Fault-tolerant Coding for Quantum Communication

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joint work with
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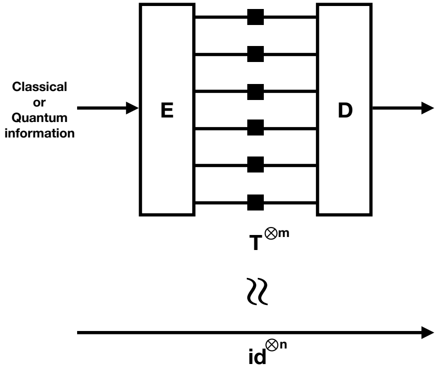
Capacities quantify the optimal rates of information transmission over noisy channels!

Capacities of quantum channels

Goal: Transmit classical or quantum information over quantum channel T.



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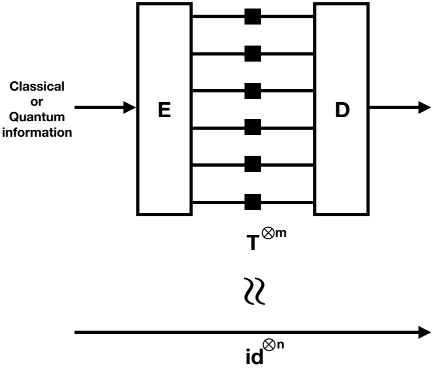

Coding error: $\epsilon = \|\mathrm{id}_2^{\otimes n} - D \circ T^{\otimes m} \circ E\| \rightsquigarrow (n, m, \epsilon)$-coding scheme.

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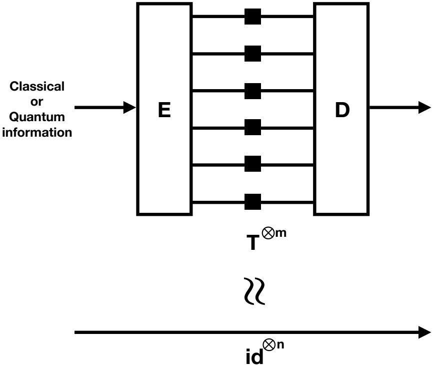

Optimal coding error: $\epsilon_m = \inf_{E_m, D_m} \|\mathrm{id}_2^{\otimes n_m} - D_m \circ T^{\otimes m} \circ E_m\|$.

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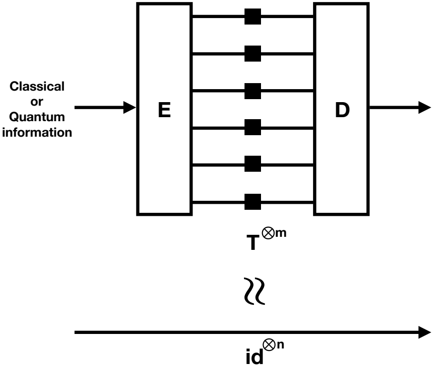


Optimal coding error: $\epsilon_m = \inf_{E_m, D_m} \|\mathrm{id}_2^{\otimes n_m} - D_m \circ T^{\otimes m} \circ E_m\|$.

Achievable rate: $R = \liminf_{m \to \infty} \frac{n_m}{m}$ such that $\epsilon_m \to 0$ as $m \to \infty$.

Capacities $C(T)$ and $Q(T)$ are the suprema of achievable rates!

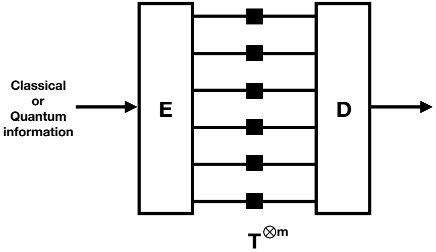
Capacity formulas

For a quantum channel $T : \mathcal{M}_{d_A} \to \mathcal{M}_{d_B}$ we have:

- $C(T) = \lim_{k\to\infty} \frac{1}{k}\chi(T^{\otimes k})$. (Holevo, Schumacher, Westmoreland)
- $Q(T) = \lim_{k\to\infty} \frac{1}{k} I_{\text{coh}}\left(T^{\otimes k}\right)$. (Lloyd, Shor, Devetak)

Provocative question: Are these formulas relevant in reality?

