3-fermion topological quantum computation

based on arXiv:2011.04693

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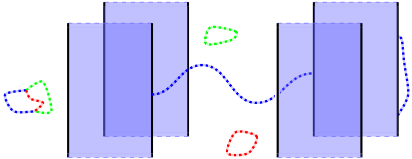
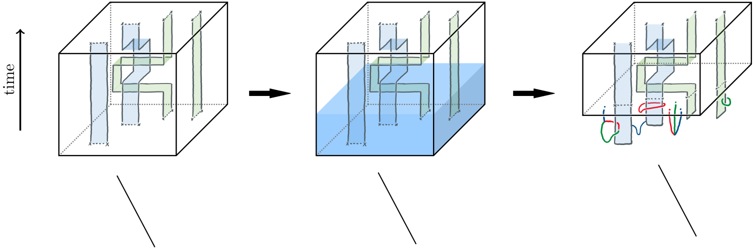
Motivation: Fault-tolerant quantum computation

- Topological quantum computation is very promising for scalable, fault-tolerant QC.
- Fault-tolerance overheads still very large.
  - Logic gates heavily restricted: Eastin–Knill '08, Bravyi–Koenig '12, Beverland *et al.* '14
- Rich topological phases in 2D and 3D still relatively unexplored.



Main result

- For an abelian anyon theory $\mathcal{C}$ (e.g. 3-Fermion) with symmetry S



Walker-Wang model with symmetry defects

Measure in local anyon basis

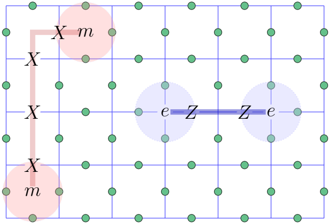
Topological quantum computation in anyonic history state

Generalises the 3D cluster state scheme of Raussendorf, Harrington, Goyal '07

Ingredients of topological quantum computation

- Anyons $=$ excitations of 2D local Hamiltonians (e.g. toric code).

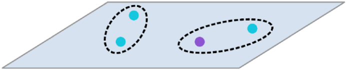
$$H_{TC} = -\sum_p \begin{smallmatrix} & Z & \\ Z & & Z \\ & Z & \end{smallmatrix} \; -\sum_v \begin{smallmatrix} & X & \\ X & + & X \\ & X & \end{smallmatrix}$$



- Strings can be interpreted as the world-lines of anyons

- Set of anyons
  $\mathcal{C} = \{a, b, c, \ldots\}$

- Fusion rules



- Braiding rules

Wen ’89. Kitaev ’97. Bravyi, Kitaev ’98. Freedman, Larsen, Wang ’00. Kitaev ’03.

The 3-Fermion theory

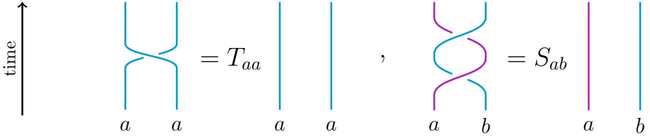
- Set of anyons:

$$\mathcal{C} = \{1, \psi_r, \psi_g, \psi_b\}$$

- Abelian ($\mathbb{Z}_2 \times \mathbb{Z}_2$) fusion rules:

$$\psi_\alpha \times \psi_\alpha = 1\,, \qquad \psi_r \times \psi_g = \psi_b\,, \qquad \alpha = r, g, b.$$

- Self (exchange) and mutual (braiding) statistics captured by modular S and T matrices (up to normalisation):



Bombín '10. Also known as $SO(8)_1$ theory - Rowell, Stong, Wang '09. Khan, Teo, Hughes '14. Teo, Hughes, Fradkin '15.

