Leaking information to gain entanglement via log-singularities

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arXiv:2003.10367 (log-singularity)
arXiv:2011.15116 (Leakage)

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✓ When, at what rate, and by what means can noisy entanglement be distilled back?

- Fertile ground for exchange of research ideas

✓ Use and make contributions to Shannon Theory, Error-correction, Optimization theory,...

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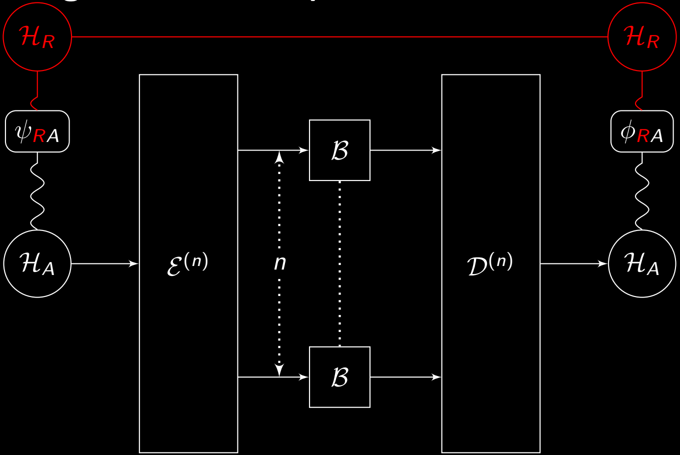
- **Theoretical tool**: Singularity in the von-Neumann entropy gives rise to positivity and non-additivity in rates for protecting entanglement

- **Conceptual surprise**: Allowing a noisy channel to leak quantum information to its environment can boosts its ability to protect entanglement.

Protection of entanglement can be quantified



$\mathcal{E}^{(n)}$ encodes and $\mathcal{D}^{(k)}$ decodes nS *qubits* of any ψ_{RA} with vanishing error as $n \mapsto \infty$

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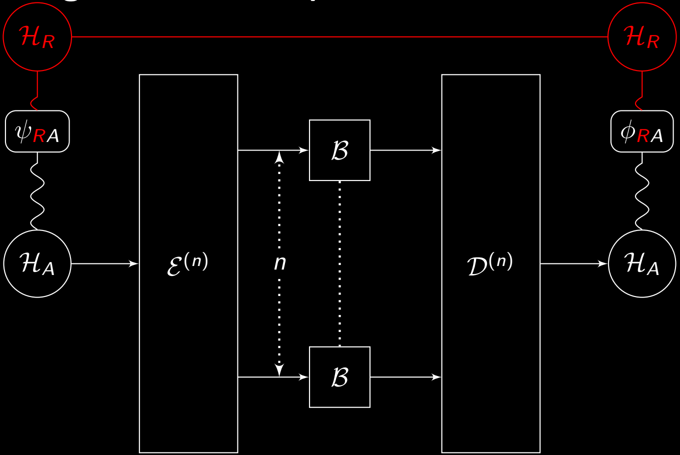

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Asymptotic Quantum Capacity $Q(\mathcal{B}) = \max S$

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- An achievable lower bound on rates for quantum key distribution

Shannon like recipe for finding an achievable rate

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$$\text{Mutual Information:} \quad I(X;Y) = H(X) + H(Y) - H(X,Y)$$

$$\text{Channel Mutual Information:} \quad C^{(1)}(\mathcal{N}) = \max_{p(x)} I(X;Y)$$

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$$\text{von-Neumann entropy:} \quad S(a) = -\mathrm{Tr}(\rho_a \log \rho_a)$$

$$\text{Coherent Information:} \quad \Delta(\mathcal{B}, \rho_a) = S(b) - S(c)$$

$$\text{Channel Coherent Information:} \quad Q^{(1)}(\mathcal{B}) = \max_{\rho_a} \Delta(\mathcal{B}, \rho_a)$$

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$$\text{Additivity:} \quad C^{(1)}(\mathcal{N} \times \mathcal{N}') = C^{(1)}(\mathcal{N}) + C^{(1)}(\mathcal{N}')$$

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$$\begin{aligned} \text{Non-Additivity:} \quad & Q^{(1)}(\mathcal{B} \otimes \mathcal{B}') \geq Q^{(1)}(\mathcal{B}) + Q^{(1)}(\mathcal{B}') \\ \text{Quantum Capacity:} \quad & Q(\mathcal{B}) = \lim_{k \mapsto \infty} Q^{(1)}(\mathcal{B}^{\otimes k})/k \geq Q^{(1)}(\mathcal{B}) \\ \text{Joint Quantum Capacity:} \quad & Q(\mathcal{B} \otimes \mathcal{B}') \geq Q(\mathcal{B}) + Q(\mathcal{B}') \end{aligned}$$

The inequalities can be strict

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Question : When and why are $\mathcal{Q}^{(1)}$ and $\mathcal{Q}$ non-additive?

