Fault-tolerant Coding for Quantum Communication

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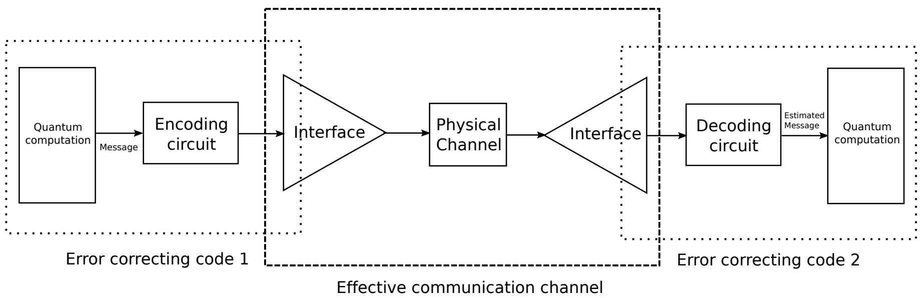
It is a basic problem in quantum Shannon theory to transmit classical or quantum messages via multiple uses of a noisy quantum channel. The famous capacity theorems by Holevo, Schumacher and Westmoreland [5, 8] in the case of classical messages, and by Lloyd, Shor and Devetak [4, 7, 9] in the case of quantum messages characterize the optimal communication rates achievable with vanishing error in the asymptotic limit of infinitely many channel uses. However, these theorems assume that encoding and decoding operations on large quantum systems can be implemented without faults. This assumption is unlikely to be satisfied in the near future, and it is even possible that the error rates of quantum logic gates will never effectively vanish, unlike the error rates of logic gates on a classical computer. While this issue has been resolved in quantum computing by using quantum error correcting codes and fault-tolerant implementations [1, 2], no such theory has been developed for the communication problems of quantum Shannon theory. Specifically, it is a priori not clear that the overall error of a coding scheme can vanish in the asymptotic limit of infinitely many channel uses at strictly positive communication rates when the quantum logic gates in the coding circuits fail with some fixed non-zero probability.

Difficulties arise when applying fault-tolerant techniques to quantum communication problems: The noise affecting a long communication line will typically be much larger than the noise affecting local gates, and special channel codes are needed to achieve communication rates close to the capacity. The encoding and decoding operations of such channel codes are large quantum circuits and to execute them reliably in the presence of gate errors they need to be implemented fault-tolerantly in a circuit code. However, the circuit code will in general not be compatible with the physical communication line (which might involve entirely different quantum hardware), and some kind of interface between this system and the circuit code will be needed. This setup is depicted in the figure below. Note that the interface is a quantum circuit itself and therefore affected by gate errors. Moreover, its execution has to leave the circuit code eventually and it will typically fail with a probability similar to that of individual gate errors.



We study the aforementioned setting of communication via quantum channels when the gates in encoding and decoding circuits are affected by a small level of noise. We focus on achievable communication rates with asymptotically vanishing overall coding error and the basic capacities in quantum Shannon theory, i.e. the classical capacity and the quantum capacity. For simplicity we focus on the noise model $\mathcal{F}_\pi(p)$ of Pauli errors affecting each location in a quantum circuit independently and which are identically distributed with a fixed probability $p \in [0,1]$. Our results can easily be adapted to local Markovian noise models. Our main contributions are:

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- We study interfaces for the concatenated 7-qubit Steane code, and determine the structure of the effective communication channel (see figure above) under the noise model $\mathcal{F}_\pi(p)$.

- We define the fault-tolerant classical capacity $C_{\mathcal{F}_\pi(p)}(T)$ of a classical-quantum or quantum channel T, and the fault-tolerant quantum capacity $Q_{\mathcal{F}_\pi(p)}(T)$ of a quantum channel T. These capacities take gate errors under the noise model $\mathcal{F}_\pi(p)$ affecting the encoder and decoder into account.

- We find a probability $\tilde{p} > 0$ and an explicit function $f : [0,1] \times \mathbb{N} \to \mathbb{R}^+$ satisfying $f(p,d) \to 0$ as $p \to 0$ such that

$$C_{\mathcal{F}_\pi(p)}(T) \geq C(T) - f(p,d)$$

  for any $0 \leq p \leq \tilde{p}$ and any classical-quantum channel $T : \mathcal{A} \to \mathcal{M}_d$.

- We find a probability $\tilde{p} > 0$ and explicit functions $g, h : [0,1] \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ satisfying $g(p, d_A, d_B, k), h(p, d_A, d_B, k) \to 0$ as $p \to 0$ such that

$$C_{\mathcal{F}_\pi(p)}(T) \geq \frac{1}{k}\chi\left(T^{\otimes k}\right) - g(p, d_A, d_B, k) \quad \text{and} \quad Q_{\mathcal{F}_\pi(p)}(T) \geq \frac{1}{k} I_{\text{coh}}\left(T^{\otimes k}\right) - h(p, d_A, d_B, k).$$

  for any $0 \leq p \leq \tilde{p}$ and any quantum channel $T : \mathcal{M}_{d_A} \to \mathcal{M}_{d_B}$.

