Classifying unitary dynamics with approximate light cones in one dimension

A converse to Lieb-Robinson bounds in one dimension using index theory

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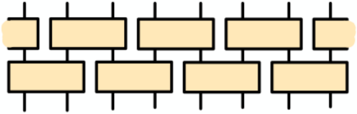
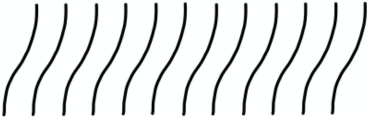
Local dynamics

Local dynamics are ubiquitous in quantum physics and computation.
Two simple one-dimensional examples:

- Quantum circuit

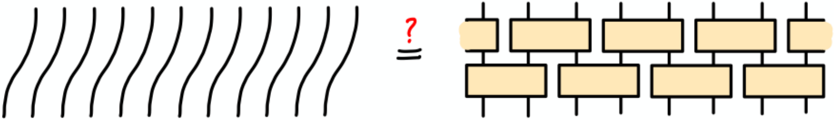


- Translation

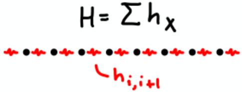


Local dynamics

Quiz question:



Local Hamiltonians and approximately local dynamics

H a local 1D Hamiltonian: 

Does e^{iHt} give local dynamics?

Use *Heisenberg picture*: $O \mapsto O(t) := e^{-iHt} O e^{iHt}$. If O_A is a local operator on a set A, how local is $O_A(t)$?

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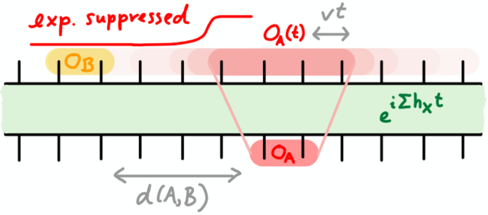
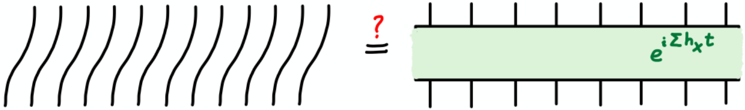


Lieb-Robinson bounds:

$$\|[O_A(t), O_B]\| \leq C e^{-a(d(A,B)-vt)} \|O_A\| \|O_B\|$$

Local Hamiltonians and approximately local dynamics

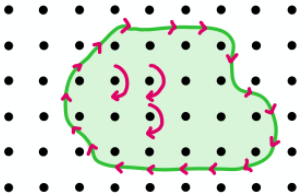
Quiz question: can a translation be *locally generated*, i.e. by some Hamiltonian evolution?



What if we allow H to be time-dependent, polynomial tails?

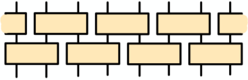
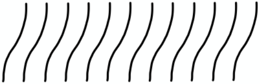
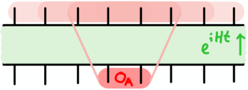
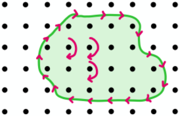
2D Floquet systems with many-body localization

2D Floquet systems with many-body localization gives rise to a 1D dynamics on the *boundary* (Po-Fidkowski-Morimoto-Potter-Vishwanath):



An example of 1D approximately local dynamics that does *not* arise from a Hamiltonian. . .

Classification of one-dimensional approximately local dynamics

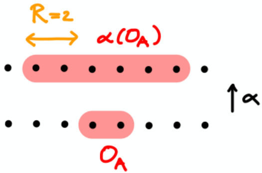
	Locally generated	Not locally generated?
Strictly local		
Approximately local		

Goal: classify one-dimensional approximately local dynamics.

Quantum cellular automata

Consider lattice of quantum spins.
A (Heisenberg picture) channel α is a quantum cellular automaton (QCA) with radius R if

1. α is an automorphism (essentially α is a unitary channel $\alpha(O) = UOU^\dagger$).
2. For any operator O_A supported on some finite set A, $\alpha(O_A)$ is supported on sites within radius R of A.

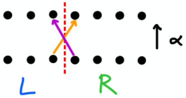


(Feynman, Deutsch, Margolus, Schumacher, Watrous, Werner, . . .)

