Entanglement bootstrap program

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February 2, 2021

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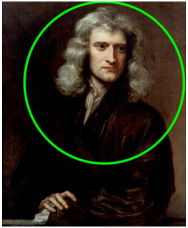

An allegory

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Inertial mass = Gravitational mass

$$m_g = m_i$$

This led to many important consequences, one example being the proof of Kepler's third law of planetary motion:

$$\frac{R^3}{T^2} = \text{Constant},$$

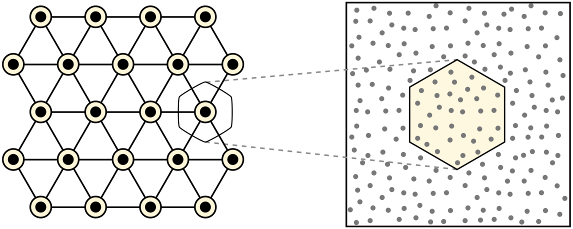
where R is the radius of the orbit of a planet and T is its period.

This talk

This talk is about simple "axioms" for many-body quantum systems.

- They are easy to verify.
- From the axioms, known equations are derived.
- New equations are also derived, just from these axioms alone.

Setup

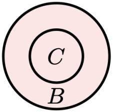


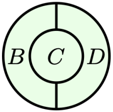
Let Λ be the set of sites. The Hilbert space has a tensor product structure:

$$\mathcal{H} = \bigotimes_{i \in \Lambda} \mathcal{H}_i,$$

where $\dim(\mathcal{H}_i) < \infty$.

Axioms



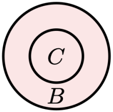


$(S_C + S_{BC} - S_B)_\sigma = 0$ $\quad$ $(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$

for every ball of $\mathcal{O}(1)$ radius, where

$$S_C := -\mathrm{Tr}(\sigma_C \log \sigma_C)$$

is the entanglement entropy of C over the global state σ.



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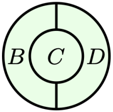
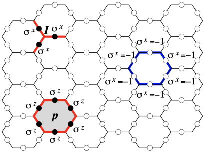
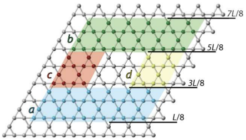


$$(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$$

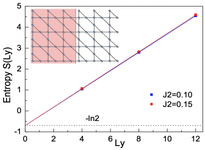
These identities look very special, but they are not.[1]



Levin and Wen (2006)

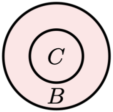


Isakov et al. (2011)



Jiang et al. (2012)

[1] At least, if we ignore the approximation error and some counterexamples.



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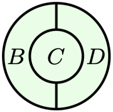
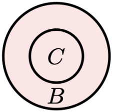


$$(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$$

What is the space of all possible quantum states consistent with these constraints?

Entanglement Bootstrap



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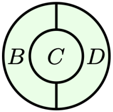

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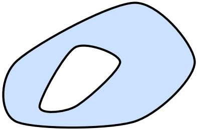

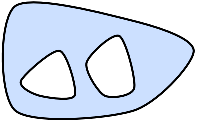

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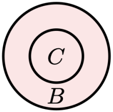

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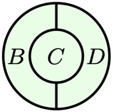

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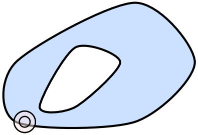

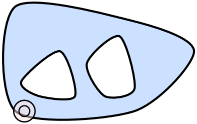

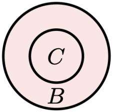

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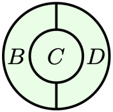

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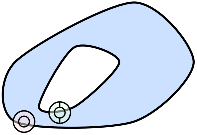

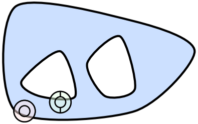

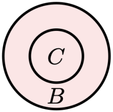

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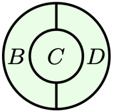

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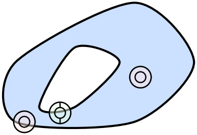
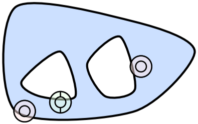




