

3-Fermion topological quantum computation [1]

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I. INTRODUCTION

Topological quantum computation (TQC) is currently the most promising approach to scalable, fault-tolerant quantum computation. In recent years, the focus has been on TQC with Kitaev's toric code [2], due to its high threshold to noise [3, 4], and amenability to planar architectures with nearest neighbour interactions. To encode and manipulate quantum information in the toric code, a variety of techniques drawn from condensed matter contexts have been utilised. In particular, some of the efficient approaches for TQC with the toric code rely on creating and manipulating gapped-boundaries, symmetry defects and anyons of the underlying topological phase of matter [5–10]. Despite great advances, the overheads for universal fault-tolerant quantum computation remain a formidable challenge. It is therefore important to analyse the potential of TQC in a broad range of topological phases of matter, and attempt to find new computational substrates that require fewer quantum resources to execute fault-tolerant quantum computation.

In this work we present an approach to TQC for more general anyon theories based on the Walker–Wang models [11]. This provides a rich class of spin-lattice models in three-dimensions whose boundaries can naturally be used to topologically encode quantum information. The two-dimensional boundary phases of Walker–Wang models accommodate a richer set of possibilities than stand-alone two-dimensional topological phases realized by commuting projector codes [12, 13]. The Walker–Wang construction prescribes a Hamiltonian for a given input (degenerate) anyon theory, whose ground-states can be interpreted as a superposition over all valid worldlines of the underlying anyons. Focusing on a particular instance of the Walker–Wang model [13] based on the 3-Fermion anyon theory (**3F** theory) [14, 15], we show that the associated ground states can be utilised for fault-tolerant measurement-based quantum computation (MBQC) [5, 16–18] via a scheme based on the braiding and fusion of lattice defects constructed from the symmetries of the underlying anyon theory. The Walker–Wang MBQC paradigm that we introduce provides a general framework for finding resource states for computation. For example, we show that the well-known topological cluster state scheme for MBQC of Ref. [5] is produced when the toric code anyon theory is used as input to the Walker–Wang construction.

II. MAIN CONTRIBUTIONS AND RESULTS

- *A topological quantum computation scheme using symmetry defects of the **3F** theory.*

We find a universal scheme for TQC where all Clifford gates can be fault-tolerantly implemented and magic states can be noisily prepared and distilled [19]. In contrast to the 2D toric code, the full Clifford group is obtained by braiding and fusing symmetry twist defects, owing to the richer symmetry group of **3F**.

The **3F** anyon theory describes superselection sectors $\{1, \psi_r, \psi_g, \psi_b\}$ with $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules $\psi_\alpha \times \psi_\alpha = 1$, $\psi_r \times \psi_g = \psi_b$, where $\alpha = r, g, b$, and modular matrices

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \quad (1)$$

The above S matrix matches the one for the anyonic excitations in the toric code, but the topological spins in the T matrix differ as ψ_r, ψ_g, ψ_b are all fermions. These modular matrices suffice to specify the gauge invariant braiding data of the theory [20], while the F symbols are trivial. The **3F** theory has an S_3 group of global symmetries corresponding to arbitrary permutations of the three fermion species, all of which leave the gauge invariant data of the theory invariant. We denote the group action on the three fermion types r,g,b, using cycle notation $S_3 \cong \{(), (rg), (gb), (rb), (rgb), (rbg)\}$ with the usual composition, e.g. $(rg) \cdot (gb) = (rgb)$. The action on the anyons is then given by $g \cdot 1 = 1, g \cdot \psi_c = \psi_{g.c}$.

Domain walls and twist-defects corresponding to these symmetries can be constructed, and can be used to encode quantum information. The main type of encoding we consider for our computational scheme is called a g -encoding, $g \in S_3$, which consists of four twist defects, as shown in Fig. 1. Logical operations on these encoded qubits are achieved by performing fusions and braids of these twists. For the $\mathbf{3F}$ theory, we can achieve all Clifford operations (including Clifford unitaries, Pauli preparations and measurements) by manipulating the symmetry defects.

Proposition 1. (Clifford universality of $\mathbf{3F}$ defect theory). For any 2-cycles $g \neq h \in S_3$ any Clifford operation can be implemented on g, h -encoded qubits by braiding and fusion of twists. A generating set of Clifford unitaries is shown in Fig. 1.

