The cost of universality: A comparative study of the overhead of state distillation and code switching with color codes

Michael E. Beverland,[1] Aleksander Kubica,[2,3,4] and Krysta M. Svore[1]

[1] Microsoft Quantum and Microsoft Research, Redmond, WA 98052, USA

[2] Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada

[3] Institute for Quantum Computing, University of Waterloo, Waterloo, ON N2L 3G1, Canada

[4] AWS Center for Quantum Computing, Pasadena, CA 91125, USA

Recent progress in demonstrating operational quantum devices [1–5] has brought the noisy intermediate-scale quantum era [6], where low-depth algorithms are run on small numbers of qubits. However, to handle the cumulative effects of noise and faults as these quantum systems are scaled, fault-tolerant (FT) schemes [7–15] will be needed to reliably implement universal quantum computation. FT schemes encode logical information into many physical qubits and implement logical operations on the encoded information, all while continually diagnosing and repairing faults. This requires additional resources, and much of the current research in quantum error correction (QEC) is dedicated toward developing FT schemes with low overhead.

The choice of FT scheme to realize universal quantum computing has important ramifications. Good schemes can significantly enhance the functionality and lifetime of a given quantum computer. Moreover, FT schemes vary in their sensitivity to the hardware architecture and design, such as qubit quality [16], connectivity, and operation speed. The understanding and choice of FT scheme will therefore influence the system design, from hardware to software, and developing an early understanding of the trade-offs is critical in our path to a scalable quantum computer.

At the base of most FT schemes is a QEC code which (given the capabilities and limitations of a particular hardware platform) should: (i) tolerate realistic noise (ii) have an efficient classical decoding algorithm to correct faults, and (iii) admit a FT universal gate set. In the search of good FT schemes we focus our attention on QEC codes which are known to achieve as many of these points as possible with low overhead. Topological codes are particularly compelling as they typically exhibit high accuracy thresholds with QEC protocols involving geometrically local quantum operations and efficient decoders; see e.g. Refs. [15, 17–31]. Two-dimensional (2D) topological codes such as the toric code [32, 33] and the color code [34] are particularly appealing for superconducting [35–37] and Majorana [38, 39] hardware, where qubits are laid out on a plane and quantum operations are limited to those involving neighboring qubits.

The FT implementation of logical gates with 2D topological codes poses some challenges. The simplest FT logical gates are applied transversally, i.e., by independently addressing individual physical qubits. These gates are automatically FT since they do not grow the support of errors. Unfortunately a QEC code which admits a universal set of transversal logical gates is ruled out by the Eastin-Knill theorem [40–42]. Furthermore, in 2D topological codes such gates can only perform Clifford operations [43–46]. There are, however, many innovative approaches to achieve universality, which typically focus on implementing non-Clifford logical gates [47–49], which achieve universality when combined with the Clifford gates.

The standard approach to achieve universality with 2D topological codes is known as *state distillation* [50–52]. It relies on first producing many noisy encoded copies of a *T state* $|\overline{T}\rangle = (|\overline{0}\rangle + e^{i\pi/4}|\overline{1}\rangle)/\sqrt{2}$, also known as a *magic state*, and then processing them using Clifford operations to output a high fidelity encoded version of the state. The high-fidelity T state can then be used to implement the non-Clifford $T = \mathrm{diag}(1, e^{i\pi/4})$ gate. Despite significant recent improvements, the overhead of state distillation is expected to be large in practice [35, 53]. A compelling alternative is *code switching* via gauge fixing [54–57] to a 3D topological code which has a transversal T gate. The experimental difficulty of moving to 3D architectures could potentially be justified if it significantly reduces the overhead compared to state distillation. To compare these two approaches and find which is most practical for consideration in a hardware design, a detailed study is required.

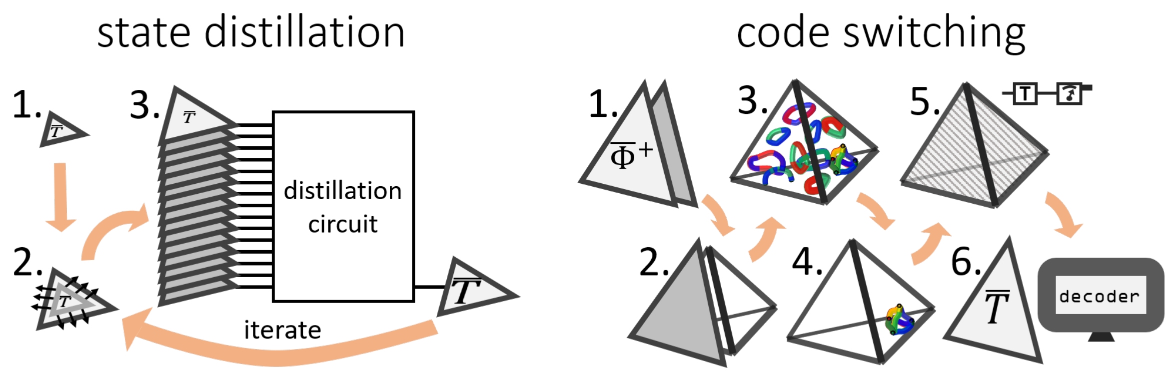

FIG. 1. Two methods of preparing high-fidelity T states encoded in the 2D color code: state distillation and code switching. Both approaches are implemented using quantum-local operations in 3D, i.e., noisy quantum operations are geometrically local, whereas ideal classical operations can be performed globally. In state distillation, many noisy encoded T states are produced and fed into a Clifford distillation circuit. In code switching, one switches to the 3D color code, where the transversal T gate is implemented.

