Entanglement bootstrap program

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1 Introduction

One of the fundamental questions in quantum many-body physics is a classification of quantum phases. In one-dimensional (1D) systems with a spectral gap, quantum information approach has provided the complete classification [1, 2, 3]. In higher dimensions, a rigorous classification of quantum phases remains an open problem.

Quantum phases (without symmetry) in two-dimensional (2D) gapped systems are expected to be classified into different *topologically ordered phases*, which cannot be described by the conventional theory of symmetry-breaking. A vast amount of work has been dedicated to classify/characterize these new phases, and nowadays, it is widely believed that all topologically ordered phases can be classified by topological quantum field theory (TQFT) [4]. In the framework of TQFT, different topological orders are distinguished by the algebraic theory of excitations called anyons. Although the existing analytical studies [5, 6] as well as numerical studies [7, 8, 9] are consistent with the prediction of this theory, we still lack a general, concrete proof that directly shows that any 2D gapped quantum phases should be classified in this framework.

In this submission, we propose the *entanglement bootstrap program* to make progress on this important problem. In this program, we posit that local reduced density matrices of such systems obey a set of entropic identities. Specifically, imagine that B, C, and D in Fig. 1 are subsystems of a two-dimensional quantum many-body system, each of which consists of many microscopic degrees of freedom e.g., qubits. These subsystems partition a ball of finite radius which is independent of the system size. We assume that, for every subsystem over a bounded-radius ball, once we partition the subsystem into the set of subsystems described in Fig. 1, the following conditions hold:

$$\begin{aligned} &S(BC) + S(C) - S(B) = 0 \\ &S(BC) - S(B) + S(CD) - S(D) = 0, \end{aligned} \tag{1}$$

where $S(X) = -\mathrm{Tr}(\rho_X \log \rho_X)$ is the entanglement entropy of a subsystem $X \in \{B, C, BC, CD\}$.

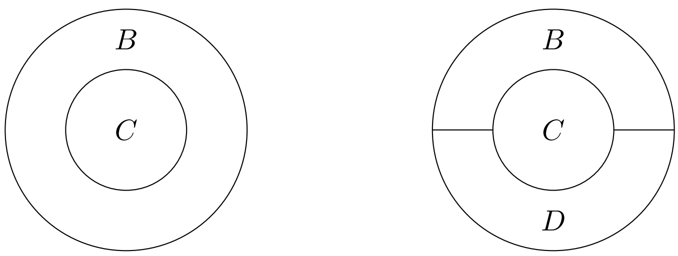

Figure 1: Subsystems relevant to our axioms.

We believe these constraints encompass a large class of many-body quantum states of interest because Eq. (1) is a simple consequence of the following equation [10, 11]:

$$S(X) = \alpha\ell - \gamma + \varepsilon \tag{2}$$

where ℓ is the boundary size of a subsystem X, α is a non-universal coefficient, and γ is a universal constant known as the topological entanglement entropy. ε is a correction term vanishing in the limit of $X \to \infty$, which we assume to be $\varepsilon = 0$ for simplicity. It should be noted that Eq. (2) has been numerically verified in many model Hamiltonians, which are both at [6] and away [7, 8] from the renormalization-fixed points. In particular, the ε term in Eq. (2) is exactly 0 at the fixed point [11]. The entropic identities we posit follow from Eq. (2) but may hold more generally. Also, while our results pertain to the case in which ε is exactly zero, we believe that these results would be naturally extended to the cases with $\varepsilon \neq 0$, since the theoretical tools we used in the proofs have approximate analogs (see Discussion of the attached papers).

From our entropic identities, we first rigorously derived the axioms of the fusion rules of anyon [12]. In addition, building upon these results, one of us has shown that the nontrivial mutual braiding statistics of anyons can be determined in our framework via the unitary S-matrix [13]. Thus, these works establish a deep connection between entanglement and the anyon theory [4]. Moreover, by applying the ideas in Ref. [12] to the *domain walls* between two different topological phases, we discovered a number of new facts that have been hitherto unknown [14]. In the presence of domain walls, some of the second axiom in Fig. 1 is relaxed, leading to a more intricate and general set of identities. Surprisingly, we discovered new kind of topological charges (or equivalently, superselection sectors) which are new to the best of our knowledge. Therefore, the entanglement bootstrap approach can lead to genuinely new discoveries in condensed matter physics.

2 Summary of Results

We consider a quantum many-body spin system $\mathcal{H}$ defined on a 2D closed manifold. In this work, we assume that there is a state σ on $\mathcal{H}$ satisfying the area law (2) locally everywhere. We refer to this state as the *reference state*. One may interpret this state as a ground state of some gapped Hamiltonian, but we do not require the existence of such Hamiltonian in our derivation.