Problem



Assumption in quantum Shannon theory:

E, D can be executed without faults.

This is not realistic:

Quantum computers are inherently noisy!

What are the optimal rates for sending information reliably over quantum channels using noisy quantum hardware?

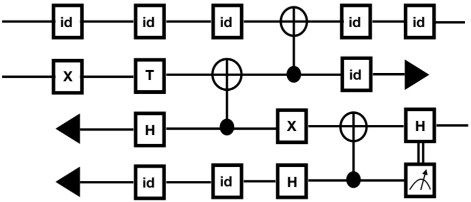
Very short introduction to fault-tolerance.

Universal set of elementary operations:

- Single-qubit gates: X, Y, Z and T.
- Two-qubit gate: $CNOT$.
- Identity gate (qubit at rest).
- Preparations and measurements in computational basis.
- Partial traces (throw qubits away).

Quantum circuits: Quantum channels build from elementary operations.

$\rightsquigarrow$ They are dense in the set of all quantum channels!



Probabilistic local noise models

In this talk: Focus on i. i. d. Pauli noise!

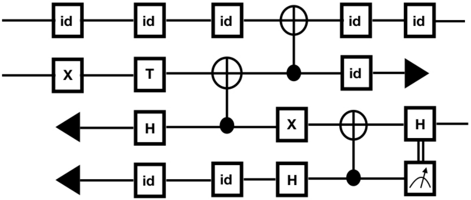
Elementary operations (locations) fail independently with probability p.

$\rightsquigarrow$ At the failing locations, we insert the noise channel

$$M(\rho) = \frac{1}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

into the circuit diagram:

- **Single qubit gate or preparation:** Insert M directly after operation.
- **CNOT gate:** Insert M directly after gate on both outputs.
- **Measurements:** Insert M directly before the measurement.



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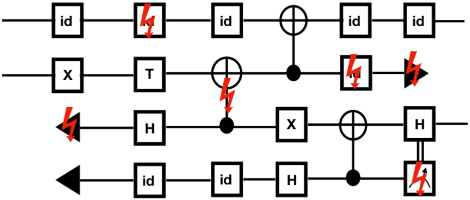

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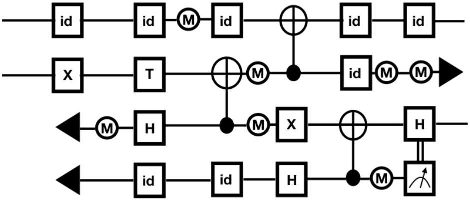

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- **Single qubit gate or preparation:** Insert M directly after operation.
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Terminology:

- $\mathcal{F}(p)$ denotes this fault model.
- For quantum circuit Γ write $[\Gamma]_{\mathcal{F}(p)}$ for noisy circuit.

The threshold theorem

Main idea: Implement quantum circuits in quantum error correcting code.
$\rightsquigarrow$ replace each elementary operation by a corresponding gadget.

Write: $\Gamma_{\mathcal{C}}$ for the implementation of a quantum circuit Γ in $\mathcal{C}$.

Theorem (Threshold Theorem by Aliferis, Gottesman, Preskill[1])

There exists a family $(\mathcal{C}_l)_{l\in\mathbb{N}}$ of QECCs with threshold $p_0 \in (0,1]$ such that:

For every quantum circuit Γ with classical input and output we have

$$\| [\Gamma_{\mathcal{C}_l}]_{\mathcal{F}(p)} - \Gamma\|_1 \leq C \left(\frac{p}{p_0}\right)^{2^l} |Loc(\Gamma)|,$$

for every $p < p_0$. Here, $Loc(\Gamma)$ denotes the set of locations in Γ.

[1] Aliferis, Gottesman, Preskill, "Quantum Accuracy Threshold for Concatenated Distance-3 Codes" Quantum Inf. and Comp. (2006)

For fault-tolerant communication we implement coding schemes in quantum error correcting codes.

Here, we focus on the classical capacity.

