Random quantum circuits anti-concentrate in log depth

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Based on joint work with **Nicholas Hunter-Jones [2]** & **Fernando Brandão [1,3]**

[1] Caltech [2] Perimeter Institute [3] Amazon Web Services

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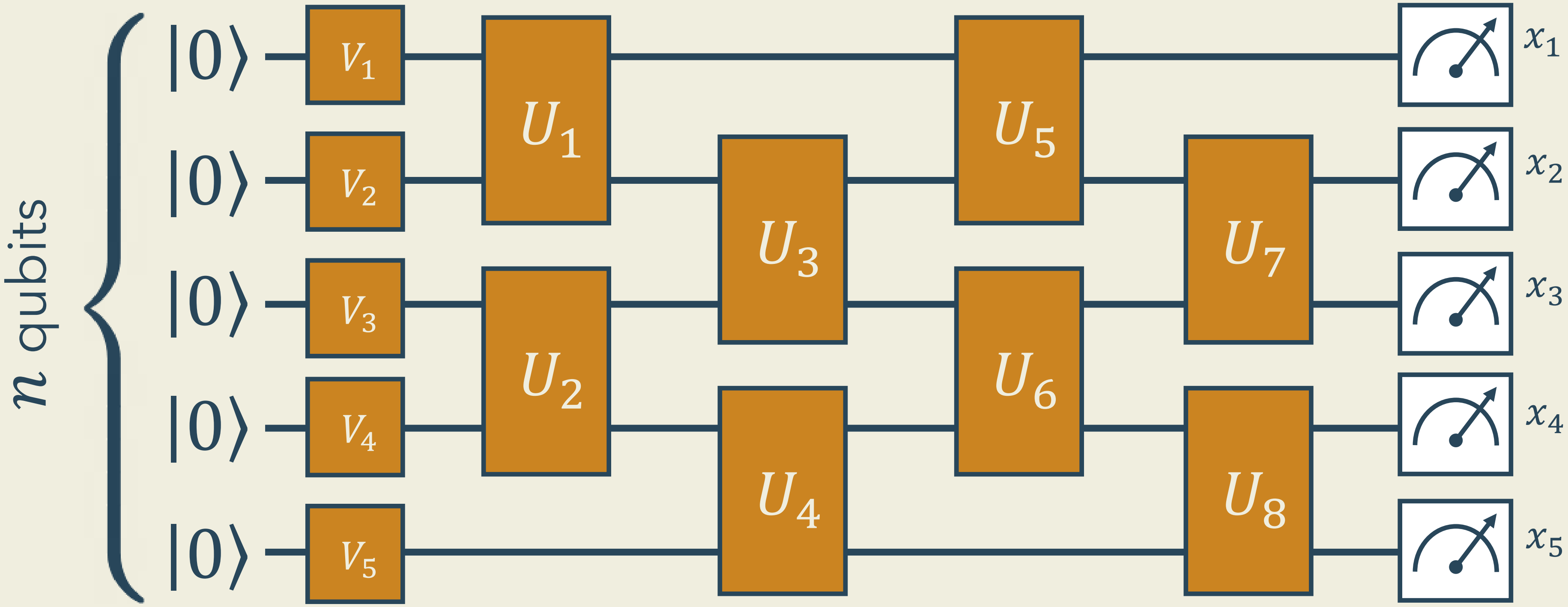
Outline

- **Definition** of anti-concentration
- **Applications** of anti-concentration for classical simulation
- Statement of **results** & interpretation
- Proof method: Mapping to a **stat mech** partition function

Random quantum circuits (RQCs)

$p_U(x) = |\langle x|U|0\rangle|^2$



Haar-random gates $V_j \sim \mathcal{U}(2)$, $U_t \sim \mathcal{U}(4)$

(sometimes consider other ensembles)

Definition of anti-concentration (AC)

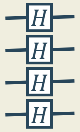
Expected Collision Probability

$Z := \mathbb{E}_U[\ \sum_x\ p_U(x)^2\ ]$ with $\mathbb{E}_U$ the average over RQC