Our framework employs the notion of *information convex set*. This is a convex set of density matrices on a region (i.e., a subsystem) that are indistinguishable from the reference state on any constant-sized ball within the region. We find that this object, under our assumptions, must obey a nontrivial set of constraints. These constraints are used to deduce the existence of globally well-defined superselection sectors and nontrivial identities that they have to satisfy.

Here is a list of our results that we find noteworthy.

1. **The isomorphism theorem**
   We show that the information convex sets associated with two topologically equivalent regions are *isomorphic* in a sense we describe in Theorem III.10 of Ref. [12]. They can be mapped into each other by a linear bijective map, and moreover, these maps preserve the distance and the entropy difference between the elements of the information convex set. Therefore, the structure of the information convex set only depends on the topology of the region associated with it. Its generalization in the presence of a domain wall is discussed in Ref. [14].

2. **A well-defined notion of topological charge**
   In Ref. [12], we show that the extreme points of an information convex set over an annulus are orthogonal to each other; any state ρ in the information convex set of an annulus must have the following form (Theorem IV.1):
   $$\rho = \bigoplus_a p_a \, \sigma^a \,,$$
   where $\{p_a\}$ is a probability distribution and $\{\sigma^a\}$ is a finite set of mutually orthogonal states (extreme points) fixed by the choice of the region.

   We can interpret the label a as an anyon type of the system. Indeed, for exactly solvable models such as the toric code, σ^a corresponds to a reduced state of an excited state with a pair of excitations between the hole and the outside. Different charges can be perfectly distinguished from each other, and they are globally well-defined. We furthermore prove that for each charge, there exists a unique antiparticle (Sec. IV. C).

In Ref. [14], we significantly extend this work in the presence of a gapped domain wall. It turns out that there are multiple types of superselection sectors, which are intimately related to each other via nontrivial identities.

3. **Extracting fusion multiplicities**
   In Ref. [12], we show that the information convex set of a 2-hole disk is isomorphic to the convex hull of orthogonal state spaces on certain finite-dimensional Hilbert spaces (Theorem IV.4, Theorem IV.5). Each orthogonal component physically corresponds to the situation that two fixed charges have a fixed total charge. This Hilbert space is interpreted as the fusion space, and their dimensions are interpreted as the fusion multiplicities. In Ref. [14], we derive the analogous identities in the presence of domain walls.

4. **Axioms of the fusion rules**
   We show that our definition of the fusion multiplicities satisfies all the axioms of the algebraic theory of anyons. The proof is based on the so-called merging technique [15] employing the structure of quantum Markov states [16]. We derived several consistency equations by "merging" two regions into another region with a different topology. The comparison before and after merging implies the fusion multiplicities should satisfy the axioms to be consistent. An analogous set of axioms is derived in the presence of domain walls as well.

5. **The topological entanglement entropy**
   It has been observed that the topological entanglement entropy γ in Eq. (2) is the logarithm of a quantity called total quantum dimension. The number only depends on the type of the topologically ordered phase. From our definition of the fusion multiplicities, we further derive this formula without making any assumption about the Hamiltonian or the underlying effective field theory as in the original papers [10, 11].

   Moreover, in Ref. [14] we derive an analog of topological entanglement entropy on domain walls. This expression is new to the best of our knowledge.

Thus, our results provide a direct link between topological order and the entanglement property of quantum many-body states in 2D systems. We believe our result is an important step towards the classification of gapped systems because, to our knowledge, it is the first work that attempts to begin with a microscopic assumption that is plausible from a physical standpoint. In contrast, in previous approaches such as the one based on tensor network [17] or the ones based on tensor category [4], one makes a nontrivial assumption about some algebraic rule that underlies the TQFT. Also, alternative traditional approaches based on operator algebra are defined for relativistic systems [18] or infinitely large spin systems [19]. The significance of our work precisely lies in the fact that we can *derive* these algebraic rules from an arguably more physical assumption, i.e., Eq. (1). Therefore our framework opens up a concrete route (i.e., proving the area law or Eq. (1)) to classify 2D gapped quantum phases without resorting to ad-hoc assumptions. Our work strongly suggests that the fundamental properties of anyons are "encoded" in a single ground state, not in the Hamiltonian nor in the degenerate ground states. Moreover, as we have shown in Ref. [14], there are even new facts we can learn about topological order from our approach. These results suggest that the entanglement bootstrap approach can be a powerful new tool to understand the rich physics of topological order.

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