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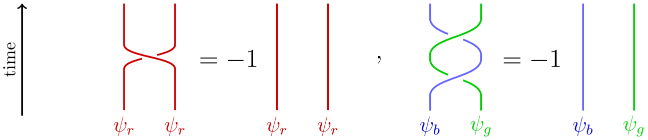

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$$T = \begin{array}{c} \begin{array}{cccc} 1 & \psi_{\mathrm{r}} & \psi_{\mathrm{g}} & \psi_{\mathrm{b}} \end{array} \\ \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ \psi_{\mathrm{r}} \\ \psi_{\mathrm{g}} \\ \psi_{\mathrm{b}} \end{array}\,, \quad S = \begin{array}{c} \begin{array}{cccc} 1 & \psi_{\mathrm{r}} & \psi_{\mathrm{g}} & \psi_{\mathrm{b}} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ \psi_{\mathrm{r}} \\ \psi_{\mathrm{g}} \\ \psi_{\mathrm{b}} \end{array}\,.$$

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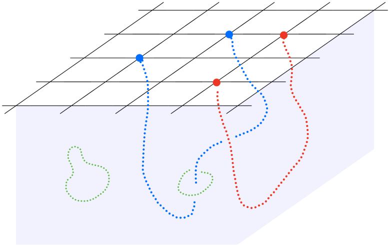
3-Fermion – where's the lattice?

- No 2D stabilizer model realising 3-Fermion
  - More generally, no 2D commuting projector model
  - Obstruction due to chiral central charge



- Can be realised on the surface of a 3D spin model (later..)

Burnell, Chen, Fidkowski, Vishwanath '13. Haah, Fidkowski, Hastings '18.

- Problem:
  - Abelian anyon theories offer no computational power.
- Solution:
  - Enrich the theory with symmetry.
  - Find non-abelian behaviour in “twist defects”.

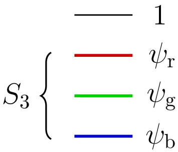
Bombín ’10. Lan, Wang, Wen ’15. Barkeshli, Bonderson, Cheng, Wang ’19.

3-Fermion anyon symmetry

- An automorphism $S : \mathcal{C} \to \mathcal{C}$ preserving braiding and fusion rules.
- For the 3-Fermion theory, all three fermions are identical – can be permuted

$$S_3 \cong \{(), (\mathrm{rg}), (\mathrm{gb}), (\mathrm{rb}), (\mathrm{rgb}), (\mathrm{rbg})\},$$

$$s \cdot 1 = 1, \quad s \cdot \psi_c = \psi_{s \cdot c}, \qquad s \in S_3, \quad c \in \{\mathrm{r}, \mathrm{g}, \mathrm{b}\}.$$



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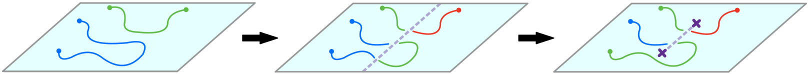
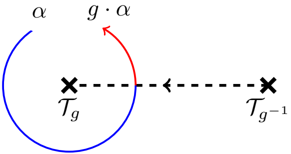
Twist defects

- Symmetry enriched theory $\mathcal{C} = \mathcal{C}_{\mathbf{3F}} \oplus \mathcal{C}_{S_3}$
- $\mathcal{C}_{S_3}$ describes theory of twists
  - Labelled by elements $s \in S_3$
  - Can be represented on the lattice as defects



- Non-abelian fusion and braiding rules.
- Interaction with 3-fermion anyons upon monodromy.

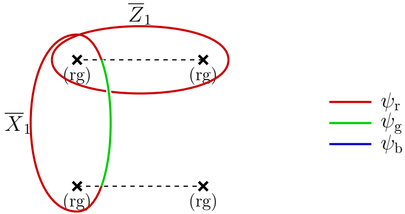


Bombín '10. Barkeshli, Bonderson, Cheng, Wang '19.