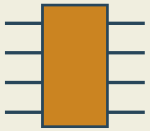
$= 2^n \mathbb{E}_U[\ p_U(0)^2\ ]$

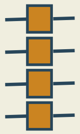
$= 2^n \mathrm{Tr}\left[\ |0\rangle\langle 0|^{\otimes 2}\ \ \mathbb{E}_U\left[U^{\otimes 2}|0\rangle\langle 0|^{\otimes 2}U^{\dagger\otimes 2}\right]\ \right]$

An RQC family is *anti-concentrated* if $Z \leq 2Z_H$ for sufficiently large n

Uniform 

$Z = \frac{1}{2^n}$

Global Haar  $Z_H := Z = \frac{2}{2^n+1}$

Single-qubit Haar  $Z = \frac{2^n}{3^n}$

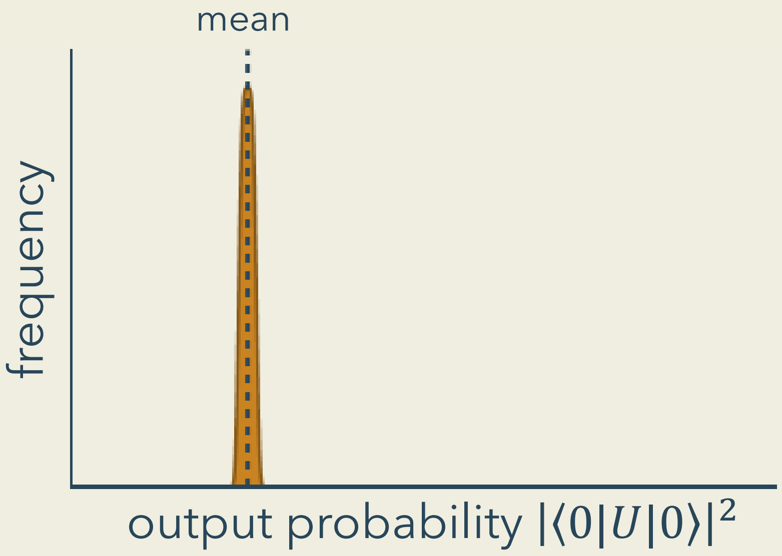
Note: If RQCs are ε-approximate 2-design, then $Z = Z_H(1 + \varepsilon/2)$

Anti-concentration means bounded variance in output probability

$$Z = 2^n \mathbb{E}_U[p_U(0)^2] \qquad \mathrm{var}\big(p_U(0)\big) = 2^{-n}Z - 2^{-2n}$$