- Our results immediately imply threshold theorems for fault-tolerant capacities:
  - For every $\epsilon > 0$ and dimension $d \geq 2$ there exists a threshold $p(\epsilon, d) > 0$ such that

$$C_{\mathcal{F}_\pi(p)}(T) \geq C(T) - \epsilon$$

    for all $0 \leq p \leq p(\epsilon, d)$ and for all classical-quantum channels $T : \mathcal{A} \to \mathcal{M}_d$.
  - For every $\epsilon > 0$ and every quantum channel $T : \mathcal{M}_{d_A} \to \mathcal{M}_{d_B}$ there exists a $p(\epsilon, T) > 0$ such that

$$C_{\mathcal{F}_\pi(p)}(T) \geq C(T) - \epsilon \quad \text{and} \quad Q_{\mathcal{F}_\pi(p)}(T) \geq Q(T) - \epsilon$$

    for all $0 \leq p \leq p(\epsilon, T)$.

Our results show that communication at strictly positive rates and with vanishing communication error is possible in non-trivial cases and in realistic scenarios where all local gates are affected by noise. To obtain our results we have to overcome several obstacles:

First, it is not immediately obvious how to define quantum communication rates in a fault-tolerant way. Fault-tolerance usually considers quantum computations with classical inputs and outputs, which are stable against errors thanks to classical error correcting codes. However, quantum communication also considers quantum messages, which are inherently prone to errors. We solve this issue by defining the fault-tolerant quantum capacity in an operational way embedding the coding scheme into an arbitrary quantum computation with classical input and output. Intuitively, the fault-tolerant quantum capacity then quantifies the optimal rates with which identity channels occuring in any quantum circuit can be approximated by the coding scheme involving a certain number of channel uses. This definition reduces to the ideal quantum capacity in the case of vanishing gate error rates. For more details see our attached article [3].

Second, we have to find fault-tolerant implementations of encoding and decoding circuits that yield efficient codes for the effective communication channel arising from an interface and the communication channel $T : \mathcal{M}_{d_A} \to \mathcal{M}_{d_B}$. For simplicity, we will focus on the special case $d_A = d_B = 2$ in the following outline of the proof strategy, and we refer to the attached article [3] for the general case. For proving our main results we use concatenated codes with the same concatenation level $l \in \mathbb{N}$ to protect quantum circuits at the sender and receiver, and we consider particular interface circuits which can be found in the attached article [3]. We show that the effective channel for the concatenated 7-qubit Steane code at noise parameter p, denoted by $T_{p,l} : \mathcal{M}_2 \otimes \mathcal{M}_2^{\otimes(7^l-1)} \to \mathcal{M}_2$, is of the form

$$T_{p,l} = (1 - cp)T \otimes \mathrm{Tr}_S + cpN_l,$$

for some quantum channel $N_l : \mathcal{M}_2 \otimes \mathcal{M}_2^{\otimes(7^l-1)} \to \mathcal{M}_2$ and where c denotes a constant. This effective channel takes as an input the ideal data before the interface circuit acts (i.e. the logical state encoded in the 7-qubit Steane code), and a syndrome state corresponding to some correctable fault pattern affecting the data (see [2] or our article [3] for more details). Here, the probability cp corresponds to the event where the interface circuit is not executed correctly. From the form of the effective channel it is clear that the coding scheme should consider the quantum channels $T_{p,l}^{\otimes m}(\cdot \otimes \sigma_S)$ for syndrome states σ_S instead of the ideal quantum channel T. Since the syndrome state σ_S depends on the coding scheme itself, and since it might be entangled across several code blocks of the concatenated code, this is not an i.i.d. coding scenario and techniques from beyond-i.i.d. quantum Shannon theory need to be applied.

Our strategy to find a fault-tolerant coding scheme works as follows: For any fixed $\delta > 0$ we can apply the Chernoff bound to show that

$$T_{p,l}^{\otimes m}(\cdot \otimes \sigma_S) \leq 2^{m(q+\delta)}\tilde{T}_q^{\otimes m} + \exp(-m\frac{\delta^2 q}{3})E \tag{1}$$

for a quantum channel $E : \mathcal{M}_2^{\otimes m} \to \mathcal{M}_2^{\otimes m}$ and where

$$\tilde{T}_q = (1 - q)T + q\frac{\mathbb{1}_2}{2}\mathrm{Tr}\,.$$

Here, we write $S_1 \leq S_2$ for linear maps S_1 and S_2 when $S_2 - S_1$ is completely positive. Finally, we apply the following strategy:

1. Find a coding scheme for classical or quantum communication respectively for the quantum channel $\tilde{T}_q$ at a fixed blocklength $k \in \mathbb{N}$.

2. Show that the coding scheme from 1. is a fault-tolerant coding scheme for the original quantum channel T, i.e. it transmits information over the channels $T_{p,l}^{\otimes m}(\cdot \otimes \sigma_S)$ when implemented in the quantum error correcting code.

3. Apply a continuity inequality (see [6]) for the quantity I_{coh}, to relate the resulting capacity bound involving $\tilde{T}_q$ to a similar bound involving the original channel T.

Note that step 1. in the previous strategy can be done using standard techniques from quantum Shannon theory (i.e. random code constructions). For step 2. we use (1) from above together with the fact that communication errors are monotone in the CP-ordering $\leq$. This leads to an error bound $\epsilon_{FT}(m) \leq 2^{m(q+\delta)}\epsilon_m$ of the fault-tolerant communication error $\epsilon_{FT}(m)$ in terms of the ideal communication error ϵ_m of the coding scheme constructed in step 1. involving m copies of the quantum channel $\tilde{T}_q$. To finish step 2. we need to determine communication rates for which $2^{m(q+\delta)}\epsilon_m \to 0$ as $m \to \infty$, which can be achieved by carefully chasing constants in the known capacity theorems from [4, 5, 7–9]. The final step 3. is straightforward.

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