Answer : Some examples, but much is unknown

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Suppose: $\lambda_1 = \epsilon, \lambda_2 = 1 - \epsilon$

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Using log-singularities, we develop

A method to check $\mathcal{Q}^{(1)} > 0$

A method to check if $\mathcal{Q}^{(1)}$ non-additive

Application 1: incomplete erasure $\mathcal{C}_g(\rho) = (1-\lambda)\mathcal{C}_1(\rho) \oplus \lambda\rho$

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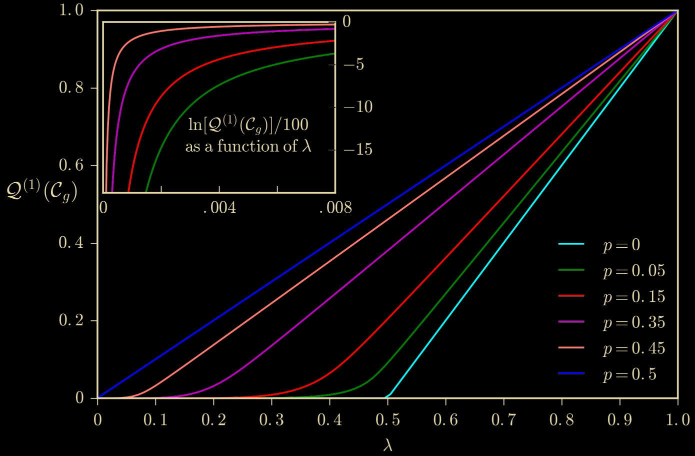
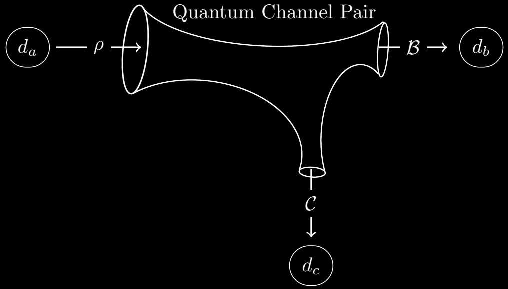


Surprisingly $\boxed{\mathcal{Q}(\mathcal{C}_1) = 0, \quad p \leq 1/2}$ but $\boxed{\mathcal{Q}^{(1)}(\mathcal{C}_g) > 0 \quad p, \lambda > 0}$

Application 2: General results about positivity of $\mathcal{Q}^{(1)}$



Theorem: If $d_c < d_b$ and $\mathcal{B}$ maps some pure state to an output of rank d_c, then $Q^{(1)}(\mathcal{B}) > 0$.

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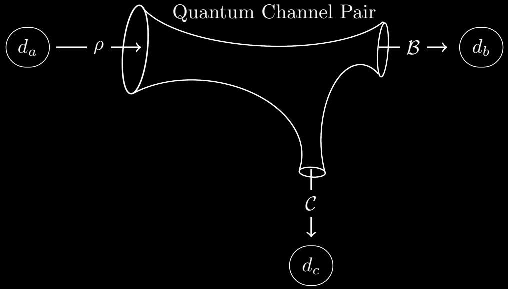

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Corollary 1: If $d_a > 1$ and $d_b > d_a(d_c - 1)$ then $\mathcal{Q}^{(1)}(\mathcal{B}) > 0$.

Corollary 2: If $\mathcal{B}$ is a qubit channel ($d_a = d_b = 2$) then $\mathcal{Q}^{(1)}(\mathcal{C}) > 0$ whenever $d_c > 2$.