Uniform

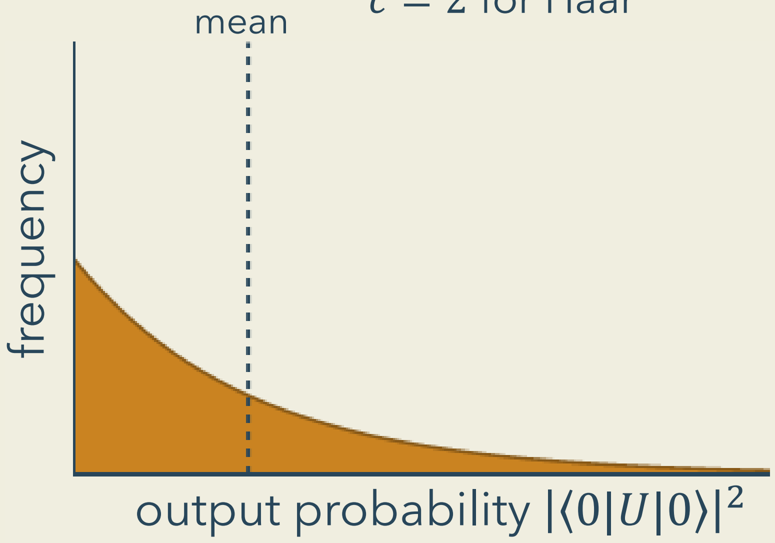
$Z \approx 2^{-n}$



Anti-concentrated

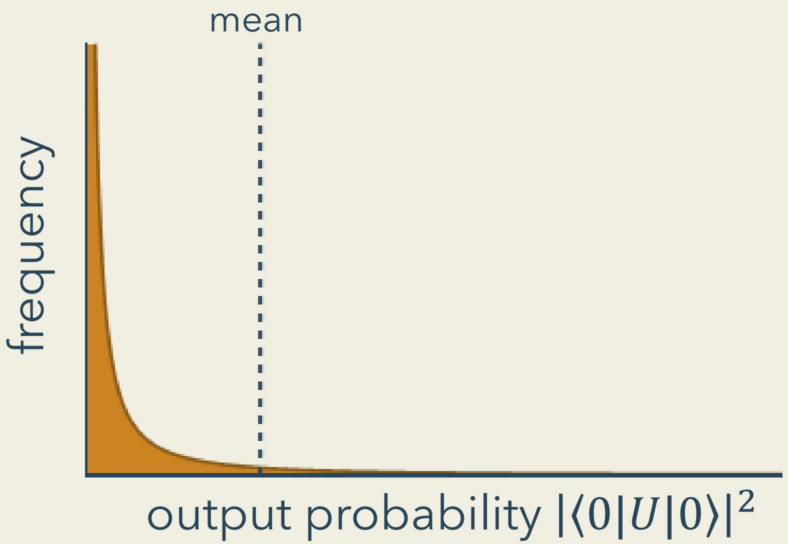
$Z \approx c \cdot 2^{-n}$

$c = 2$ for Haar



Not anti-concentrated

$Z \geq e^{n^c} \cdot 2^{-n}$



Application: When are quantum circuits hard to simulate classically?

- **Anti-concentration implies hard-to-simulate**: AC is an ingredient for some mathematical arguments underlying "quantum computational supremacy" proposals
  - Idea: To connect approximately sampling from p_U to #P-hard computational problems, need most outputs to be close to the mean

[Aaronson, Arkhipov '11] [Bremner, Montanaro, Shepherd '15] [Bouland, Fefferman, Nirkhe, Vazirani '18]

- **Anti-concentration implies easy-to-simulate**: In certain situations, there are efficient simulation algorithms assuming AC
  - Classical simulation of IQP circuits with depolarizing noise

    [Bremner, Montanaro, Shepherd '16]
  - Random ensembles of circuits with local depolarizing noise of strength ϵ approach uniform distribution allowing for easy simulation

    [Gao, Duan '18]

Anti-concentration and "spoofing" RQC experiments

- Google implemented 2D RQCs on a superconducting device and verified its outputs had non-negligible score on the Linear Cross-Entropy Benchmarking (Linear XEB) metric
- Can "spoof" Linear XEB on shallow RQCs if output is anti-concentrated
  - Depth-d circuits in D spatial dimensions can achieve ε score in time
    $$(2^n Z)\exp\left(\varepsilon 15^{-d}\right)\operatorname{poly}(n, 2^{d^D})$$
  - For $D = 1$, polynomial time if AC in $\log(n)$-depth (proved)
  - For $D = 2$, polynomial time if AC in $\sqrt{\log(n)}$-depth (conjectured)

[Barak, Chou, Gao '20]

Definition: RQC architectures

- An "RQC architecture" is a method of choosing circuit layout
- We develop a general framework that applies to any architecture
- 1D architecture: gates act on nearest-neighbor qubits arranged on a ring
- Complete-graph architecture: each gate is chosen to act on a random pair of qubits
- Higher dimensional local architectures are harder to analyze rigorously

Results: Sharp bounds on circuit size for anti-concentration

- We show upper and lower bounds on Z as a function of circuit size s
- In 1D, depth $d = 2s/n$, for complete-graph, $d = O(s \ln(n)/n)$ whp
- Using bounds on Z, we compute bounds on the "AC size" s_{AC}

Definition: s_{AC} is the minimum circuit size for which $Z \leq 2Z_H$ holds

Architecture	Upper bound on s_{AC}	Lower bound on s_{AC}
1D on a ring	[Barak, Chou, Gao '20] $\frac{1}{2\ln(5/4)} n \ln(n) + O(n)$	$\frac{1}{2\ln(5/4)} n \ln(n) - O(n)$
Complete graph	$\frac{5}{6} n \ln(n) + O(n)$	$\frac{5}{6} n \ln(n) - O(n)$
General	$O(n^2)$ **conj.** $O(n \log(n))$	$\Omega(n \log(n))$