Fault-tolerant classical communication

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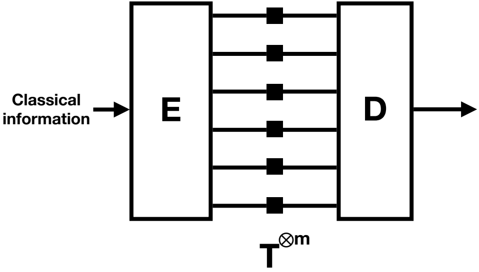

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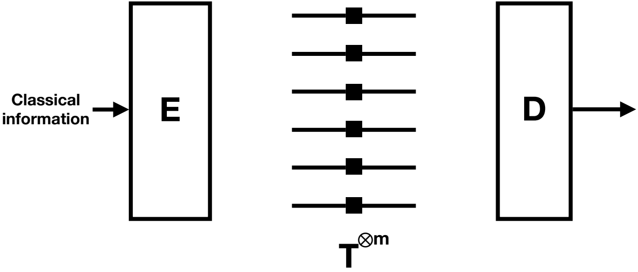

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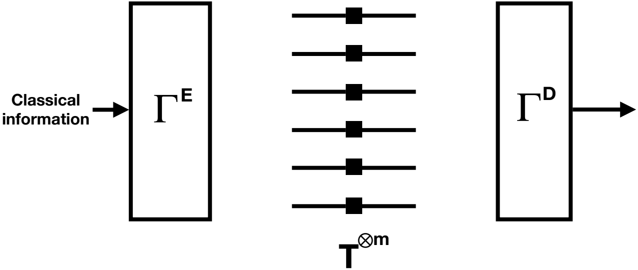

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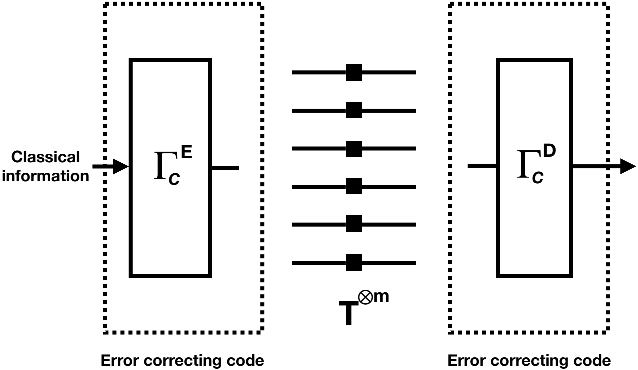

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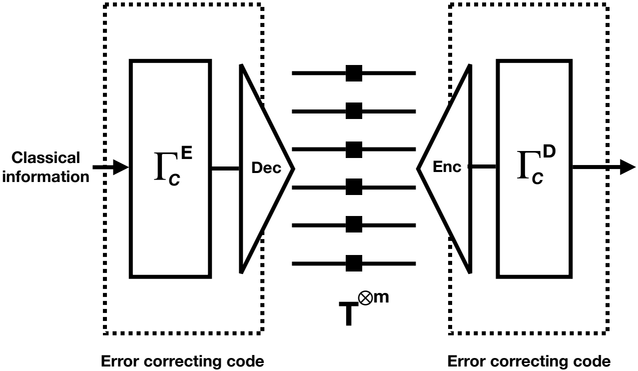


Problem: How do we get the logical states into and out of the channel?

$\rightsquigarrow$ Need interface quantum circuits! $\rightsquigarrow$ **This is noisy as well!**

Let's formalize this!

The fault-tolerant classical capacity

For simplicity: Consider qubit channel $T : \mathcal{M}_2 \to \mathcal{M}_2$.

For classical channel S on $\{1, \ldots, N\}$ define

$$\epsilon_{cl}(S) = \frac{1}{N} \sum_{i=1}^{N} \text{Prob}\left(S(i) \neq i\right),$$

the average classical communication error.