Encoding in symmetry defects

- Twists $\mathcal{T}_s$ labelled by $s \in S_3$.
- Encode logical information in two pairs of twists, e.g.

$$\mathcal{T}_{(\mathrm{rg})} \times \mathcal{T}_{(\mathrm{rg})} = 1 + \psi_{\mathrm{b}},$$
$$\mathcal{T}_{(\mathrm{rgb})} \times \mathcal{T}_{(\mathrm{rbg})} = 1 + \psi_{\mathrm{r}} + \psi_{\mathrm{g}} + \psi_{\mathrm{b}}.$$



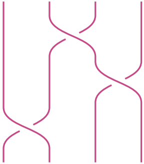
- Encode one logical qubit if $s = (\mathrm{rg}), (\mathrm{gb}), (\mathrm{rb})$
- Encode two logical qubits if $s = (\mathrm{rgb}), (\mathrm{rbg})$

Bombín ’10.

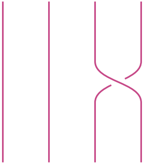
3-Fermion Clifford completeness

Clifford completeness of symmetry enriched 3-Fermion theory.

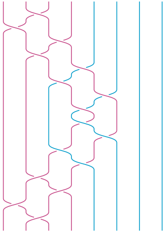
- All *n*-qubit Clifford operations can be generated by braiding and fusing twist defects of the 3-Fermion theory.



Hadamard



S-gate



Controlled-*Z* gate

- Universal fault-tolerant gateset via distillation $\rho_T^{\otimes n} \xrightarrow[\text{Clifford}]{} \rho'_T \approx |T\rangle\langle T|$.
- Ask later for comparison to toric-code / Raussendorf topological cluster scheme

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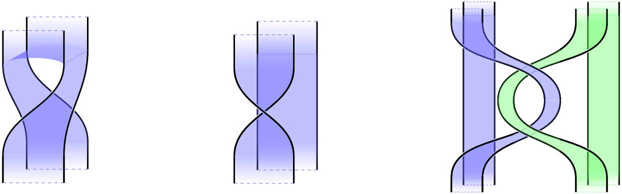

Hadamard | S-gate | Controlled-Z gate

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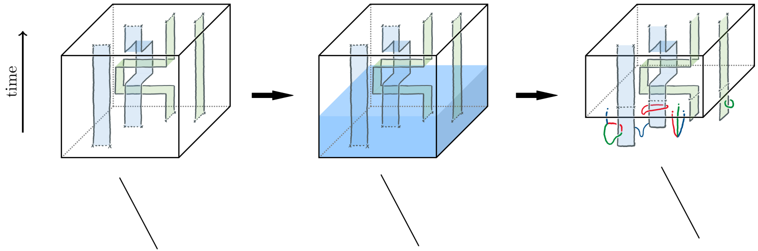
Walker–Wang measurement-based quantum computation



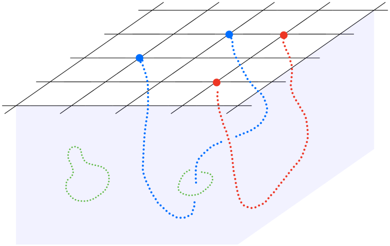
Walker-Wang model with symmetry defects

Measure in local anyon basis

Topological quantum computation in anyonic history state

Walker–Wang states

- Input:
  - Anyon theory $\mathcal{C}$ with braiding and fusion rules.
  - 3-Manifold discretized on a lattice
- Output: commuting projector Hamiltonian H with
  - Ground state consists of superposition over anyon trajectories.
  - $\mathcal{C}$ appearing as excitations on the boundary.



Walker, Wang, '11.

The 3-Fermion Walker–Wang Hamiltonian

- Two qubits per link on a cubic lattice
- Stabilizer model

$\sigma \ \tau$

$1 = |{+}{+}\rangle$

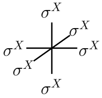
$\psi_{\mathrm{r}} = |{-}{+}\rangle$

$\psi_{\mathrm{g}} = |{+}{-}\rangle$

$\psi_{\mathrm{b}} = |{-}{-}\rangle$

$$H_{\mathbf{3F}} = -\sum_{v} A_v - \sum_{P} B_p$$

$A_v^{(\psi_r)} =$



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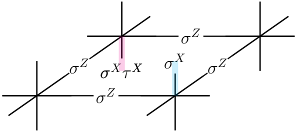
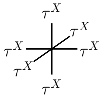


