymptotic scaling. This fact has important potential implications for hardness-of-simulation arguments underlying quantum supremacy experiments based on random circuit sampling. Anti-concentration is a necessary ingredient for these arguments and our bounds show that the depth needed is much lower than the conventional wisdom, which has been based on results on 2-designs.

Second of all, there has been some uncertainty and confusion surrounding anti-concentration in previous literature that our results manage to clarify, settling two statements formally conjectured in previous work. We explicitly compute the $n\log(n)$ constant prefactor c_{cg} for the complete-graph architecture, which had been conjectured by Harrow & Mehraban [14] (QIP 2019). More recently, Barak, Chou, & Gao [11] conjectured that 2D circuits anti-concentrate in $O(\sqrt{\log(n)})$ circuit depth, or $O(n\sqrt{\log(n)})$ circuit size. Our general $\Omega(n\log(n))$ lower bound refutes this conjecture and implies that their method relying on the collision probability is not sufficient to prove that their algorithm is able to spoof 2D RQCs in polynomial time.

Finally, the fact that we can prove upper and lower bounds that match even up to the constant prefactor is a testament to the power of the method we use, and suggests a broader utility for it and the way we apply it. Our approach, discussed further below, follows previous work in mapping RQCs to classical statistical mechanical models, and goes beyond by performing a successful combinatorial analysis of the resulting quantities. In the end, this yields not only rigorous bounds, but also an appealing heuristic picture of the situation for anti-concentration.

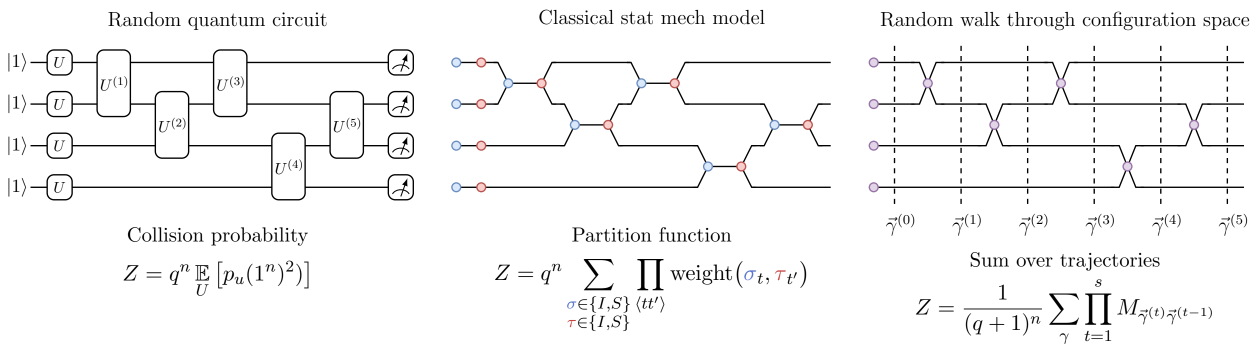

Figure 1: The equivalent ways to interpret the RQC collision probability Z. The RQC collision probability can be reinterpreted as the partition function of a stat mech model and as a stochastic process of evolving configurations.

Technical contributions

Our method combines ideas from statistical mechanics and stochastic processes. We utilize a statistical mechanical mapping for RQCs, where the collision probability can be exactly re-expressed as the partition function of an Ising-like statistical mechanics model. This mapping has been applied to RQCs in several other contexts, but in the context of anti-concentration we observe that one can go further by reinterpreting this description as a Markov chain, where the number of gates needed for anti-concentration ultimately translates into the time needed for certain expectation values to converge under the dynamics of the Markov chain. Figure 1 depicts these equivalent descriptions.

Once the general framework for this Markov chain has been derived, it is possible to succinctly show that $\Omega(n \log(n))$ gates are necessary for anti-concentration, no matter how they are arranged. Then, focusing on the cases of 1D and complete graph architectures, we are able to give more specific upper and lower bounds on the anti-concentration size. For the 1D upper bound, the approach builds off the techniques in [15, 16]. The lower bound, on the other hand, requires more additional ideas specific to the anti-concentration problem. For the complete-graph analysis, our solution requires application of several techniques in a novel way. The proof idea first involves reworking the quantity of interest into a sum over all possible paths of a random walker on a line weighted by the probability that the walker is successfully able to finish the path prior to the end of the circuit. This probability is bounded with a Chernoff bound, which is combined with a subtle observation on how to perform the sum over paths.

The method not only yields sharp quantitative bounds, it also produces an appealing qualitative explanation on how and why the collision probability reaches its limiting value, which allows for effective heuristic reasoning even in architectures that we have not explicitly considered here. We further anticipate that our method may be used to study and improve bounds for other phenomena exhibited by RQCs, such as thermalization and decoupling.

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