- Understanding many-body physics
- Model of computation

Quantum cellular automata and index theory

In one dimension QCAs are completely classified by the GNVW index (Gross-Nesme-Vogts-Werner).
Informally:



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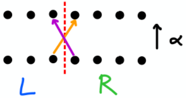
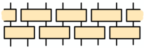
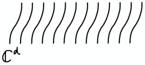


$$\mathrm{ind}(\alpha) = \#\text{qubits moving right} - \#\text{qubits moving left}$$

- $\mathrm{ind}(\alpha) = 0$ if and only if $\alpha =$ 
- $\mathrm{ind}(\alpha) = \log(d)$ for $\alpha =$ 

- $\mathrm{ind}(\alpha \circ \beta) = \mathrm{ind}(\alpha) + \mathrm{ind}(\beta)$ and $\mathrm{ind}(\alpha \otimes \beta) = \mathrm{ind}(\alpha) + \mathrm{ind}(\beta)$.

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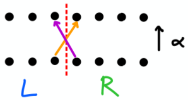

$$\text{ind}(\alpha) = \#\text{qubits moving right} - \#\text{qubits moving left}$$

Consequence: Every 1D QCA can be written as a composition of a tensor product of translations and a circuit.

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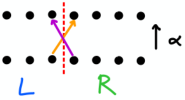

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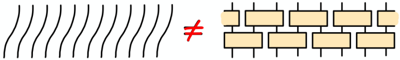
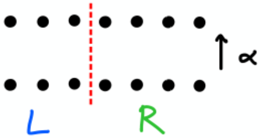
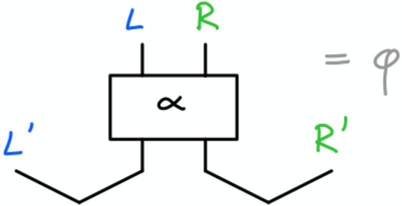
Answer to first quiz question:



The GNVW index as an information flow

Let ϕ be the *Choi state* of α:





$$\mathrm{ind}(\alpha) = \frac{1}{2}\left[I(L' : R)_\phi - I(L : R')_\phi\right]$$

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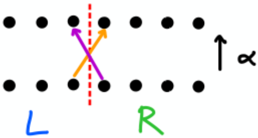

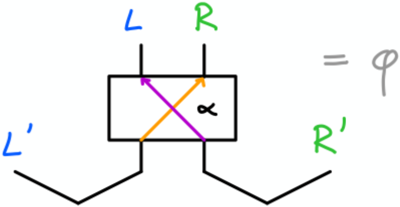

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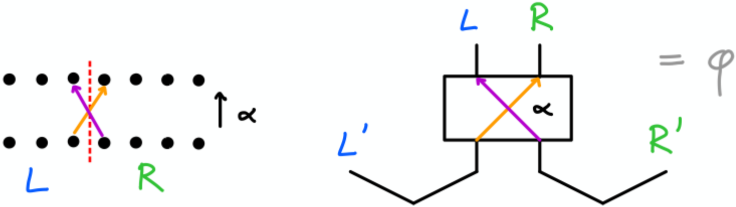


$$\mathrm{ind}(\alpha) = \frac{1}{2}\left[I(L' : R)_\phi - I(L : R')_\phi \right]$$

This expression in principle also makes sense for approximately local dynamics. However, it is not at all clear that it takes discrete values!

Approximately local dynamics

Idea: take the definition of a QCA and replace

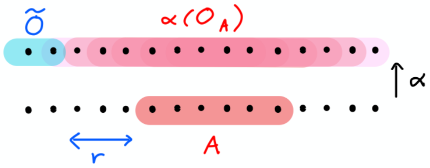
strict locality $\Rightarrow$ Lieb-Robinson type bounds.

Approximately local dynamics

Let $f : \mathbb{N} \to \mathbb{R}$ be some $o(1)$ function (e.g. $f(r) = e^{-ar}$ or $f(r) = \frac{1}{r^a}$).

We define an approximately locality preserving unitary (ALPU) with $f(r)$-tails on a spin chain to be an automorphism α (of the quasi-local algebra of the spin chain) which is such that for any operator O_A supported on some interval A, and any operator $\tilde{O}$ supported on a set at least r sites away from A

$$\|[\alpha(O_A), \tilde{O}]\| \leq f(r)\|O_A\|\|\tilde{O}\|.$$



Results: approximation by QCAs

Theorem

Suppose α is an ALPU with $f(r)$-tails. Then there exists a sequence $\alpha^{(r)}$ of QCAs of radius $2r$ such that

$$\|\alpha(O_A) - \alpha^{(r)}(O_A)\| = \mathcal{O}(f(r)|A|)$$

for any operator O_A with $\|O_A\| = 1$ supported on some finite set A.

Idea is as in digital simulation of Hamiltonian evolution, but now without a Hamiltonian and only Lieb-Robinson bounds...

