On the entropic convergence of quantum Gibbs samplers

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This joint submission is based on the papers [1] and [2].

Introduction. In any realistic setting, a quantum system undergoes unavoidable interactions with its environment. These interactions lead to alterations of the information initially contained in the system. Within the current context of emerging quantum information-processing devices, a proposed solution to the problem of decoherence is to encode the quantum logical information into a highly entangled many-body state in order to protect it from the action of local noise [3, 4]. Such a state will typically belong to the ground space of a Hamiltonian modeling the noiseless, unitary evolution of the system in the absence of an environment. When the environmental noise can be modeled by a Markovian evolution and below some critical temperature, the resulting self-correcting quantum memory should survive for a time which scales at least polynomially with the size of the system. Conversely, faster decoherence was recently used as a viable method for the preparation and control of relevant phases of matter [5–9], as well as to estimate the run-time of algorithms based on the efficient preparation of a Gibbs state [10]. The variety of the aforementioned applications indicates the importance of finding easy criteria for the study of the speed at which quantum lattice spin systems thermalize.

Since the seminal works of Dobrushin-Shlosman and Stroock-Zegarlinski, equilibrium and out-of-equilibrium properties of classical lattice spin systems are known to be closely related: in their attempt to answer the problem of the analytical dependence of a Gibbs measure to its corresponding potential, Dobrushin and Shlosman introduced twelve equivalent statements, one of which we refer to as the condition of *exponential decay of correlations*: the correlations between two separated regions A and B of a lattice spin system decay exponentially in the distance separating A from B. On the other hand, given a potential, one can construct a Markov process, usually called *Glauber dynamics*, whose reversing state coincides with the Gibbs state for the given potential. For these dynamics, Holley and Stroock [11, 12] made the key observation that systems thermalizing in times scaling logarithmically in the system size, a property known as *rapid mixing*, satisfy exponential decay of correlations at equilibrium. The converse implication, namely that exponential decay of correlations implies rapid mixing, was investigated later on in a series of articles [13–15] by Zegarlinski and Stroock, who proved the stronger condition of an exponential entropic decay of the dynamics towards the limiting Gibbs measure, also known as *logarithmic Sobolev inequality*. Moreover, any of these equivalent conditions occurs above some critical temperature, hence rigorously establishing the equivalence between dynamical and static phase transitions.

The extension of the above unifying theory to quantum spin systems is still far from being well understood. Based on previous work of [16], Temme and Kastoryano recently showed that, above a critical temperature, any heat-bath dynamics associated with a commuting Hamiltonian satisfies the rapid mixing property [17]. Previously, the uniform positivity of the spectral gap for these Markov processes was shown in [18] to be equivalent to a stronger condition of clustering of the correlations. More recently, exponential clustering of correlations of a Gibbs state was proved to imply its efficient preparation on a quantum [19] or classical [20] computer. In other words, the transition in the phase of a quantum system is also accompanied by a transition in the hardness of approximation [21].

Logarithmic Sobolev inequalities are by now one of the most powerful tools available in the study of classical spin systems [22], and are still the subject of active research [23–25]. They have also found numerous applications in optimization, information theory and probability theory [26–29], just to name a few. Further developing these tools to quantum systems is likely to find a smorgasbord of applications, as already illustrated by the ones contained in this submission. Prior to this work, only very few quantum systems were known to satisfy a logarithmic Sobolev inequality [30–32], and establishing it has been an open problem for years [18].

Summary of results. In this submission, we make significant progress towards a generalization of the theory to the quantum setting by proving the equivalence between (i) entropic exponential convergence to equilibrium and (ii) exponential decay of correlations, for quantum spin systems evolving under a classical process. This result constitutes the first unconditional proof of the entropic convergence to equilibrium for quantum lattice spin systems. We also make progress towards answering the question for quantum evolutions by proving (ii)⇒(i) for Gibbs states of 2-local commuting Hamiltonians. Moreover, our results hold independently of the dimension of the lattice. We emphasize that proving such a result for quantum systems is nontrivial, even in the case of systems thermalizing to a classical state. This is because the initial state could be highly entangled, and it is a-priori not clear whether entanglement could be used as a resource to substantially slow down the thermalization. Our analysis rigorously proves this is not the case. Entangled initial states also pose significant technical challenges, as most proofs for classical systems rely on concepts that do not generalize to the quantum settings, such as conditioning on the boundary or coupling. From a mathematical point of view, our main result constitutes the first complete proof of the existence of a functional inequality called the *modified logarithmic Sobolev inequality* (MLSI) [33] for interacting quantum spin systems independently of the lattice size.