Definition (Fault-tolerant coding schemes for classical communication)

An (n, m, ϵ) fault-tolerant coding scheme for classical communication over T under the noise model $\mathcal{F}(p)$ is a pair of quantum circuits Γ^E and Γ^D with:

- $\Gamma^E : \mathbb{C}^{2^n} \to \mathcal{M}_2^{\otimes m}$ *has classical input.*
- $\Gamma^D : \mathcal{M}_2^{\otimes m} \to \mathbb{C}^{2^n}$ *has classical output.*
- *We have*

$$\inf_{\mathcal{C}_1, \mathcal{C}_2} \epsilon_{cl}\left(\left[\Gamma^D_{\mathcal{C}_1} \circ \text{Enc}_{\mathcal{C}_1} \circ T^{\otimes m} \circ \text{Dec}_{\mathcal{C}_2} \circ \Gamma^E_{\mathcal{C}_2}\right]_{\mathcal{F}(p)}\right) \leq \epsilon,$$

with infimum over QECCs $\mathcal{C}_1$ and $\mathcal{C}_2$, and $\text{Enc}_{\mathcal{C}_1}$ and $\text{Dec}_{\mathcal{C}_2}$.

$\rightsquigarrow$ **FT capacity:** $C_{\mathcal{F}(p)}(T)$ supremum of fault-tolerantly achievable rates.

Main result

Threshold theorem for capacity

FT capacity: $C_{\mathcal{F}(p)}(T)$ supremum of fault-tolerantly achievable rates.

Theorem (Christandl, AMH 2020)

For every quantum channel $T : \mathcal{M}_{d_A} \to \mathcal{M}_{d_B}$ *and every* $\delta > 0$*, there exists a threshold* $p(\delta, T) > 0$ *such that*

$$C_{\mathcal{F}(p)}(T) \geq C(T) - \delta.$$

for all $0 \leq p \leq p(\delta, T)$*. In particular, we have*

$$\lim_{p \searrow 0} C_{\mathcal{F}(p)}(T) = C(T),$$

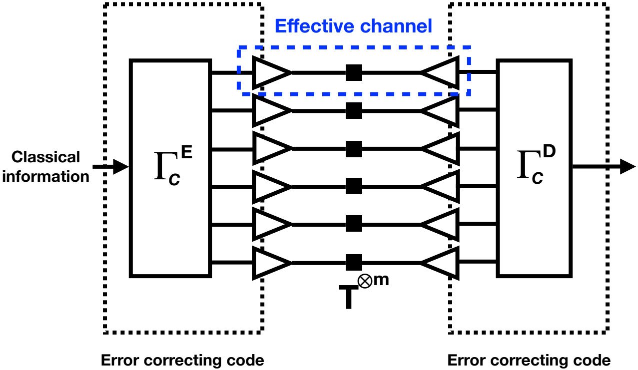
for all quantum channels $T : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$.

Sketch of the proof and some difficulties.

For simplicity, we consider qubit channel $T : \mathcal{M}_2 \to \mathcal{M}_2$**.**

Step 1: Choose a circuit code and interface circuit.

Circuit code and interface



Obvious choice: Use codes $(\mathcal{C}_l)_{l\in\mathbb{N}}$ from threshold theorem.

⇝ **Concatenated** 7**-qubit Steane code:** Encode 1 qubit into 7^l qubits.

⇝ **We can use a product interface.** ⇝ What is the effective channel?

Step 2: Identify effective channel.

Short intermezzo

How to analyze fault-tolerant circuits under noise?

Following Aliferis, Gottesman and Preskill[2]

[2]Following Aliferis, Gottesman, Preskill, "Quantum Accuracy Threshold for Concatenated Distance-3 Codes" Quantum Inf. and Comp. (2006)

QECCs $(\mathcal{C}_l)_{l\in\mathbb{N}}$ from threshold theorem

We have

$$\mathcal{C}_l = \text{span}\left(|\bar{0}\rangle, |\bar{1}\rangle\right) \subset (\mathbb{C}^2)^{\otimes 7^l}.$$

There exists direct-sum decomposition

$$(\mathbb{C}^2)^{\otimes 7^l} = \bigoplus_{s\in\{0,1\}^{7^l-1}} E_s(\mathcal{C}_l)$$

for product-Pauli operator E_s.

E_s is the Pauli error associated with syndrome s.

To correct errors, we measure syndrome and apply corresponding E_s!