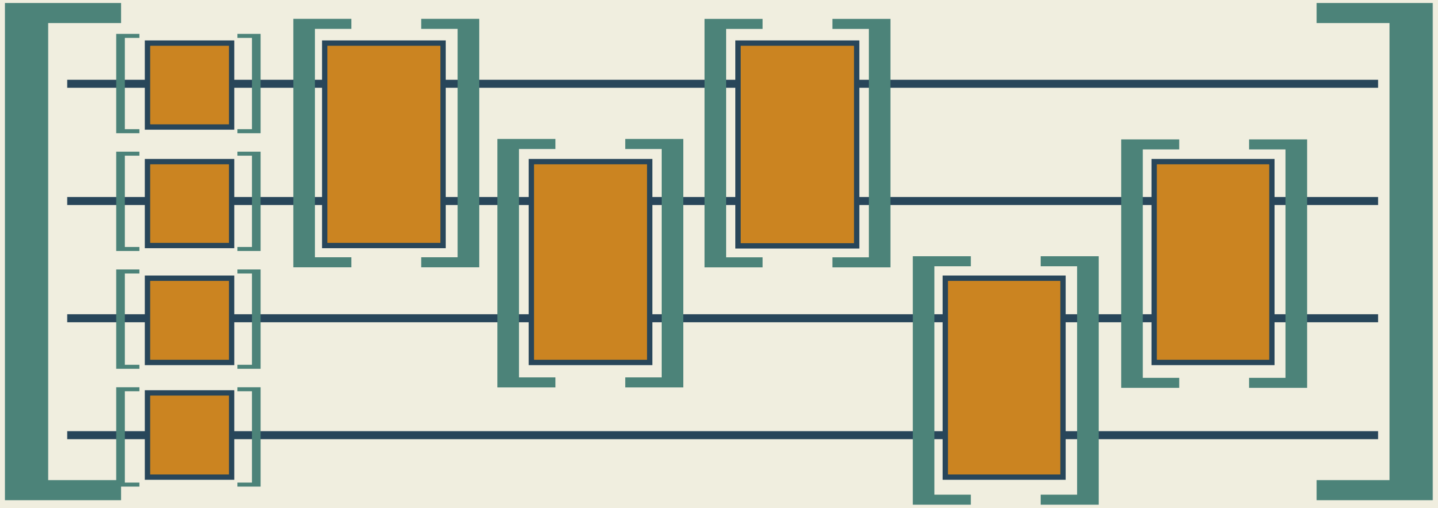
Takeaways: Anti-concentration occurs before approximate 2-design

- Upper and lower bounds match up to subleading correction, indicates power of method based on stat mech map
- Anti-concentration achieved much faster than approximate 2-design property
- Hard regime for quantum computational supremacy potentially attainable at smaller depth than previously thought
- Conjectures settled
  - [Harrow, Mehraban '18]: Constant pre-factor for AC in complete-graph is $5/6$
  - [Barak, Chou, Gao '20]: 2D circuits reach AC in $O(\sqrt{\log(n)})$ depth --- We show $\Omega(\log(n))$

Method: Perform Haar expectation over each gate individually

Recall: $Z = 2^n \mathrm{Tr}\left[|0\rangle\langle 0|^{\otimes 2} \;\; \mathbb{E}_U \left[U^{\otimes 2} |0\rangle\langle 0|^{\otimes 2} U^{+\otimes 2} \right] \right]$