$A_v^{(\psi_g)} =$



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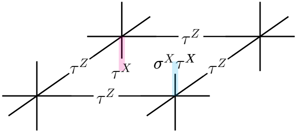
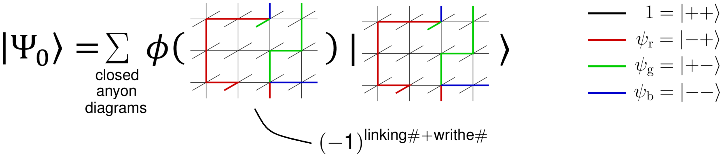
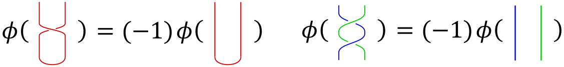


Burnell, Chen, Fidkowski, Vishwanath ’13. Haah, Fidkowski, Hastings ’18.

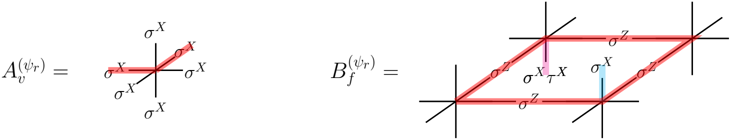
The 3-Fermion Walker–Wang ground state



- Consistent with the 3-fermion braid relations



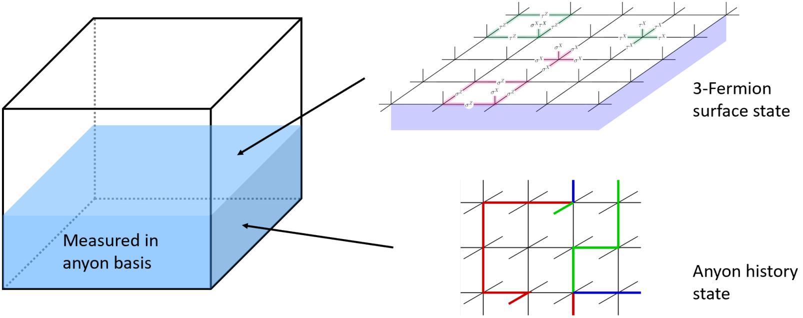
- Enforced by star and plaquette terms



3-Fermion Walker–Wang MBQC

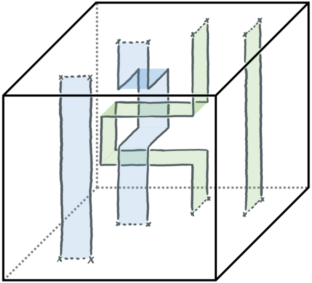
- To prepare and propagate surface states, we measure in the local anyon basis on each link
  - Measure σ^X and τ^X, random outcome from

$$\{|\mathbf{1}\rangle = |++\rangle, |\psi_r\rangle = |-+\rangle, |\psi_g\rangle = |+-\rangle, |\psi_r\rangle = |--\rangle\}$$



- Prepare surface state with random configurations of fermions.

3-Fermion Walker–Wang MBQC



- For logical gates

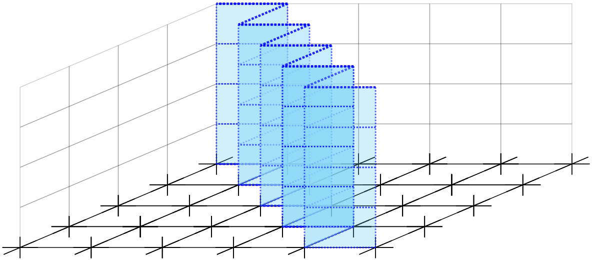
  Quantum Circuit $C \rightarrow$ Defect Hamiltonian $H_{\mathbf{3F}}(C)$