Theorem 1 (MLSI for quantum lattice spin systems (informal))**.** Given the Gibbs state σ_Λ of a local commuting Hamiltonian H_Λ on the lattice Λ, there exists a local quantum Markov semigroup $(\mathrm{e}^{t\mathscr{L}_\Lambda})_{t\geqslant 0}$ converging to σ_Λ exponentially fast in relative entropy distance if (a) H_Λ is classical and $\beta < \beta_c$, or (b) H_Λ is a nearest neighbour Hamiltonian, and $\beta < \beta_c$. Here, $\beta_c < (5\,\mathrm{e}\,gh\kappa)^{-1}$ is a critical inverse temperature depending on the locality κ, the interaction strength h and the growth constant g of H_Λ. In the case of a classical Hamiltonian, we further prove the equivalence between (i) MLSI, (ii) exponential decay of correlations in σ_Λ, (iii) uniform positivity of the spectral gap and (iv) rapid mixing.

Applications. We present three applications of our results in which the convergence in relative entropy is crucial. First, we show that the output energy of an Ising quantum annealer subject to finite range classical thermal noise at high enough temperature outputs a state whose energy is close to that of the thermal state of the noise after an annealing time that is constant in system-size. Although the results of [34] also allow us to make a similar analysis based on our MLSI, here we take a new approach by exploiting quantum optimal transport techniques [35–37], showcasing the potential of such techniques for quantum computation. Secondly, we apply our results to quantum asymmetric hypothesis testing. There we show a decay estimate on the type II error for two Gibbs states corresponding to commuting potentials in the finite blocklength regime. Finally, we also apply our main result to obtain efficient quantum Gibbs samplers for certain Gibbs states corresponding to commuting potentials. Our methods only require the implementation of a circuit of local quantum channels of logarithmic depth, in contrast to previous results [19] that required quasi-local quantum channels.

Our proof of Theorem 1 is adapted from a modern strategy by [38]. It splits into three parts:

Strengthened exponential decay of correlations First, we prove a strengthened exponential decay of correlations below the critical inverse temperature β_c. For a classical Gibbs state, this condition is precisely the one of Dobrushin-Shlosman. We provide an extension to the commuting, nearest neighbour setting. Our construction of the conditional expectations involved in the result relies on a Schmidt decomposition of the local interactions, which was already used in the study of the local Hamiltonian problem in [39]. We refer to our main article [2] for more details.

Theorem 2 (Conditioned $L_1 - L_\infty$ exponential decay of correlations (informal))**.** Let σ_Λ be the Gibbs state of a commuting nearest neighbour Hamiltonian H_Λ at inverse temperature $\beta \leqslant \beta_c$. Then, for any two overlapping regions $C, D \subset \Lambda$, any boundary condition $\omega \equiv \omega_{\partial C\cup D}$ and any observable X^ω conditioned on the boundary of $C \cup D$,

$$\langle (E^\omega_C - E^\omega_{C\cup D})(X^\omega),\, (E^\omega_D - E^\omega_{C\cup D})(X^\omega)\rangle_{\sigma^\omega} \leqslant c\,|C\cup D|\;\mathrm{e}^{-\,\mathrm{dist}(D\setminus C,C\setminus D)/\xi}\;\|X^\omega\|_\infty\,\|X^\omega\|_{L_1(\sigma^\omega)}\,,$$

where $\{E_A^\omega\}$, $A \in \{C, D, C \cup D\}$, is a family of conditional expectations with respect to σ_Λ, and where σ^ω is the local Gibbs state conditioned on the boundary of $C \cup D$.