Each gate is chosen independently so the expectation factorizes

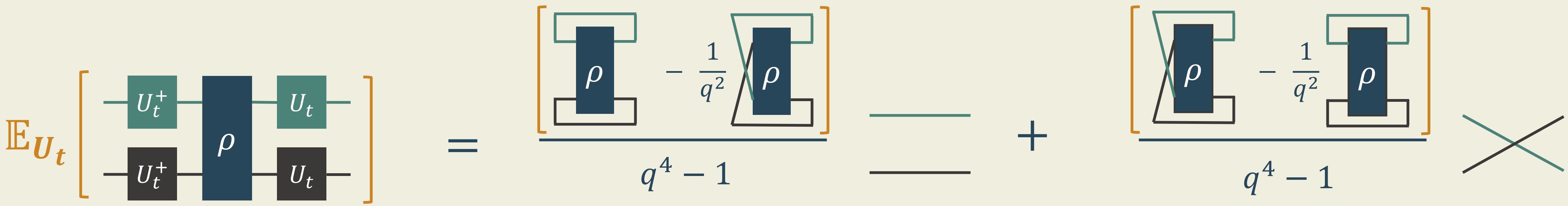


Above: $[\; U \;]$ denotes $\mathbb{E}_U[U^{\otimes 2} \otimes U^{+\otimes 2}]$

Method: Second-moment Haar formula

Key idea: Expectation formula for a single Haar-random $q^2 \times q^2$ unitary U_t expresses action of $U_t^{\otimes 2}$ as linear combination of identity (I) and swap (S) operations on two copies of the system

$$\mathbb{E}_{U_t}\left[U_t^{\otimes 2}\rho\, U_t^{+\otimes 2}\right] = \left(\frac{\mathrm{Tr}(\rho) - \mathrm{Tr}(\rho S)/q^2}{q^4 - 1}\right) (I) \quad + \quad \left(\frac{\mathrm{Tr}(\rho S) - \mathrm{Tr}(\rho)/q^2}{q^4 - 1}\right) (S)$$

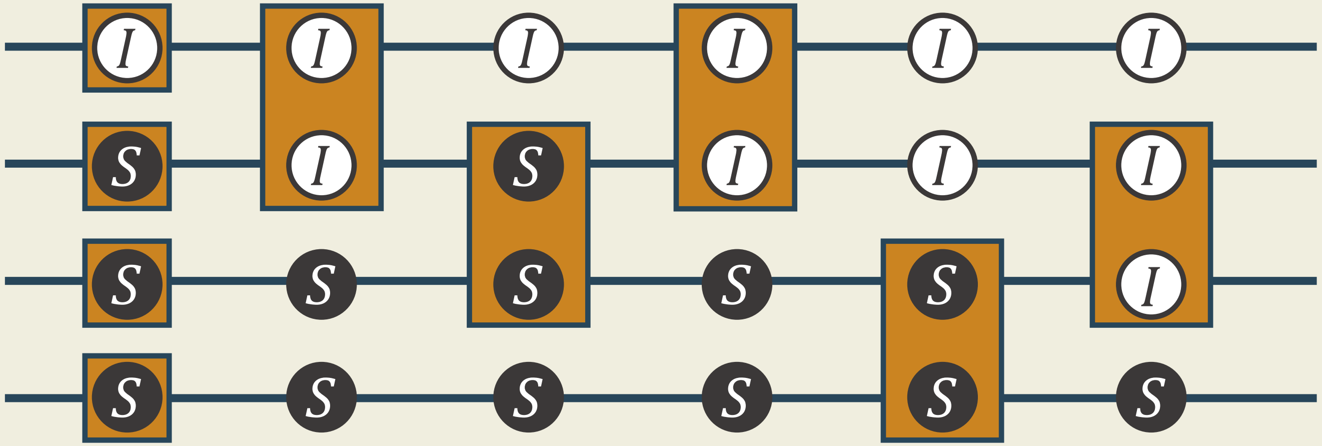


Example of a trajectory

Expectation over each gate yields linear combination of I and S. Z is given by weighted sum over I/S assignments. These assignments are organized into **trajectories** of length-n bit strings.

Rules for each gate

1. Both locations must agree
2. At most one flip
3. No gate, no flip
4. Each flip decreases weight by 2/5



$$\text{Weight} = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \times 1 \times \frac{2}{5} = \frac{16}{625}$$

Example of a trajectory that reaches a fixed point

Expectation over each gate yields linear combination of I and S. Z is given by weighted sum over I/S assignments. These assignments are organized into **trajectories** of length-n bit strings.