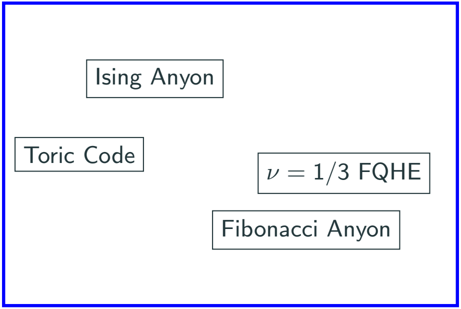
Topological Order

Certain gapped quantum many-body systems in two spatial dimensions (2D) can exhibit **topological order**. The following signatures of topological order are well-known.

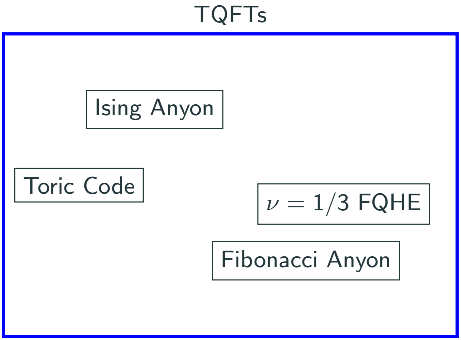
1. Locally indistinguishable degenerate ground states [Wen and Niu (1990)]
2. Anyons [Leinaas and Myrheim (1977), Wilczek (1982)]
3. Fractionalized excitations (with symmetry) [Laughlin (1983)]

Currently, the unifying mathematical language that describes 2D topological order is Topological Quantum Field Theory (**TQFT**). [Witten (1998)]

TQFTs



- Each TQFT is defined in terms of a discrete set of data.
  $d_a, N_{ab}^c, F_u^{abc}, R_{ab}, \ldots$

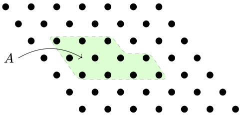


- Each TQFT is defined in terms of a discrete set of data.
  $d_a, N_{ab}^c, F_u^{abc}, R_{ab}, \ldots$
- These data satisfy a set of equations which hold for **all** TQFTs. **ex)**

$$d_a d_b = \sum_c N_{ab}^c d_c,$$

$$\sum_i N_{ab}^i N_{ic}^d = \sum_j N_{aj}^d N_{bc}^j, \ldots$$

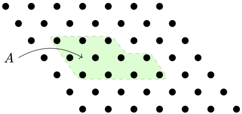
Kitaev and Preskill (2006) and Levin and Wen (2006) derived the following equation for 2D ground states, assuming TQFT is a valid description.



$$S(A) = a|\partial A| - \gamma, \quad \gamma = \ln \sqrt{\sum_a d_a^2},$$

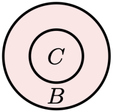
where d_a is the **quantum dimension** of an anyon a.[2]

[2] See [Zhang et al. (2011), Haah (2016), Shi (2020)] for related works.



$$S(A) = a|\partial A| - \gamma.$$

This equation actually implies our axioms.



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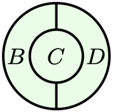
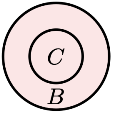


$(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$

However, these axioms themselves may hold more generally.

Entanglement bootstrap: Entanglement $\to$ TQFT



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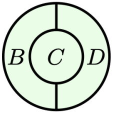

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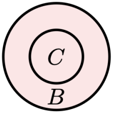
What is the space of all possible quantum states consistent with these constraints?



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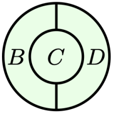

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What is the space of all possible quantum states consistent with these constraints?
A: We don't know yet, but this space may be, in some sense, equivalent to the space of all possible 2D TQFTs!

Entanglement bootstrap $\leftrightarrow$ TQFT?

If true, this would be interesting for two reasons.

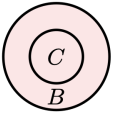
- Fundamental reasons:
    - As a theory, TQFT obeys certain charge conservation laws. Normally, charge conservation follows from symmetry. Here, we only have entanglement, no symmetry, no Hamiltonian.
    - We may be able to learn new things about TQFTs.
- Practical reason: Detecting concrete signatures of topological order from ground states.