Our result extends on the recent quantum generalization of Dobrushin-Shlosman [20] in two ways: First, we get a bound in terms of the product of an L_1 and an L_∞ norm, as opposed to the standard albeit weaker $L_\infty - L_\infty$ bound. Secondly, our construction in this specific 2-local setting allows for a local bound in any subregion $C \cup D$ conditioned on its boundary, as opposed to the global bounds found in [20]. This local refinement is crucial to our subsequent proof of MLSI.

Approximate tensorization of the relative entropy. Cesi's tour de force was to realize that Dobrushin and Shlosman's $L_1 - L_\infty$ exponential decay of correlations could be used to prove the following generalization of the strong subadditivity (SSA) of the entropy, here written with quantum notations for sake of clarity: for any classical state ρ,

$$D(\rho\|E_{C\cup D*}(\rho)) \leqslant \left(1 + c\,|C \cup D|\ \mathrm{e}^{-\operatorname{dist}(D\backslash C, C\backslash D)/\xi}\right)\left(D(\rho\|E_{C*}(\rho)) + D(\rho\|E_{D*}(\rho))\right). \tag{$*$}$$

Indeed, when σ_Λ is the maximally mixed state, i.e. at $\beta = 0$, the (dual) conditional expectation E_{A*} reduces to the partial trace Tr_A on any region A, and $c = 0$ so that $(*)$ reduces to the celebrated SSA [40]: taking C and D non-overlapping, and $B = \Lambda\backslash CD$, $S(BCD)_\rho + S(B)_\rho \leqslant S(BC)_\rho + S(BD)_\rho$. Using the multivariate trace inequalities recently derived in [41], we extend the result of Cesi to quantum states in [1], informally stated below in a more general von Neumann algebraic setting. Our result has more applications than the one of proving Theorem (1). For instance, we derive tightenings of the entropic uncertainty relations. We refer to [1] for more details.

Theorem 3 (Approximate tensorization of the quantum relative entropy (informal))**.** Let $\mathscr{M} \subset \mathscr{N}_1, \mathscr{N}_2 \subset \mathscr{N}$ be finite-dimensional von Neumann algebras, with corresponding conditional expectations $E_{\mathscr{M}}$, E_1 and E_2. Under a condition of $L_1 - L_\infty$ clustering of correlations, the following inequality holds: there exists a constant c depending on the clustering, such that for all quantum state ρ,

$$D(\rho\|E_{\mathscr{M}*}(\rho)) \leqslant c\left(D(\rho\|E_{1*}(\rho)) + D(\rho\|E_{2*}(\rho))\right) + d(\rho)\,,$$

where $d(\rho)$ is a ρ-dependent additive error term that measures the deviation of ρ from being diagonal in the block decomposition of the matrix algebra $\mathscr{M}$.

Modified logarithmic Sobolev inequality Theorem (1) states the existence of a constant rate $\alpha > 0$, independent of the size of Λ, such that for any initial state ρ evolving according to the semigroup, $D(\mathrm{e}^{t\mathscr{L}_\Lambda}(\rho)\|\sigma_\Lambda) \leqslant \mathrm{e}^{-\alpha\, t}\, D(\rho\|\sigma_\Lambda)$. It turns out that this exponential convergence is equivalent to its derivative with respect to t at $t = 0$. The resulting inequality is known as a *modified logarithmic Sobolev inequality* (MLSI): for any state ρ,

$$\alpha\, D(\rho\|\sigma_\Lambda) \leqslant -\left.\frac{d}{dt}\right|_{t=0} D(\mathrm{e}^{t\mathscr{L}_\Lambda}(\rho)\|\sigma_\Lambda) = \mathrm{EP}_{\mathscr{L}_\Lambda}(\rho)\,. \tag{MLSI}$$

The right-hand side of the MLSI has the useful property of being linear in the generator $\mathscr{L}_\Lambda$. Moreover, under the $L_1 - L_\infty$ clustering property of σ_Λ, the approximate tensorization of the relative entropy for classical spins can be used to prove that the left-hand side of MLSI is approximately sub-additive. These two crucial properties led Cesi to formulate the idea of decomposing the problem into regions of a small fixed size, where the MLSI constant α is known to exist. However, the non-vanishing of the constant d on quantum states found in Theorem 3 is responsible for the failure of Cesi's argument in the quantum regime. In [2], we devise an original argument, which we refer to as *peeling*, in order to manage our way around this issue. In layman's terms, our idea consists in proving that any initial state ρ will very quickly converge into a state γ whose constant $d(\gamma)$ vanishes on appropriately chosen regions $C \cup D$. For these states, we recover $(*)$, which allows us to conclude our proof.