Rules for each gate

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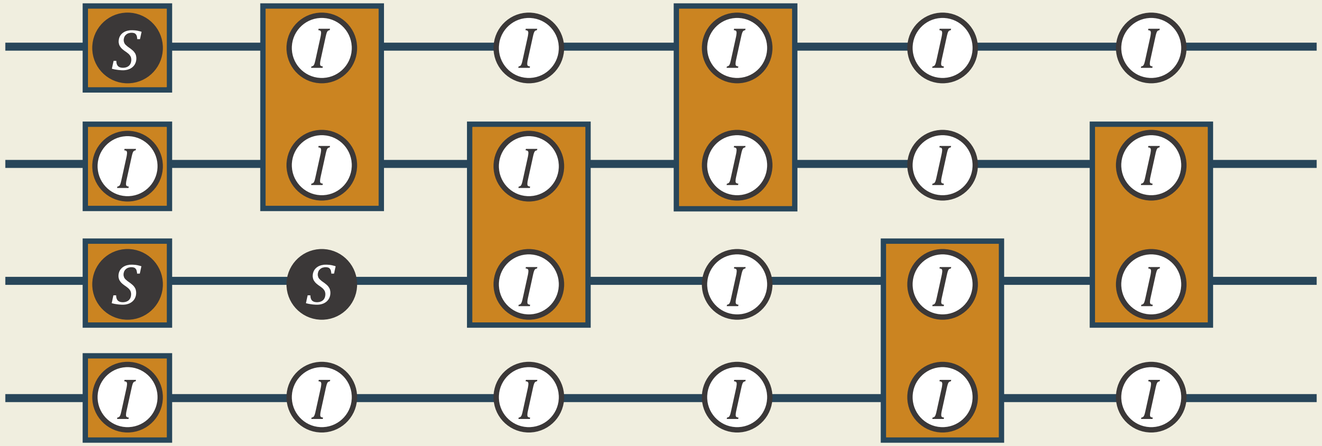


$$\text{Weight} = \frac{2}{5} \times \frac{2}{5} \times 1 \times 1 \times 1 = \frac{4}{25}$$

Collision probability as a partition function

$$Z = \left(\frac{1}{3}\right)^n \sum_{\text{trajectories}} \left[\frac{2}{5}\right]^{\#\text{ of bits flipped during trajectory}}$$

$$= \sum_{\sigma \in \{I,S\}^{n(s+1)}} \exp(-H(\sigma))$$

where $H(\sigma)$ is the sum of local interaction terms

Summing over trajectories

Lemma: A Sanity Check

Let $\mathcal{T}$ be the set of all trajectories (of any finite length) that reach a fixed point. Then, regardless of architecture