This talk

1. Important concepts
    - Information convex set
    - Structure theorems (for information convex set)
    - Merging theorem
2. Applications

Information convex set

[Kim (2015), Shi, Kato, and Kim (2020)]



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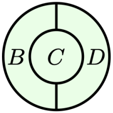
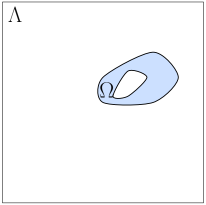
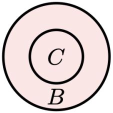


$(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$

Consider a subsystem, say $\Omega \subseteq \Lambda$.



[Kim (2015), Shi, Kato, and Kim (2020)]



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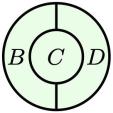

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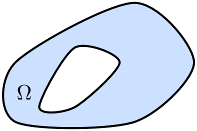

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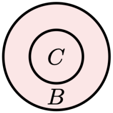

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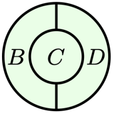

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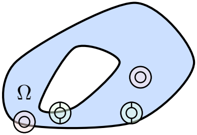
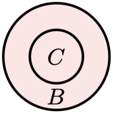
Demand our axioms on every $\mathcal{O}(1)$-sized ball that intersects with this subsystem.



Information convex set

[Kim (2015), Shi, Kato, and Kim (2020)]



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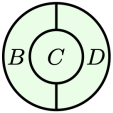
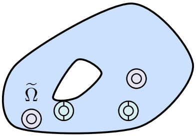


$$(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$$

To demand such constraint, we need to "thicken" Ω slightly, say to $\widetilde{\Omega}$. Let $\widetilde{\Sigma}(\Omega)$ be the set of **all** density matrices satisfying this constraint.



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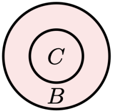

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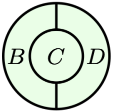
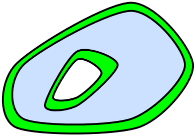


$$(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$$

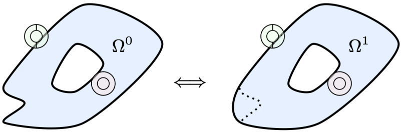
Trace out $\widetilde{\Omega} \setminus \Omega$ (the green region).



What one then obtains is the *information convex set*, denoted as:

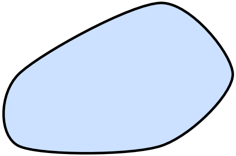
$$\boxed{\Sigma(\Omega).}$$

Structure theorems



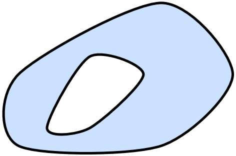
$$\Sigma(\Omega^0) \cong \Sigma(\Omega^1)$$

if there is a "smooth" path between Ω^0 and Ω^1.



Disk

For any disk D, $\Sigma(D)$ contains a unique element. [Kim (2014)]



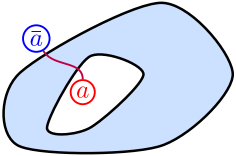
Simplex theorem (A special instance)

For any annulus X,

$$\Sigma(X) = \left\{ \bigoplus_n p_a \rho_a : \mathrm{Tr}(\rho_a) = 1, \rho_a \geq 0 \right\},$$

where different ρ_as are orthogonal to each other.

Extreme points of the simplex = Topological charges



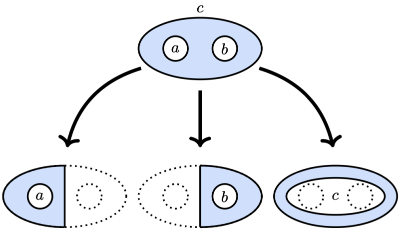
Simplex theorem (A special instance)

For any annulus X,

$$\Sigma(X) = \left\{ \bigoplus_n p_a \rho_a : \mathrm{Tr}(\rho_a) = 1, \rho_a \geq 0 \right\},$$

where different ρ_as are orthogonal to each other.