[1] Ivan Bardet, Á Capel, and Cambyse Rouzé. Approximate tensorization of the relative entropy for noncommuting conditional expectations. *arXiv:2001.07981*, 2020.

[2] Ángela Capel, Cambyse Rouzé, and Daniel Stilck França. The modified logarithmic sobolev inequality for quantum spin systems: classical and commuting nearest neighbour interactions. *arXiv preprint arXiv:2009.11817*, 2020.

[3] Peter W Shor. Scheme for reducing decoherence in quantum computer memory. *Physical Review A*, 52(4):R2493, 1995.

[4] Eric Dennis, Alexei Kitaev, Andrew Landahl, and John Preskill. Topological quantum memory. *Journal of Mathematical Physics*, 43(9):4452–4505, 2002.

[5] Frank Verstraete, Michael M Wolf, and J Ignacio Cirac. Quantum computation and quantum-state engineering driven by dissipation. *Nature physics*, 5(9):633, 2009.

[6] Sebastian Diehl, A Micheli, A Kantian, B Kraus, HP Büchler, and P Zoller. Quantum states and phases in driven open quantum systems with cold atoms. *Nature Physics*, 4(11):878, 2008.

[7] Barbara Kraus, Hans P Büchler, Sebastian Diehl, Adrian Kantian, Andrea Micheli, and Peter Zoller. Preparation of entangled states by quantum Markov processes. *Physical Review A*, 78(4):042307, 2008.

[8] Niels Syassen, Dominik M Bauer, Matthias Lettner, Thomas Volz, Daniel Dietze, Juan J Garcia-Ripoll, J Ignacio Cirac, Gerhard Rempe, and Stephan Dürr. Strong dissipation inhibits losses and induces correlations in cold molecular gases. *Science*, 320(5881):1329–1331, 2008.

[9] Dirk Witthaut, F Trimborn, and S Wimberger. Dissipation-induced coherence and stochastic resonance of an open two-mode Bose-Einstein condensate. *Physical Review A*, 79(3):033621, 2009.

[10] Fernando GSL Brandão and Krysta M Svore. Quantum speed-ups for solving semidefinite programs. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 415–426. IEEE, 2017.

[11] Richard A. Holley and Daniel W. Stroock. Applications of the stochastic Ising model to the Gibbs states. *Communications in Mathematical Physics*, 48(3):249–265, 1976.

[12] Richard A Holley and Daniel W Stroock. Uniform and L^2 convergence in one dimensional stochastic Ising models. *Communications in Mathematical Physics*, 123(1):85–93, 1989.

[13] Daniel W Stroock and Boguslaw Zegarlinski. The logarithmic Sobolev inequality for continuous spin systems on a lattice. *Journal of Functional Analysis*, 104(2):299 – 326, 1992.

[14] Daniel W. Stroock and Boguslaw Zegarlinski. The equivalence of the logarithmic Sobolev inequality and the Dobrushin-Shlosman mixing condition. *Comm. Math. Phys.*, 144(2):303–323, 1992.

[15] Daniel W. Stroock and Boguslaw Zegarlinski. The logarithmic Sobolev inequality for discrete spin systems on a lattice. *Comm. Math. Phys.*, 149(1):175–193, 1992.

[16] A. W. Majewski and B Zegarlinski. Quantum stochastic dynamics I: Spin systems on a lattice. *Mathematical Physics Electronic Journal*, 1:37, 1995.

[17] Kristan Temme and Michael J Kastoryano. How fast do stabilizer Hamiltonians thermalize? *arXiv:1505.07811*, 2015.

[18] Michael J Kastoryano and Fernando GSL Brandao. Quantum gibbs samplers: the commuting case. *Communications in Mathematical Physics*, 344(3):915–957, 2016.