In our work, we estimate the resources needed to prepare high-fidelity T states encoded in the 2D color code, via either state distillation or code switching. We assume that both approaches are implemented using quantum-local operations [58] in 3D, i.e., quantum operations are noisy and geometrically local, whereas classical operations can be performed globally and perfectly (although they must be computationally efficient). In particular, we simulate these two approaches by implementing them with noisy circuits built from single-qubit state preparations, unitaries and measurements, and two-qubit unitaries between nearby qubits. For state distillation, this 3D setting allows a stack of 2D color code patches, whereas for code switching it allows to implement the 3D color code; see Fig. 1. We then seek to answer the following question: *to prepare T states of a given fidelity, are fewer resources required for state distillation or code switching?*

Our main finding is that code switching does not offer substantial savings over state distillation in terms of both *space overhead*, i.e., the number of physical qubits required, and *space time overhead*, i.e., the space overhead multiplied by the number of physical time units required; see Fig. 2. State distillation significantly outperforms code switching over most of the circuit noise error rates $10^{-4} \leq p \leq 10^{-3}$ and target T state infidelities $10^{-20} \leq p_{\text{fin}} \leq 10^{-4}$, except for the smallest values of p, where code switching slightly outperformed state distillation. In our analysis we carefully optimize each step of code switching, and also investigate the effects of replacing each step by an optimal version to account for potential improvements. On the other hand we consider only a standard state distillation scheme, and using more optimized schemes such as [59–61] would give further advantage to a state distillation approach. We also find asymptotic expressions which support our finding that state distillation requires lower overhead than code switching for $p \ll 1$ and $\log p_{\text{fin}}/\log p \gg 1$. In particular, the space and space-time overhead scale as $(\log p_{\text{fin}}/\log p)^{\Gamma_*}$ and $(\log p_{\text{fin}}/\log p)^{\Gamma_*+1}$ respectively, where $\Gamma_{\text{CS}} = 3$ for code switching and $\Gamma_{\text{SD}} = \log_3 15 = 2.46\ldots$ for the distillation scheme we implement.

To arrive at our main simulation results, we accomplish the intermediate goals below.

2D color code optimization and analysis.—We first adapt the projection decoder [30] to the setting where the 2D color code has a boundary and syndrome extraction is imperfect, as well as optimize the stabilizer extraction circuits. We find a circuit noise threshold greater than