Results: definition of index for ALPUs

Theorem

Suppose α is an ALPU with $f(r)$-tails, and $\alpha^{(r)}$ a sequence of approximating QCAs as in the previous theorem. Then

1. For sufficiently large r the value of $\mathrm{ind}(\alpha^{(r)})$ is independent of r and of the choice of approximation. We define

$$\mathrm{ind}(\alpha) := \lim_{r\to\infty} \mathrm{ind}(\alpha^{(r)}).$$

2. If $f(r) = \mathcal{O}(\frac{1}{r^{1+\delta}})$ for $\delta > 0$

$$\mathrm{ind}(\alpha) = \frac{1}{2}\big[I(L' : R)_\phi - I(L : R')_\phi\big].$$

Results: properties of the index for ALPUs

Theorem

Suppose α, β are ALPUs. Then

1. $\mathrm{ind}(\alpha \circ \beta) = \mathrm{ind}(\alpha) + \mathrm{ind}(\beta)$ and $\mathrm{ind}(\alpha \otimes \beta) = \mathrm{ind}(\alpha) + \mathrm{ind}(\beta)$.
2. $\mathrm{ind}(\alpha) = \mathrm{ind}(\beta)$ if and only if there exists a path through the space of ALPUs with $g(r)$-tails for some $g(r) = o(1)$ from α to β.

In fact, if α and β have $f(r)$-tails, we may take the path between them to be generated by a some Hamiltonian evolution for unit time with

$$H(t) = \sum_X H_X(t) \qquad \|H_X\| \approx \mathcal{O}(f(\mathrm{diam}(X))).$$

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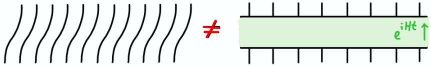
Answer to the second quiz question:



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All the key properties of the GNVW index generalize under the replacements

$$\text{QCA} \Rightarrow \text{ALPU}$$
$$\text{circuit} \Rightarrow \text{quasi-local Hamiltonian evolution.}$$

This classifies ALPUs modulo Hamiltonian evolutions.

Proof techniques

Suppose $\mathcal{A}, \mathcal{B} \subseteq B(\mathcal{H})$ are algebras. We say that $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}$ if for every $O \in \mathcal{A}$

$$\inf_{\tilde{O} \in \mathcal{B}} \|O - \tilde{O}\| \leq \varepsilon \|O\|.$$

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Theorem (Christensen, 1980)

Suppose $\mathcal{A}, \mathcal{B} \subseteq B(\mathcal{H})$ are hyperfinite von Neumann algebras with $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}$ for $\varepsilon < \frac{1}{8}$. Then there exists a unitary $u \in B(\mathcal{H})$ such that

$$u\mathcal{A}u^{\dagger} \subseteq \mathcal{B} \qquad \|u - I\| \leq 12\varepsilon.$$

Can be seen as a type of *Ulam stability* for von Neumann algebras.

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We provide a self-contained proof in the appendix and prove additional properties of u.

Proof techniques

For the approximation of an ALPU α with $f(r)$-tails by QCAs:

1. By blocking r sites, α is nearest neighbor up to error $\varepsilon = f(r)$.
2. Apply the near-inclusion theorem to a sequence of half chain algebras to obtain α_n such that α_n local around site $2n$ and $\|\alpha - \alpha_n\| = \mathcal{O}(\varepsilon)$.

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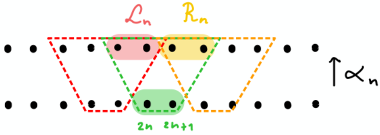

3. This implies a factorization property (GNVW):

$$\alpha_n(\mathcal{A}_{2n} \otimes \mathcal{A}_{2n+1}) = \mathcal{L}_n \otimes \mathcal{R}_n.$$

Use this to glue together the different α_n to $\alpha^{(r)}$.

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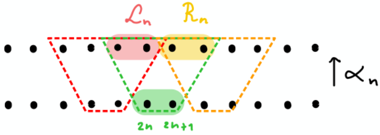

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Outlook

- Finite systems: Also works for open chain, not yet for periodic chain.
- Higher dimensions: Fascinating recent work on classification of higher dimensional QCAs (Fidkowski, Freedman, Haah, Hastings). Full classification in 2D, exotic examples in 3D. What about ALPUs in higher dimensions?
- Noisy QCAs: Can also consider strictly local channels which are not unitary (Piroli, Cirac).
  Almost unitary and strictly local $\Rightarrow$ close to a QCA?
- Algebra stability applications? Are there other interesting QI applications for the type of algebra stability result we discuss?

Thank you for your attention!