Emergent classicality in general multipartite states and channels

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By the monogamous nature of entanglement, a single quantum system cannot be highly entangled with many others. From a dynamical perspective, this monogamy constrains the spreading of information. The no-cloning theorem provides a simple example of such a constraint; more generally, quantum information cannot be widely distributed with high fidelity.

Constraints on information spreading also shed light on the quantum-to-classical transition. Many questions remain about precisely how and when classical behavior emerges from quantum many-body systems. When a small system interacts with a large environment, the environment often acts as a measuring apparatus, decohering the system in some basis. This paradigm is further elaborated by research programs on decoherence and "quantum Darwinism," describing how certain observables of the system are "selected" by the environment [1–6].

Brandão et al. [7] proved a powerful monogamy theorem constraining the spread of quantum information. They show that some aspects of the decoherence process must exist for *any* quantum channel. They consider general time-evolutions of a system A initially uncorrelated with a large multipartite environment $B_1 \otimes \cdots \otimes B_n$. Their result states that for a large fraction of environmental subsystems B_i, the only information about A that is accessible on B_i must be classical, i.e. it must be obtainable from a fixed measurement on A. Crucially, they show that the relevant measurement on A is independent of the subsystem B_i of interest. Thus the system A must "appear classical" to an observer at B_i, in the sense that the only accessible information about A is classical.

However, the abovementioned result only constrains a large *fraction* of environmental subsystems. For a fixed error tolerance, the number of subsystems left unconstrained by the theorem increases arbitrarily with the total size of the environment. Intuitively, this growth seems to contradict the monogamy of entanglement, which suggests the fragment of the environment with non-classical information about A must have bounded extent. In other words, monogamy suggests the results of [7] can be greatly improved.

We obtain this stronger constraint on quantum information spreading. In particular, we show that for large environments, for everywhere in the environment excluding some $O(1)$-sized subsystem Q, the locally accessible information about A must be approximately classical, i.e. obtainable from some fixed measurement on A. This result corroborates the above intuition from monogamy. The statement is totally general, applicable to arbitrary quantum channels and quantum states. We call the excluded region Q the "quantum Markov blanket," or simply the Markov blanket, following the terminology in classical statistics [8].

The proof of our result may be framed constructively as an optimization procedure, allowing numerical applications on numerically tractable systems. As a demonstration, we analyze a numerical example involving a small spin chain. The numerical algorithm that results from our proof identifies the quantum Markov blanket and the effective measurement induced on a subsystem by the spin chain dynamics.

PRIOR WORK

As mentioned above, the result is both inspired by and significantly strengthens the related result [7]. More generally, the ideas in the proof are inspired by principles in [9, 10].

Quantum de Finetti theorems characterize the marginals of permutation-invariant states, which are approximately separable for large systems [11]. Thus de Finetti theorems corroborate the monogamy of entanglement. Our result may be seen as a quantum de Finetti-type theorem for non-permutation-invariant systems. For instance, the result about k-extendible states in Corollary 2 of [9] may be seen as a special case of our Theorem 1 when specialized to permutation-invariant states. Likewise, compare to Theorem 1 of [12].

Early work in the direction of de Finetti-type results for non-permutation-invariant systems includes the "decoupling" theorems of [10]. These show that for large multipartite states, after conditioning the state on a measurement of a small random subset of qudits, the marginals on most other small subsets are approximately product states. (The measurement "decouples" them.) The result of [7] and our Theorem 1 (when applied to Choi states rather than channels) may also be seen as decoupling theorems in this sense.

The technique of using small conditional mutual information $I(X,Y|Z)$ to show ρ_{XY} is close to separable was developed by [9], where they use the one-way LOCC norm. The use of the one-way LOCC norm is a technique inspired by [12], where it was applied to obtain de Finetti theorems. The method is further developed by [7, 12, 13].

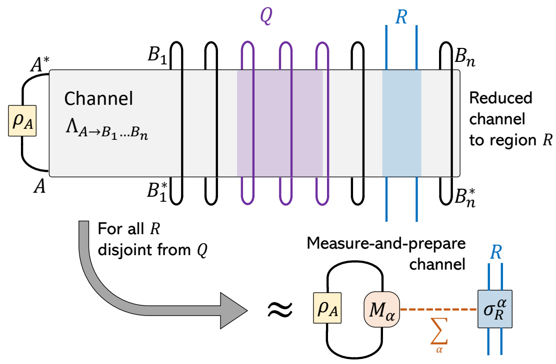

Figure 1. Illustration of main result, Theorem 1. The quantum channel Λ is shown acting on a state ρ_A, with a partial trace over the complement of the output region R. For any R that does not overlap the "Markov blanket" Q, the reduced channel is approximately a "measure-and-prepare" channel. Importantly, the measurements M_α on A are *independent* of the choice of region R.

TECHNICAL RESULTS

We considers arbitrary channels with multiple outputs, and we characterizes the reduced channels onto small subsets of outputs. We prove that for all small subsets of outputs except those overlapping some fixed $O(1)$-sized excluded subset, the corresponding reduced channels are measure-and-prepare, and moreover they use the same measurement. We denote this excluded region Q, the quantum Markov blanket.