Extreme points **define the topological charges/superselection sectors** of the underlying theory.
ex) The electric/magnetic particles of the toric code.



Hilbert space theorem (A special instance)

For a two-holed disk Y, consider a subset of $\Sigma(Y)$, denoted as $\Sigma_{ab}^{c}(Y)$ by demanding the state to be consistent with the extreme points corresponding to a, b, and c.

$$\Sigma_{ab}^{c}(Y) \cong \text{State space of some finite-dimensional Hilbert space.}$$

Structure theorems $\to$ TQFT data

- Shi, Kato, and Kim (2019): The extreme points of the simplex and the abstract Hilbert spaces correspond to the known topological charges and their fusion spaces.

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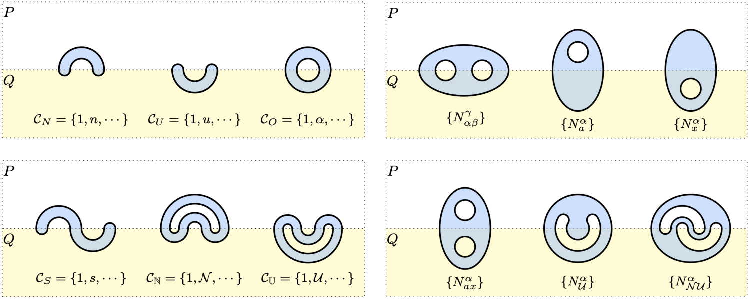
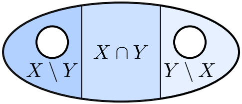
- Shi and Kim (2020): Some of the extreme points and the abstract Hilbert spaces seem to correspond to hitherto **unknown** topological charges and their fusion spaces! (They all follow from simply **removing** some of the axioms.)



Merging

Note: We are **not** talking about THE quantum state merging...

[Horodecki, Oppenheim, and Winter (2005)]



Merging theorem (informal)

Let $\rho_X \in \Sigma(X)$ and $\lambda_Y \in \Sigma(Y)$. If these states are consistent over $X \cap Y$ and are the "maximum-entropy states" (subject to some linear constraints),

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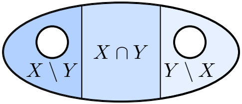

Merging theorem (informal)

Let $\rho_X \in \Sigma(X)$ and $\lambda_Y \in \Sigma(Y)$. If these states are consistent over $X \cap Y$ and are the "maximum-entropy states" (subject to some linear constraints), there exists a density matrix

$$\rho_X \bowtie \lambda_Y \in \Sigma(X \cup Y),$$

which is consistent with both ρ_X and λ_Y.

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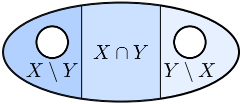

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$$\rho_X \bowtie \lambda_Y \in \Sigma(X \cup Y),$$

which is consistent with both ρ_X and λ_Y. Moreover,

$$S(\rho_X \bowtie \lambda_Y) = S(\rho_X) + S(\lambda_Y) - S(\rho_{X \cap Y}),$$

which impies that $\rho_X \bowtie \lambda_Y$ is again a "maximum-entropy state."

* A related lemma first proven by Kato et al. (2016).

Structure theorems and merging

Structure theorems

1. Annulus: $\Sigma(\Omega) \cong$ Simplex. Moreover, the extreme points are orthogonal to each other.
2. Disk with two holes: $\Sigma(\Omega) \cong$ Convex hull of state spaces of finite-dimensional Hilbert spaces.

Structure theorems and merging

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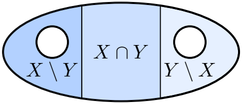
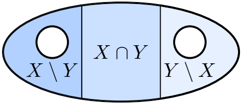


$$S(\underbrace{\rho_X \bowtie \lambda_Y}_{\text{Max Entropy}}) = S(\rho_X) + S(\lambda_Y) - S(\rho_{X\cap Y}),$$

Using the structure theorems, the maximum entropies can be expressed in a closed form, in terms of the entropies of the extreme points, leading to nontrivial **identities**.