Symmetries and defects – domain walls

- First lift anyon symmetry to (Clifford) lattice symmetry

$$s \in S_3 : \{\psi_{\mathrm{r}}, \psi_{\mathrm{g}}, \psi_{\mathrm{b}}\} \to \{\psi_{\mathrm{r}}, \psi_{\mathrm{g}}, \psi_{\mathrm{b}}\} \;:$$
$$U(s) H_{\mathbf{3F}} U(s)^\dagger = H_{\mathbf{3F}}.$$

- Domain wall creation – apply restriction of symmetry $U(s)|_R$



$$H_{\mathbf{3F}} \to U(s)|_R H_{\mathbf{3F}} U(s)^\dagger|_R$$

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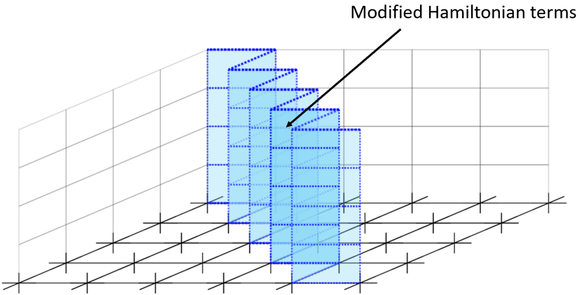

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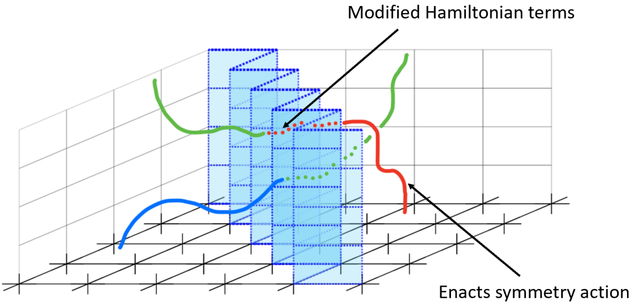
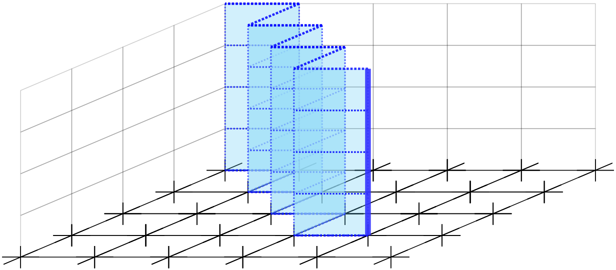


$$H_{3F} \to U(s)|_R H_{3F} U(s)^\dagger|_R$$

Symmetries and defects – twist defects

- Twist defect creation – create domain wall and gap out 1D boundary



- Twist can be made translationally invariant.
  - Utilizing theorem of Haah, Fidkowski, Hastings '18.

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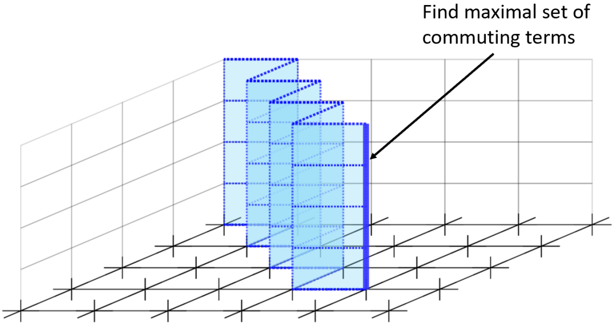

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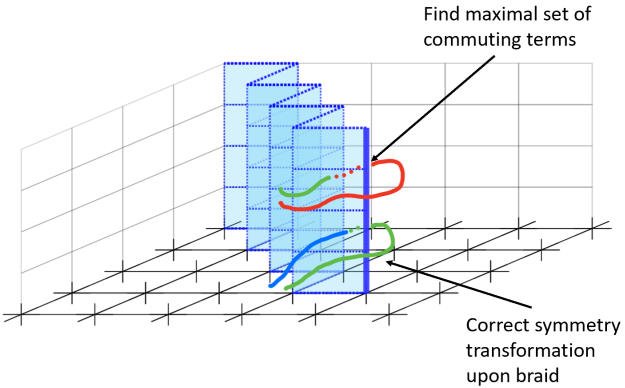