[19] Fernando GSL Brandão and Michael J Kastoryano. Finite correlation length implies efficient preparation of quantum thermal states. *Communications in Mathematical Physics*, 365(1):1–16, 2019.

[20] Aram W Harrow, Saeed Mehraban, and Mehdi Soleimanifar. Classical algorithms, correlation decay, and complex zeros of partition functions of quantum many-body systems. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 378–386, 2020.

[21] Allan Sly. Computational transition at the uniqueness threshold. In *2010 IEEE 51st Annual Symposium on Foundations of Computer Science*, pages 287–296. IEEE, 2010.

[22] Fabio Martinelli. *Lectures on Glauber Dynamics for Discrete Spin Models*, pages 93–191. Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.

[23] Zongchen Chen, Kuikui Liu, and Eric Vigoda. Optimal mixing of Glauber dynamics: Entropy factorization via high-dimensional expansion. *arXiv preprint arXiv:2011.02075*, 2020.

[24] Mary Cryan, Heng Guo, and Giorgos Mousa. Modified log-Sobolev inequalities for strongly log-concave distributions. In *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1358–1370. IEEE, 2019.

[25] Yin Tat Lee and Santosh S Vempala. Stochastic localization+ Stieltjes barrier= tight bound for log-Sobolev. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 1122–1129,

2018.
[26] Maxim Raginsky and Igal Sason. Concentration of measure inequalities in information theory, communications and coding. *arXiv preprint arXiv:1212.4663*, 2012.
[27] Cédric Villani. *Optimal transport: old and new*, volume 338. Springer Science & Business Media, 2008.
[28] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348. Springer Science & Business Media, 2013.
[29] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities: A nonasymptotic theory of independence*. Oxford university press, 2013.
[30] Kristan Temme, Fernando Pastawski, and Michael J Kastoryano. Hypercontractivity of quasi-free quantum semigroups. *Journal of Physics A: Mathematical and Theoretical*, 47(40):405303, 2014.
[31] Salman Beigi, Nilanjana Datta, and Cambyse Rouzé. Quantum reverse hypercontractivity: its tensorization and application to strong converses. *Communications in Mathematical Physics*, 376(2):753–794, 2020.
[32] Ángela Capel, Angelo Lucia, and David Pérez-García. Quantum conditional relative entropy and quasi-factorization of the relative entropy. *Journal of Physics A: Mathematical and Theoretical*, 51(48):484001, 2018.
[33] Michael J. Kastoryano and Kristan Temme. Quantum logarithmic Sobolev inequalities and rapid mixing. *Journal of Mathematical Physics*, 54(5), 2013.
[34] Daniel Stilck França and Raul Garcia-Patron. Limitations of optimization algorithms on noisy quantum devices, 2020. arXiv:2009.05532v1.
[35] Cambyse Rouzé and Nilanjana Datta. Concentration of quantum states from quantum functional and transportation cost inequalities. *Journal of Mathematical Physics*, 60(1):012202, jan 2019.
[36] Li Gao, Marius Junge, and Nicolas LaRacuente. Fisher information and logarithmic Sobolev inequality for matrix valued functions. *arXiv: 1807.08838*, 2018.
[37] Eric A. Carlen and Jan Maas. Gradient flow and entropy inequalities for quantum Markov semigroups with detailed balance. *Journal of Functional Analysis*, 273(5):1810–1869, 2017.
[38] Filippo Cesi. Quasi-factorization of the entropy and logarithmic Sobolev inequalities for Gibbs random fields. *Probability Theory and Related Fields*, 120(4):569–584, 2001.
[39] Sergey Bravyi and Mikhail Vyalyi. Commutative version of the local Hamiltonian problem and common eigenspace problem. *Quantum Information & Computation*, 5(3):187–215, 2005.
[40] Elliott H Lieb and Mary Beth Ruskai. Proof of the strong subadditivity of quantum-mechanical entropy. *Les rencontres physiciens-mathématiciens de Strasbourg-RCP25*, 19:36–55, 1973.
[41] David Sutter, Mario Berta, and Marco Tomamichel. Multivariate trace inequalities. *Communications in Mathematical Physics*, 352(1):37–58, 2017.