Topological Defect Networks for Fractons of all Types

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I. MAIN IDEAS

Fractons are topological excitations that are immobile due to superselection rules, they occur in exotic topological phases of matter which defy expectations based on more conventional topological quantum field theories (TQFTs). Interest in these phases was driven by Haah's discovery of the cubic code[1], and demonstration that it supports no stringlike logical operators and forms a partially self-correcting topological quantum memory at finite temperature[2,3]. The study of these phases has grown into an active subfield at the interface of quantum codes and condensed matter theory, with particular focus on exotic topological order[4–10], slow quantum dynamics[11–14] and subsystem symmetries[15–22]. Fracton topological phases present a novel challenge to the classification of quantum matter, a scheme which has otherwise largely succeeded for topological phases of matter[23]. This is especially true for topological orders in 2+1D, whose classification in terms of modular tensor categories[24,25] is widely accepted to be complete and has important applications to the theory of topological quantum computation.

To date there have been many attempts at developing a classification and understanding of all fracton phases of matter. It has proven deceptively difficult to find a unified classification of even the relatively simple class of translation-invariant, exactly solvable, gapped fracton models. This raises the following natural questions: does there exist a *unified* framework which captures *all* types of gapped fracton phases, and if so, does this framework fit within the existing TQFT landscape? In this work, we answer both in the affirmative. Rather than abandoning the TQFT framework, we instead espouse the idea of seeking out modifications to TQFTs which are fundamentally sensitive to geometry. Drawing inspiration from the field of defect TQFTs[26–31], as well as from the recent classification of crystalline SPTs[32–34], we introduce *topological defect networks* as a unified framework for the description and classification of all types of gapped fracton phases.

II. OVERVIEW OF RESULTS

In this work, we demonstrate that *all* the known types of gapped fracton models can be realized by topological defect networks. We proceed by constructing concrete examples of topological defect networks for well-known gapped fracton models, including the X-Cube (a foliated type-I model) and Haah's B code (a type-II model), as well as a fractal type-I model. We also present a new non-Abelian panoptic fracton model based on 3+1D D_4 gauge theory, which hosts fully immobile non-Abelian excitations. We further argue that topological defect networks give rise to no stable translation invariant gapped fracton phases in 2+1D.

The central ingredients in our construction are topological defects. A topological defect can be embedded into a D+1 dimensional TQFT by introducing new interactions and possibly new degrees of freedom, which are spatially localized on some lower $d < D$ dimensional region, into the system without closing the bulk gap. For a Hamiltonian, this corresponds to modifying its terms near the region specified by the defect while maintaining the energy gap. Consequently, the behavior of bulk topological excitations is modified in the vicinity of the defect. For instance, some bulk excitations may condense on the defect; or, the defect could nontrivially permute the topological superselection sectors of excitations passing through it. Both kinds of defects—anyon condensing and anyon permuting—have been introduced into the 2+1D toric code Hamiltonian, and are of interest for their applications in topological quantum computation[35,36].

A 3+1D *topological defect network* is a particular instance of a TQFT containing defects that live on a *stratified* 3-manifold $\mathcal{M}$. A stratification consists of collections S_j ($j \in [0,3]$) of j-dimensional submanifolds that decompose the manifold, as in Fig. 1. Elements of S_j are referred to as j-strata. We assign a 3+1D TQFT to each 3-strata and associate a topological defect with each

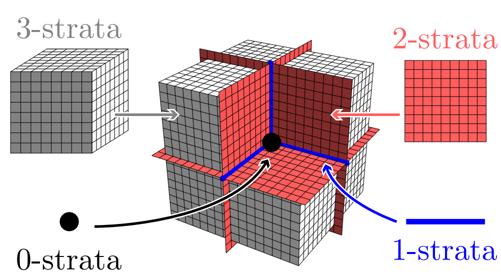

FIG. 1. A stratification of 3d space into 3-strata (white regions), 2-strata (red squares), 1-strata (blue lines), and 0-strata (black vertex). A 3+1D TQFT lives on each 3-strata, and they are coupled together along the lower-dimensional 2- and 1-strata defects. This defect TQFT could be described by a lattice model with qubits on, e.g., the black cubic lattice.