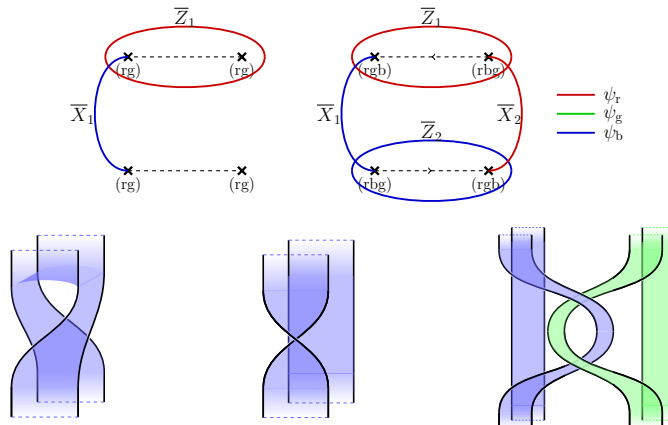


FIG. 1. (Top row) Representative fermionic string operators for logical Pauli operators for a g -encoding. (top left) A single qubit is encoded in four twists defined by $(rg) \in S_3$. (top right) Two qubits are encoded in four twists defined by $(rgb), (rbg) \in S_3$. (Bottom row) Space-time defect configurations for Clifford unitaries, with simulated time moving upwards. (bottom left) The Hadamard gate. (bottom middle) The S gate. (bottom right) The two qubit CZ gate. Domain walls for $g \in S_3$ are colored according to the fermion that they leave invariant.

- *Fault-tolerant measurement based quantum computation with Walker–Wang ground-states.*

We give a general prescription for using ground states of Walker–Wang models based on abelian anyon theories as resources for fault-tolerant measurement-based quantum computation. The Walker–Wang construction prescribes a Hamiltonian for a given input (degenerate) anyon theory, whose ground-states can be interpreted as a superposition over all valid worldlines of the underlying anyons. For abelian anyon theories, we show how symmetries of the anyon theory can be promoted to symmetries of the Walker–Wang Hamiltonian, and subsequently used to construct symmetry defects.

To perform a computation using Walker–Wang resource states, we first prepare the states with symmetry defects for a specific computation. To drive the computation, measurements in the local anyon basis are performed. In this computational framework, errors that occur during the computation (e.g. errors on the state or measurement) manifest as violations of the local anyon conservation law in the post measured state (called the history state). We can decode such errors using standard decoding methods such as minimum weight perfect matching. Errors and the corresponding syndromes are shown in Fig. 3. The scheme can be achieved in a 2D architecture by preparing (and measuring) the state layer by layer. This process is shown in Fig. 2.

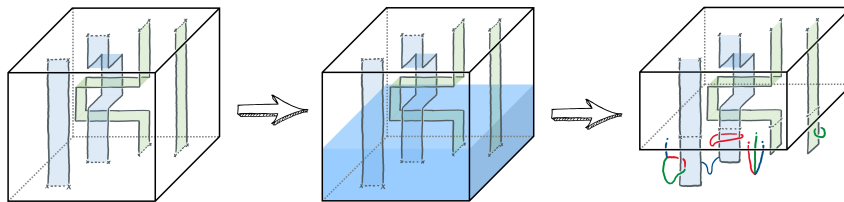


FIG. 2. Fault-tolerant MBQC with the $\mathbf{3F}$ Walker–Wang model. (left) We prepare the ground state of the Walker–Wang model with defects and twists. The ground state is given by a weighted superposition of all braided anyon worldlines (middle) The computation is driven by performing local measurements in the local anyon basis (as shaded in blue). In the case of 3-fermion, the measurement that achieves this is the Pauli- X basis. (right) The post measured state is given by a fixed anyon worldline string-net called a history state.

- *Performance of fault-tolerant measurement-based quantum computation with $3\mathbf{F}$ Walker–Wang ground-states.*

We explicitly provide a prescription for how to implement symmetry defects for the $3\mathbf{F}$ Walker–Wang model – which can be realised as a stabilizer model in 3-dimensions – and how to use it for fault-tolerant MBQC. For a simple (toy) error model consisting of IID Pauli errors and measurement errors with rate p , we show that the threshold for the scheme is identical to the well-known toric code memory threshold in with the same error model [3] (equivalently the threshold for topological cluster state MBQC [5]).