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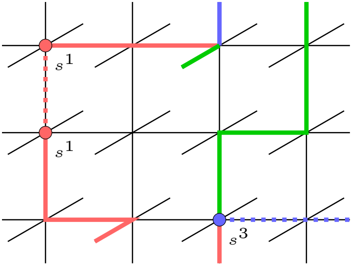
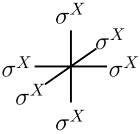
Errors and error-correction

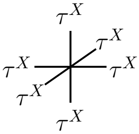
- Pauli errors and measurement errors lead to creation of fermion string segments
    - Detected by violations of fermion parity



$A_v^{(\psi_r)} =$



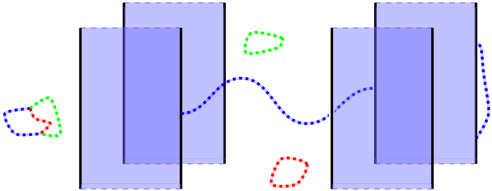
$A_v^{(\psi_g)} =$



- Decode by restoring anyon parity: vertex matching
    - Minimum weight pair matching Dennis, Kitaev, Landahl, Preskill ’02

Errors and error-correction

- Logical error $\Longleftrightarrow$ Error $\cup$ Recovery alters twist charge
  - Suppressed exponentially in twist separation.

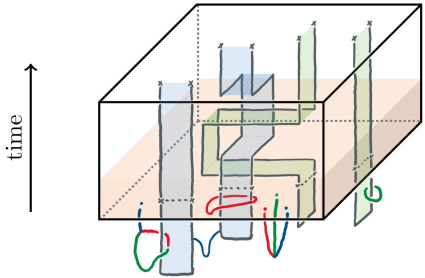


- Bulk decoding problem identical to that of the toric code and topological cluster model.
  - Threshold of 4.4% for depolarizing noise.

Wang, Harrington, Preskill '03. Raussendorf, Harrington, Goyal '07.

Remark: implementable in a 2D architecture

- Only a constant width window of the 3D Walker–Wang ground state needs to active at any time-step.



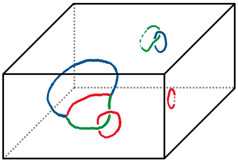
Raussendorf, Harrington '06

Summary

- New classes of models for topological MBQC.
- Stepping stone to exploring more interesting phases for computation.
- Can embed defect theory in 2D subsystem code of Bombín.

Future work

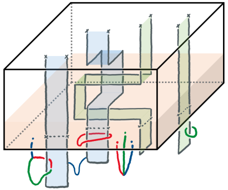
- Non-abelian anyon theories?
  - talk on Fibonacci by Schotte, Zhu, Burgelman, Verstraete.
- Adiabatic measurement-based quantum computation.
- Domain walls between different phases for non-Clifford?
- Comparison of distillation overheads to toric-based.
- MBQC throughout 1-form symmetry-protected topological phase.

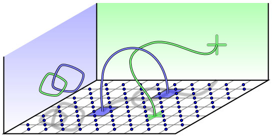


Questions?

Comparison with toric code Walker–Wang

- Can reproduce the Raussendorf *et al.* cluster-state model using toric code anyon theory $\mathcal{C}_{TC} = \{1, e, m, \epsilon\}$ as input.





3-Fermion Walker–Wang

- Surface state = 3-fermion
- Global symmetry S_3
- 1-form symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$
- No $(1+1)$D gapped boundaries
- Clifford complete braiding and fusion of twists

Toric Code Walker–Wang

- Surface state = Toric code
- Global symmetry $\mathbb{Z}_2$
- 1-form symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$
- Two types of $(1+1)$D gapped boundaries
- Topological charge projections required to complete Cliffords

Raussendorf, Harrington, Goyal '07