j-stratum (for $j < 3$), thereby coupling together the ambient TQFTs. This coupling, mediated by the defect network, underpins the flexibility of topological defect networks as it directly imposes the mobility constraints characteristic of fracton models. In particular, this coupling dictates the set of topological excitations that condense on a given j-strata and hence, along with the braiding data already encoded in the 3+1D TQFT, determines the set of excitations which cannot pass through that strata—any excitations which braid non-trivially with any of the condensed excitations on a defect are prohibited from passing through it. We utilize these properties to construct defect TQFTs that "trap" certain topological excitations and hence describe fracton topological orders.

By comparison, previous fracton constructions essentially take the defect network length scale to be the same as the length scale of the microscopic lattice. As emphasized in Fig. 1, in this work we are considering a microscopic lattice length scale that is much smaller than the defect network length scale. The particular lattice cellulation used for the microscopic lattice does not affect the long distance physics (e.g., the particle mobility constraints). This allows us to take a continuum limit of the microscopic lattice, yielding a field theoretic description of the resulting fracton models in terms of defect TQFTs.

III. IMPACT AND IMPORTANCE

In this work, we have introduced topological defect networks and demonstrated that they can describe a comprehensive variety of gapped fracton phases. Based on our ability to fit the broad typology of gapped fractonic matter into our framework, we expect that all fracton phases admit a defect network description. As such, we conjecture that topological defect networks realize all zero-temperature gapped phases of matter. As a byproduct of this conjecture, we have also argued that no fracton phases exist in 2+1D gapped systems, thereby demonstrating the potential of topological defect networks as tools for the classification of phases of matter. This provides a promising new route to approach the classification of all gapped phases of matter in the near future.

Another promising application of topological defect networks is to construct new fracton topological codes beyond the stabilizer formalism, which are not bound by known restrictions on topological stabilizer codes[2,3]. We have already constructed one such model that supports non-Abelian fractons based on D_4 gauge theory. Our formalism also provides a framework within which universal results on the best achievable performance of general fracton topological codes may be established by utilizing the existing theory of defect TQFTs.

[1] Jeongwan Haah, "Local stabilizer codes in three dimensions without string logical operators," Phys. Rev. A **83**, 042330 (2011).

[2] Sergey Bravyi and Jeongwan Haah, "Quantum self-correction in the 3d cubic code model," Phys. Rev. Lett. **111**, 200501 (2013).

[3] Sergey Bravyi and Jeongwan Haah, "Energy landscape of 3d spin hamiltonians with topological order," Phys. Rev. Lett. **107**, 150504 (2011).

[4] Han Ma, Ethan Lake, Xie Chen, and Michael Hermele, "Fracton topological order via coupled layers," Phys. Rev. B **95**, 245126 (2017).

[5] Sagar Vijay, "Isotropic Layer Construction and Phase Diagram for Fracton Topological Phases," (2017), arXiv:1701.00762.

[6] Sagar Vijay and Liang Fu, "A Generalization of Non-Abelian Anyons in Three Dimensions," (2017), arXiv:1706.07070.

[7] Wilbur Shirley, Kevin Slagle, and Xie Chen, "Fractional excitations in foliated fracton phases," Annals of Physics **410**, 167922 (2019).

[8] Hao Song, Abhinav Prem, Sheng-Jie Huang, and M. A. Martin-Delgado, "Twisted fracton models in three dimensions," Phys. Rev. B **99**, 155118 (2019).

[9] D. Bulmash and T. Iadecola, "Braiding and Gapped Boundaries in Fracton Topological Phases," Phys. Rev. B **99**, 125132 (2019).

[10] Arpit Dua, Dominic J. Williamson, Jeongwan Haah, and Meng Cheng, "Compactifying fracton stabilizer models," Phys. Rev. B **99**, 245135 (2019).

[11] Claudio Chamon, "Quantum glassiness in strongly correlated clean systems: An example of topological overprotection," Phys. Rev. Lett. **94**, 040402 (2005).

[12] Claudio Castelnovo and Claudio Chamon, "Topological quantum glassiness," Philosophical Magazine **92**, 304–323 (2012), arXiv:1108.2051.