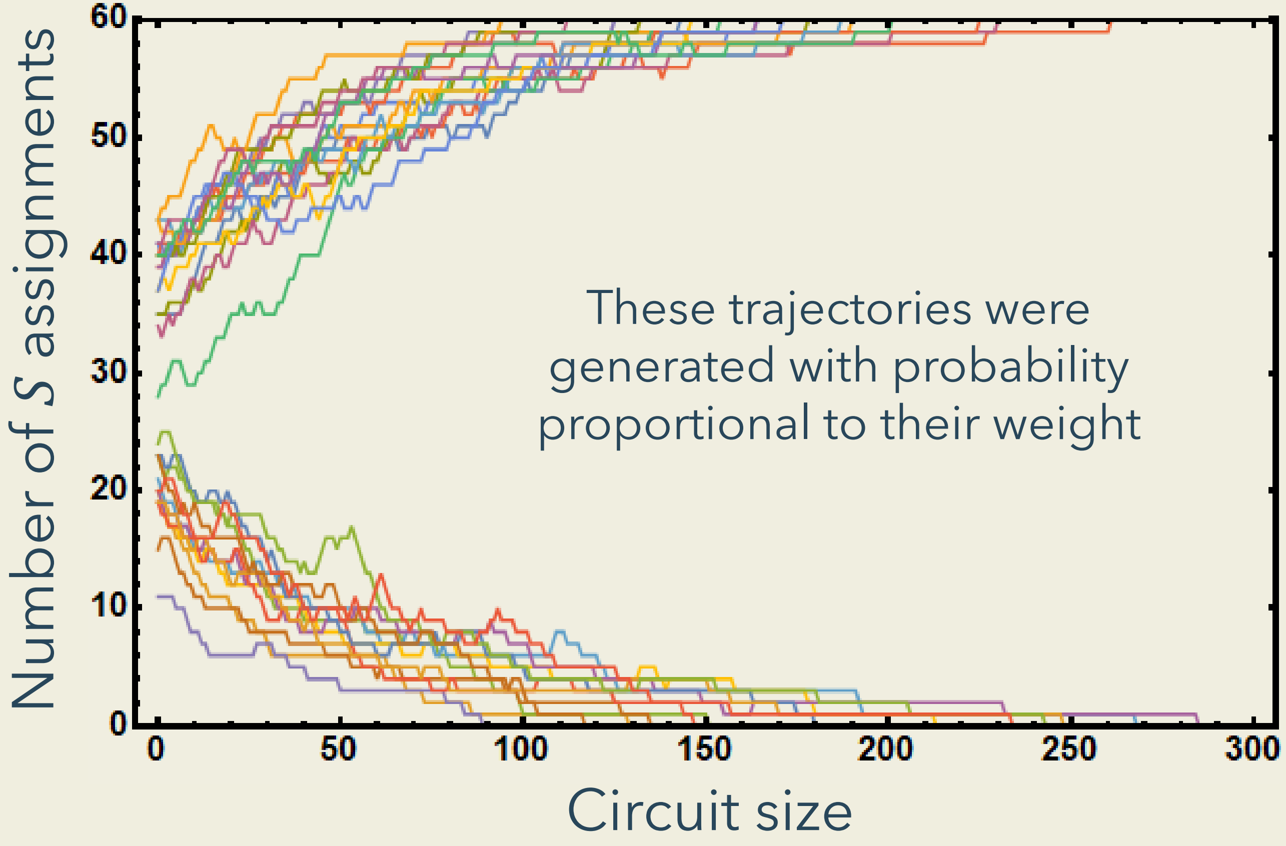
$$Z = \left(\frac{1}{3}\right)^n \sum_{\gamma \in \mathcal{T}} \mathrm{weight}(\gamma) = Z_H = \frac{2}{2^n + 1}$$

- Local RQCs of infinite size converge to global Haar
- Local RQCs of finite size have $Z > Z_H$ because some trajectories have not reached a fixed point and are overweighted

Complete graph: Typical trajectories and convergence of Z

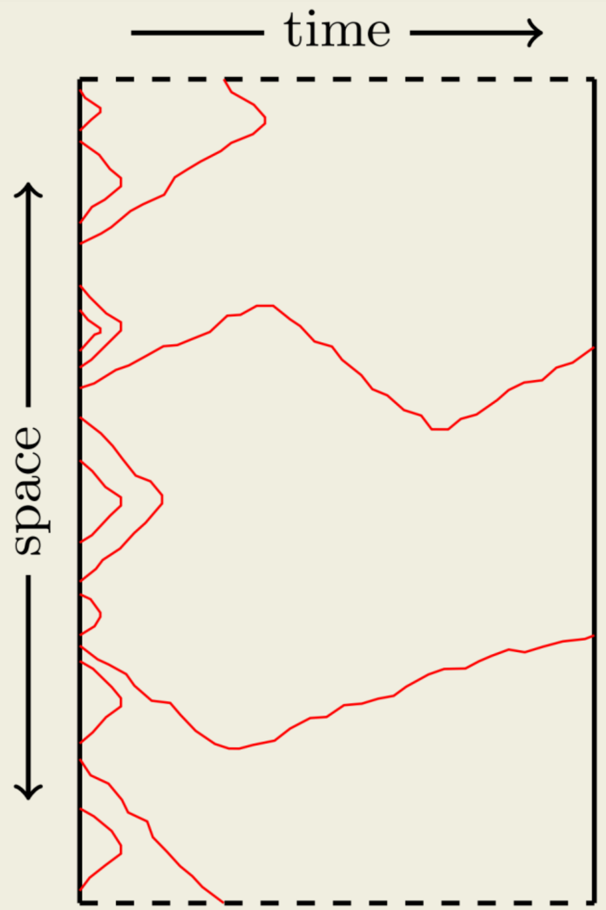
- Weight decreases with each disagreeing pair
- Typical evolutions quickly move toward fixed points I^n or S^n
- Each interval of n steps → constant fraction closer to fixed point → AC after $O(n \log(n))$ steps