New type of non-additivity using log-singularity

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$\mathcal{B}$: Qubit amplitude damping channel with $\mathcal{Q}(\mathcal{B}) = 0$

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V.Siddhu, arXiv:2003.10367 (2020)

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A simple zero capacity channel boosts error correction rates

V.Siddhu, arXiv:2003.10367 (2020)

Non-additivity can give rise to conceptual surprises

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Belief: Allowing leakage of quantum information to a channel's environment is detrimental for its ability to preserve entanglement.

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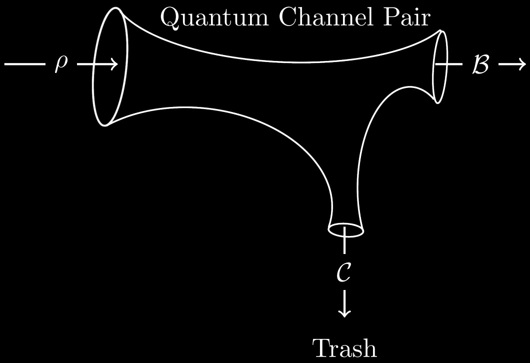
Non-additivity can give rise to conceptual surprises

Belief: Allowing leakage of quantum information to a channel's environment is detrimental for its ability to preserve entanglement.

Result: Quantum capacity can be boosted by allowing leakage of quantum information to a channel's environment.

V.Siddhu, arXiv:2011.15116 (2020)

What is the leakage procedure?



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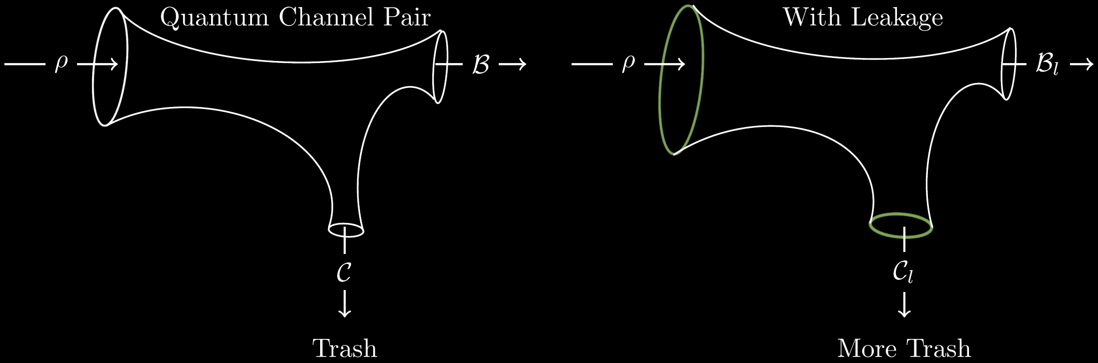
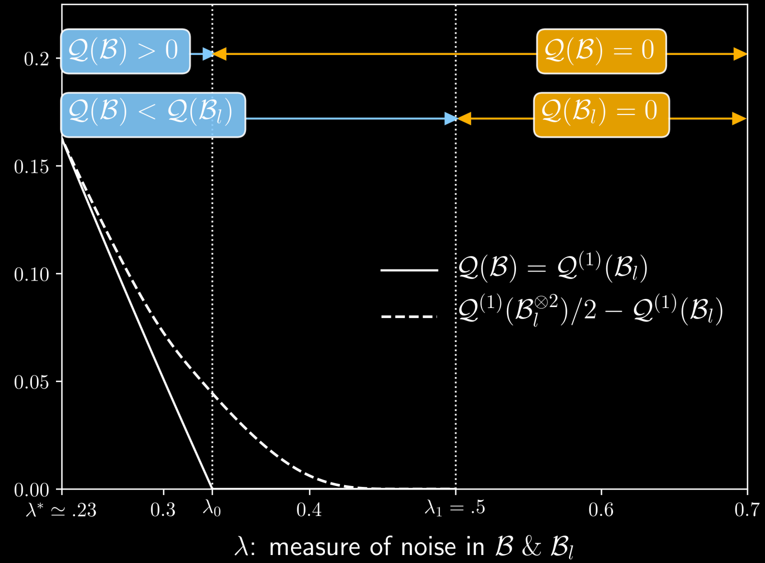


Allowing qutrit channel $\mathcal{B}$ to leak takes $\mathcal{B} \mapsto \mathcal{B}_l$



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- **Message**: Study of quantum capacities/non-additivity is a fundamental and fertile area for developing new theoretical and conceptual tools

THANK YOU