[13] Isaac H. Kim and Jeongwan Haah, "Localization from superselection rules in translationally invariant systems," Phys. Rev. Lett. **116**, 027202 (2016).

[14] Abhinav Prem, Jeongwan Haah, and Rahul Nandkishore, "Glassy quantum dynamics in translation invariant fracton models," Phys. Rev. B **95**, 155133 (2017).

[15] Sagar Vijay, Jeongwan Haah, and Liang Fu, "Fracton topological order, generalized lattice gauge theory, and duality," Phys. Rev. B **94**, 235157 (2016).

[16] Dominic J. Williamson, "Fractal symmetries: Ungauging the cubic code," Phys. Rev. B **94**, 155128 (2016).

[17] Yizhi You, Trithep Devakul, F. J. Burnell, and S. L. Sondhi, "Subsystem symmetry protected topological order," Phys. Rev. B **98**, 035112 (2018).

[18] Wilbur Shirley, Kevin Slagle, and Xie Chen, "Foliated fracton order from gauging subsystem

symmetries," SciPost Physics **6**, 041 (2019).

[19] Albert T. Schmitz, "Gauge structures: From stabilizer codes to continuum models," Annals of Physics **410**, 167927 (2019).

[20] Trithep Devakul, Wilbur Shirley, and Juven Wang, "Strong planar subsystem symmetry-protected topological phases and their dual fracton orders," Phys. Rev. Research **2**, 012059 (2020).

[21] J. P. Ibieta-Jimenez, L. N. Queiroz Xavier, M. Petrucci, and P. Teotonio-Sobrinho, "Fracton-like phases from subsystem symmetries," (2019), arXiv:1908.07601.

[22] Nathanan Tantivasadakarn and Sagar Vijay, "Searching for fracton orders via symmetry defect condensation," Phys. Rev. B **101**, 165143 (2020).

[23] Xiao-Gang Wen, "Colloquium: Zoo of quantum-topological phases of matter," Rev. Mod. Phys. **89**, 041004 (2017).

[24] G Moore and N Seiberg, "Classical and quantum conformal field theory," Communications in Mathematical Physics **123**, 177–254 (1989).

[25] Alexei Kitaev, "Anyons in an exactly solved model and beyond," Annals of Physics **321**, 2–111 (2006).

[26] Scott Morrison and Kevin Walker, "The blob complex," (2010), arXiv:1009.5025.

[27] Anton Kapustin, Kevin Setter, and Ketan Vyas, "Surface operators in four-dimensional topological gauge theory and Langlands duality," (2010), arXiv:1002.0385.

[28] Alexei Davydov, Liang Kong, and Ingo Runkel, "Field theories with defects and the centre functor," (2011), arXiv:1107.0495.

[29] Nils Carqueville, "Lecture notes on 2-dimensional defect TQFT," (2016), arXiv:1607.05747.

[30] Nils Carqueville, Ingo Runkel, and Gregor Schaumann, "Line and surface defects in Reshetikhin-Turaev TQFT," (2017), arXiv:1710.10214.

[31] Nils Carqueville, Catherine Meusburger, and Gregor Schaumann, "3-dimensional defect TQFTs and their tricategories," (2016), arXiv:1603.01171.

[32] Hao Song, Sheng-Jie Huang, Liang Fu, and Michael Hermele, "Topological phases protected by point group symmetry," Phys. Rev. X **7**, 011020 (2017).

[33] Sheng-Jie Huang, Hao Song, Yi-Ping Huang, and Michael Hermele, "Building crystalline topological phases from lower-dimensional states," Phys. Rev. B **96**, 205106 (2017).

[34] Dominic V. Else and Ryan Thorngren, "Crystalline topological phases as defect networks," Phys. Rev. B **99**, 115116 (2019).

[35] S. B. Bravyi and A. Yu. Kitaev, "Quantum codes on a lattice with boundary," (1998), arXiv:quant-ph/9811052.

[36] H. Bombin, "Topological order with a twist: Ising anyons from an abelian model," Phys. Rev. Lett. **105**, 030403 (2010).