Typical trajectories for $N = 60$



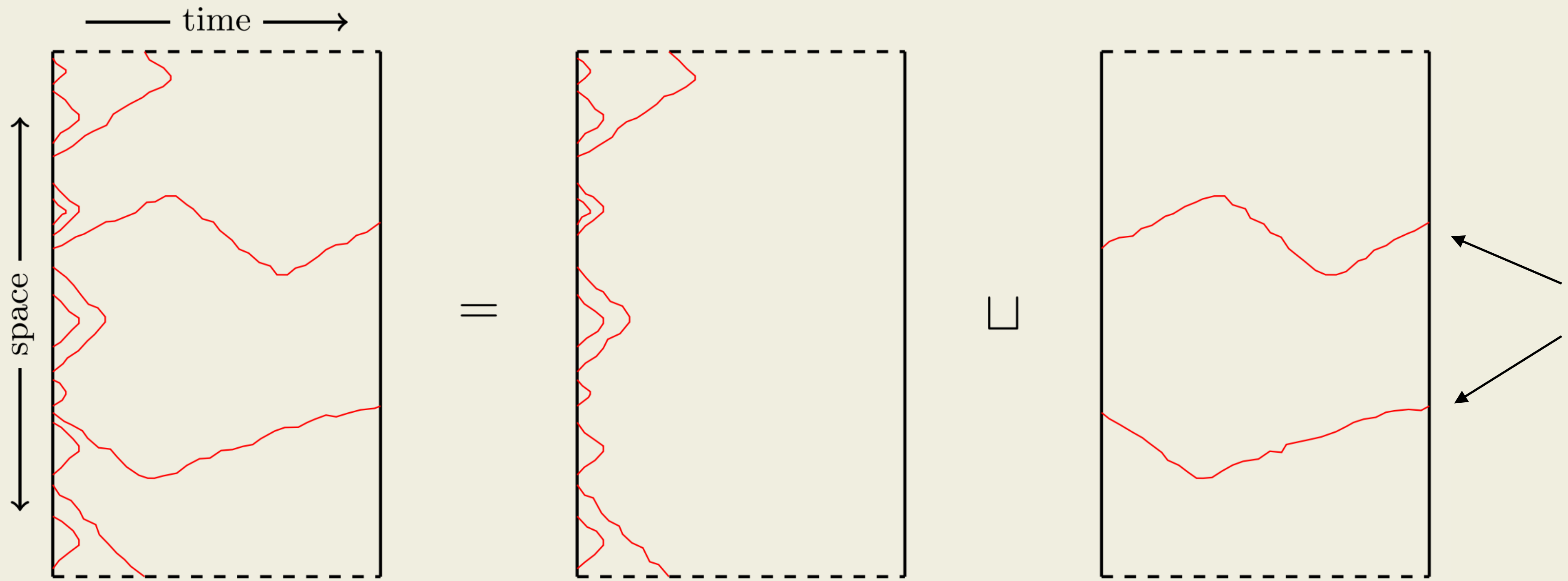
1D analysis: Focus on domain walls

- In 1D, configurations have I and S domains
- Gates cause domain walls to move one way or the other
- Domain wall pairs can annihilate but cannot be created
- Energy penalty for each domain wall



Approach builds off [Hunter-Jones '19] study of 1D 2-design time

1D analysis: Decomposing trajectories



k domain walls survive until the end

Contribution from trajectories where everything annihilates is at most Z_H

Contribution decays exponentially in kt

Note: [Barak, Chou, Gao '20] showed 1D upper bound using the same method. We also use it to show lower bound.

Conclusions and open problems

Conclusions

- $\boldsymbol{\Theta}(n\log(n))$ gates is necessary & sufficient for AC in both geometrically local and non-local architectures
- Anti-concentration faster to achieve than appr. 2-design
- Stat mech map effective for second moment calculation

Open problems

- Prove $\boldsymbol{\Theta}(n\log(n))$ scaling for 2 and higher dimensional RQCs
- Study approximate 2-design time for complete-graph architecture using stat mech map
- What can stat mech map say about RQCs with noise?