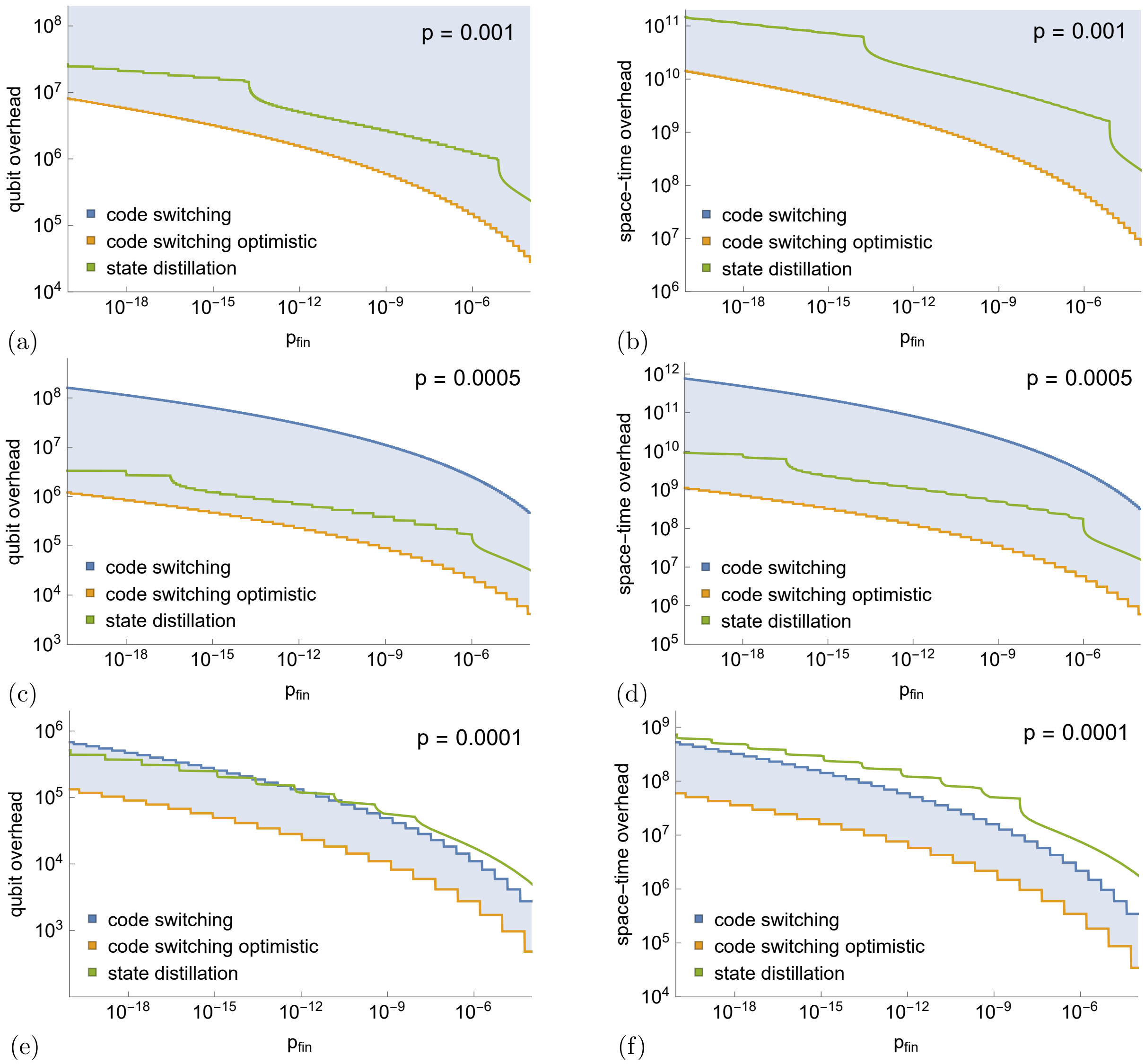

FIG. 2. The comparison of (a) the qubit and (b) space-time overhead as a function of the infidelity p_{fin} of the output T state for state distillation and code switching. Possible future improvements of any steps of our code switching protocol would be included within the shaded region. Note there is no code switching curve for $p = 0.001$ without assuming optimistic improvements to the protocol as this is higher than the observed threshold for code switching.

0.37(1)%, which is the highest to date for the 2D color code, narrowing the gap to that of the surface code. We also analyze the noise equilibration process during logical operations in the 2D color code and provide an effective logical noise model.

Noisy state distillation analysis.—Using the effective logical noise model, we carefully analyse the overhead of state distillation. We strengthen the bounds on failure and rejection rate by explicitly calculating the effect of faults at each location in the Clifford state distillation circuits rather than simply counting the total number of locations [35, 53, 62–64]. We remark that we stack 2D color codes in the third dimension to implement logical operations such as the CNOT in constant time, whereas strictly 2D approaches such as lattice surgery would require a time proportional to code distance. The circuit-noise threshold for this state distillation scheme with the 2D color code is equal to the error correction threshold of 0.37(1)%.

Further insights into 3D color codes.—We provide a surprisingly direct way to switch between the 2D color code and the 3D subsystem color code. Our method exploits a particular gauge-fixing of the 3D subsystem color code for which the code state admits a local tensor product structure in the bulk and can therefore be prepared in constant time. We also adapt the restriction decoder [31] to the setting where the 3D color code has a boundary and optimize it, which results in a threshold of 0.80(5)% and a better performance for small system sizes.

End-to-end code switching simulation.—This is the culmination of our work, where building upon results from the previous sections we provide a simplified recipe for code switching, detailing each step and specifying important optimizations. In our simulation, we exploit the special structure of the 3D subsystem color code to develop a method of propagating noise through the T gates in the system, despite the believed computational hardness of simulating general circuits with many qubits and T gates. We numerically find the failure probability of implementing the T gate with code switching as a function of the code distance and the circuit noise strength, which, in turn, allows us to estimate the T gate threshold to be 0.07(1)%. We not only find numerical estimates of the overhead of the fully specified protocol, but also bound the minimal overhead of a code switching protocol with various conceivable improvements, such as using optimal measurement circuits, and optimal classical algorithms for decoding and gauge fixing of the 3D color code.

This work provides a much-needed comparative study of the overhead of state distillation and code switching, and enables a deeper understanding of these two approaches to FT universal quantum computation. More generally, careful end-to-end analyses with this level of detail will become increasingly important to identify the most resource-efficient FT schemes and, in turn, to influence the evolution of quantum hardware. Although our study focuses on color codes, we expect our main finding, i.e., that code switching does not significantly outperform state distillation, to hold for other topological codes such as the toric code as considered in Ref. [49]. Furthermore, we believe that state distillation will not be outperformed by code switching exploiting either 2D subsystem codes [65–67] or emulation of a 3D system with a dynamical 2D system [68–71] since these schemes are even more constrained than when 3D quantum-local operations are allowed. We remark that there are other known FT techniques for implementing a universal gate set [12, 72–76], however they are not immediately applicable to large-scale topological codes. Nevertheless, we are hopeful that there are still new and ingenious FT schemes to be discovered that could dramatically reduce the overhead and hardware requirements for scalable quantum computing.

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