Merging → TQFT equations



$$S(\underbrace{\rho_X \bowtie \lambda_Y}_{\text{Max Entropy}}) = S(\rho_X) + S(\lambda_Y) - S(\rho_{X\cap Y}),$$

We can apply the merging theorem by choosing $\rho_X \in \Sigma(X)$ and $\lambda_Y \in \Sigma(Y)$ to be the extreme points, (say ρ_a and ρ_b), deriving

$$\boxed{d_a d_b = \sum_c N_{ab}^c d_c,}$$

where

$$d_a = \exp\left(\frac{S(\rho_a) - S(\sigma)}{2}\right).$$

Recall that $\Sigma_{ab}^c(Y)$ is isomorphic to the state space of some Hilbert space. N_{ab}^c is the dimension of that Hilbert space.

This equation **defines** $d_a, d_b, \ldots$, also known as the *quantum dimensions.*

Merging → Known TQFT equations

We can recover the **fusion rule** of anyons. Denote

$$\mathcal{C} = \{1, a, b, c, \dots\}$$

be the set of topological charges.

$$\begin{aligned} N_{ab}^{c} &= N_{ba}^{c} \\ N_{a1}^{c} = N_{1a}^{c} &= \delta_{a,c} \\ \forall a \in \mathcal{C}, \exists \bar{a} \in \mathcal{C} \quad \text{s.t.} \quad N_{ab}^{1} &= \delta_{b\bar{a}} \\ N_{ab}^{c} &= N_{\bar{b}\bar{a}}^{\bar{c}} \\ \sum_{i \in \mathcal{C}} N_{ab}^{i} N_{ic}^{d} &= \sum_{j \in \mathcal{C}} N_{aj}^{d} N_{bc}^{j} \end{aligned}$$

* Topological S-matrix and the Verlinde formula can be also obtained.

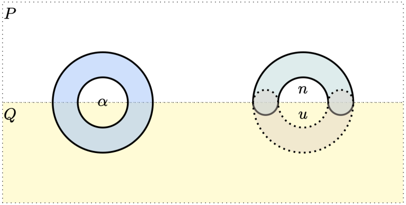
[Shi (2020)]

* However, we do not recover *all* the equations of the TQFT (aka Unitary modular tensor category). This remains as an open problem.

Merging → New TQFT equations

We can also obtain identities, which, to the best of our knowledge, are new.

ex)



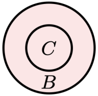
$$d_n^2 d_u^2 = \frac{\sum_{\alpha \in \mathcal{C}^{[n,u]}} d_\alpha^2}{\sum_{\alpha \in \mathcal{C}^{[\mathbf{1},\mathbf{1}]}} d_\alpha^2},$$

where

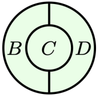
$$d_a = \exp\left(\frac{S(\rho_a) - S(\rho_1)}{2}\right),$$

and many more...

Summary



$(S_C + S_{BC} - S_B)_\sigma = 0$

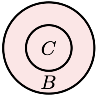


$(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$

Our axioms impose highly nontrivial restrictions on the information convex set.

1. Structure theorems determine the "shape" of the information convex sets.
2. Merging theorem leads to nontrivial identities between the elements of the information convex sets, which are precisely the TQFT identities.

Summary



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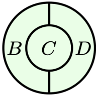

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1. Structure theorems determine the "shape" of the information convex sets.
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Applications

1. Deriving the known TQFT identities just from entanglement.
2. Deducing the existence of new topological charges.
3. Deriving new TQFT identities for the new topological charges.

Future directions

- Generalizations
  - Higher dimensions
  - Codimension-2 defects
  - Symmetry
- The merging theorem is very powerful. What else can we do?
  - One possibility: Constructing a nontrivial solution to the quantum marginal problem, in a related context. [See the poster session **C.2.12**.]
- Is there a more general theorem/method that deduces conservation laws from entanglement?
- Can you reproduce our result just using TQFT?