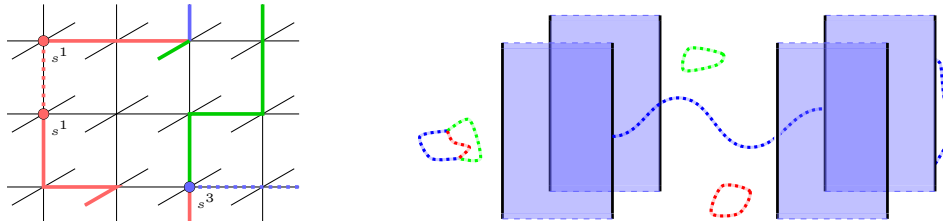


FIG. 3. Errors and error-correction in Walker–Wang MBQC. Syndromes observed in Walker–Wang MBQC. (left) Syndromes are observed as violations of local anyon conservation. For the $3\mathbf{F}$ model, lines are color-coded according to the observed measurement outcomes corresponding to the basis $|\mathbf{1}\rangle := |++\rangle$, $|\psi_r\rangle := |-\rangle$, $|\psi_g\rangle := |+-\rangle$, $|\psi_b\rangle := |--\rangle$. Possible errors producing the observed syndrome are displayed by dashed lines. Nontrivial syndromes $s_v = (a, b) \in \mathbb{Z}_2^2$ on each vertex are observed due to violations of local anyon conservation. For example, $s^1 = (1, 0)$ and $s^3 = (1, 1)$ arises from ψ_r and ψ_b string errors, as depicted. (right) Undetectable errors in Walker–Wang MBQC depicted by dashed lines. The homologically trivial loops do not result in a logical error. The central error depicted in blue that extends between different twists results in a logical error.

- *Symmetry-protected computational phases of matter*

We ground our computational framework in the context of symmetry-protected topological (SPT) phases of matter. In particular, we explore the relationship between the fault-tolerance properties of our MBQC scheme and the underlying 1-form symmetry-protected topological order of the Walker–Wang resource state. While the 3D topological cluster state (of Ref. [5]) has the same \mathbb{Z}_2^2 1-form symmetries as the $3\mathbf{F}$ Walker–Wang ground state, they belong to distinct SPT phases. These examples provide steps toward a more general understanding of fault-tolerant, computationally universal phases of matter [21–29].

- *Realisation of $3\mathbf{F}$ defects in a 2D subsystem code*

Finally, we find another setting for the implementation of our computation scheme by demonstrating how symmetry defects can be introduced into the 2D subsystem color code of Bombín [30, 31], which supports a $3\mathbf{F}$ 1-form symmetry and a $3\mathbf{F}$ anyon phase. By demonstrating how the symmetries of the emergent anyons are represented by lattice symmetries, we open up the possibility of an alternative formulation of the $3\mathbf{F}$ TQC scheme based on deformation of a subsystem code in (2+1)D – this may be of practical advantage for 2D architectures where 2-body measurements are preferred. Our construction of symmetry defects in this subsystem code may be of independent interest.

III. IMPACT AND FUTURE DIRECTIONS

We have presented a new universal scheme for topological quantum computation based on the $3\mathbf{F}$ state. We give an explicit recipe for how to realise this scheme, and other based on abelian anyon theories and their symmetries in the context of MBQC using Walker–Wang ground states. Our computation scheme based on the defects of the $3\mathbf{F}$ anyon theory provides a nontrivial example of the power of the Walker–Wang approach, as the $3\mathbf{F}$ anyon theory is chiral and cannot be realized as the emergent anyon theory of a 2D commuting projector model. Further, our results lead to a wide class of potential resource states for fault-tolerant MBQC.

Looking forward, these results provide a new avenue to find more resource efficient schemes for quantum computation. We hope that this example provides an intriguing step into topological quantum computation using more general anyon schemes and a launch point for the study of further non-stabilizer models. For instance, using a recent extension of the Walker–Wang model which is capable of realizing an arbitrary symmetry-enriched topological order on the boundary under a global on-site symmetry action [32].

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