Theorem 1. (Emergent classicality for channels.) *Consider a quantum channel $\Lambda : \mathcal{D}(A) \to \mathcal{D}(B_1 \otimes ... \otimes B_n)$. For output subsets $R \subset \{B_1, ..., B_n\}$, let $\Lambda_R \equiv \mathrm{Tr}_{\bar{R}} \circ \Lambda : \mathcal{D}(A) \to \mathcal{D}(R)$ denote the reduced channel onto R, obtained by tracing out the complement $\bar{R}$. Then for any $|Q|, |R| \in \{1, ..., n\}$, there exists a measurement, described by a POVM $\{M_\alpha\}$, and an "excluded" output subset $Q \subset \{B_1, ..., B_n\}$ of size $|Q|$, such that for all output subsets R of size $|R|$ disjoint from Q,*

$$\|\Lambda_R - \mathcal{E}_R\|_\diamond \le d_A^3 \sqrt{2 \ln (d_A) \frac{|R|}{|Q|}} \tag{1}$$

using a measure-and-prepare channel

$$\mathcal{E}_R(X) := \sum_\alpha \mathrm{Tr}(M_\alpha X) \sigma_R^\alpha \tag{2}$$

for some states $\{\sigma_R^\alpha\}_\alpha$ on R, where $d_A = dim(A)$ and $\|...\|_\diamond$ is the diamond norm on channels. The measurement $\{M_\alpha\}$ does not depend on the choice of R, while the prepared states σ_R^α may depend on R.

The theorem is illustrated in Fig. 1. The result is true for any $|Q|$, but to guarantee smaller error in the approximation, one needs $|Q|$ large compared to $|R|$. Nonetheless, we refer to Q as $O(1)$-sized because for a fixed size $|R|$, any fixed error tolerance on the RHS of Eq. 1 only requires some fixed $|Q|$, independent of both the total number of outputs n and the dimensions $\dim(B_i)$ of each output. Physically, the region Q is where any locally accessible quantum information about A must be stored. Therefore by no-cloning or monogamy of entanglement, no quantum information about A can be locally accessible outside this region. Meanwhile, Q will also contain any locally accessible classical information about A. However, unlike the quantum information, the classical information may also be present in copies outside of Q.

An essential point is that the measurement $\{M_\alpha\}$ in this theorem does not depend on R, so that apart from the $O(1)$-sized region Q, different "observers" in different parts of the system can only receive classical information about the input in the same "generalized basis," i.e. resulting from the same POVM on A. (The observers may also receive no information at all.) This supports the "objectivity" of the emergent classical description of quantum systems.

The ability to approximate the reduced channels Λ_R as measure-and-prepare channels $\mathcal{E}_R(X)$ using the *same* measurement is nontrivial, and it does not follow merely from the fact that the reduced channels are compatible [14, 15] in the sense of being reduced from the same parent channel. Indeed, we separately show there exist compatible measure-and-prepare channels that cannot be written using the same measurement.

By channel-state duality, we can also formulate the result for states rather than channels. When the analogous result is expressed for states, one may use the one-way LOCC norm [9, 16], allowing one to remove the exponential dependence on the size of A in the error bound.

The central idea of the proof is to imagine expanding a small region of the environment to gradually encompass the entire system. During this process, one learns gradually more about the input system A. Through a greedy algorithm, one calculates an optimized path of expansion that extracts the most information from A. By strong subadditivity, even an optimal path must reach some "bottleneck" such that further expanding the region does not yield additional information about A. Analyzing this bottleneck gives rise to the result.

Because the theorem applies to any channel, it will be helpful to consider a few very different cases. Take A to be a single qubit, and take B to consist of n qubits $B_1, ..., B_n$. Then we have the following simple cases:

1. Let Λ be a Haar-random isometry. Then for A fixed and n large, the reduced channels on small subsets will be approximately constant channels, and moreover they are measure-and-prepare channels in a trivial sense: they can be expressed as any measurement on A followed by a preparation of some constant state, independent of the outcome of the measurement. In this case Theorem 1 easily holds, and in fact the approximation has zero error; one could even take the excluded region Q to be the empty set.

2. Let Λ faithfully transmit A to some B_i, while preparing an arbitrary state on the remaining outputs. Then the reduced channel Λ_{B_i} is the identity channel, and the excluded region Q must consist of at least B_i. The remaining reduced channels are constant channels, and thus the error in Theorem 1 is already zero for $|Q| = 1$.

3. Let Λ be the isometry $|0\rangle_A \mapsto |0...0\rangle_B$, $|1\rangle_A \mapsto |1...1\rangle_B$. Then every reduced channel Λ_{B_i} is a measure-and-prepare channel, measuring in the $\{|0\rangle, |1\rangle\}$ basis and preparing the corresponding $|0\rangle$ or $|1\rangle$ state. Thus the error in Theorem 1 is already zero for empty Q.

These examples constitute three loose categories of dynamics: either (1) no information about A becomes available to small regions of B, (2) quantum information about A becomes available to a small region, and remains locally inaccessible elsewhere, or (3) the only locally accessible information about A is classical, distributed through a decoherence process, so that each B_i becomes classically correlated with A via a consistent basis on A. Our Theorem 1 demonstrates that heuristically, these three categories and combinations thereof exhaust the possibilities of quantum dynamics. That is, while some information about A may become accessible on a small region of B, all remaining small regions of B must be only classically correlated with A, and moreover in some consistent basis.

Finally, the existence of a numerical algorithm for determing the quantum Markov blanket and associated measurement on A appears promising for applications of these results to many-body physics. In simple many-body examples, calculating the measurement A yields the effective classical variables being decohered within the system. In more complicated many-body systems where these effective classical variables may not be obvious, our results provide a new venue for analysis.

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