Separating data and noise

We have

$$(\mathbb{C}^2)^{\otimes 7^l} = \bigoplus_{s\in\{0,1\}^{7^l-1}} E_s(\mathcal{C}_l) = \text{span}\left(\bigcup_{s\in\{0,1\}^{7^l-1}} \{E_s|\bar{0}\rangle, E_s|\bar{1}\rangle\}\right).$$

Define unitary map $D : (\mathbb{C}^2)^{\otimes 7^l} \to \mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes 7^l-1}$ by

$$D\left(E_s|\bar{i}\rangle\right) = |i\rangle \otimes |s\rangle.$$

Ideal decoder: $\text{Dec}^* : \mathcal{M}_2^{\otimes 7^l} \to \mathcal{M}_2 \otimes \mathcal{M}_2^{\otimes 7^l-1}$ given by

$$\text{Dec}^* = \text{Ad}_D.$$

Ideal encoder: $\text{Enc}^* : \mathcal{M}_2 \otimes \mathcal{M}_2^{\otimes 7^l-1} \to \mathcal{M}_2^{\otimes 7^l}$ given by

$$\text{Enc}^* = \text{Ad}_{D^\dagger} = (\text{Dec}^*)^{-1}$$

How to use these ideal operations?

Most basic case:

$$\mathrm{Dec}^*\left(E_{s_1}|\overline{\psi}\rangle\langle\overline{\psi}|E_{s_2}\right) = |\psi\rangle\langle\psi| \otimes |s_1\rangle\langle s_2|$$

Advanced case:

For noise channel $R(X) = \sum_k \mathrm{Ad}_{A_k}$ with each $A_k \in \mathrm{span}\,(E_s)$ we have

$$\mathrm{Dec}^*\left(R\left(|\overline{\psi}\rangle\langle\overline{\psi}|\right)\right) = |\psi\rangle\langle\psi| \otimes \sigma_S$$

for syndrome state σ_S depending on R.

Correctable errors transform to products under Dec*!

Step 2: Identify effective channel.

Identify effective channel

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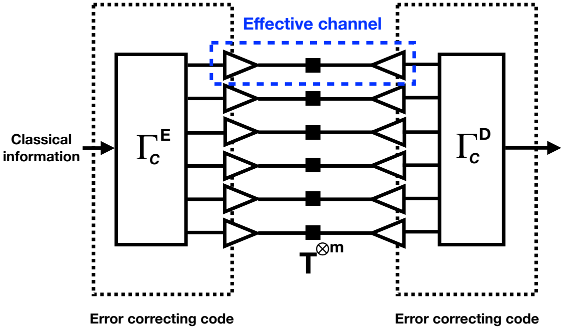

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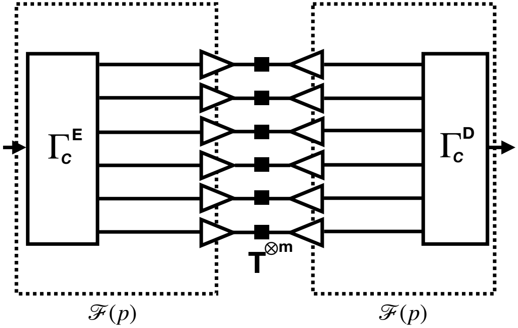

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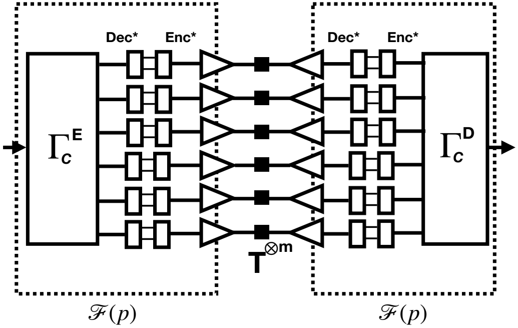

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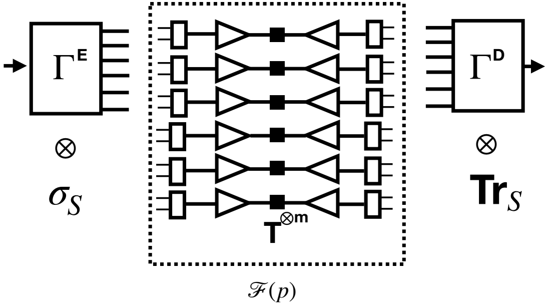

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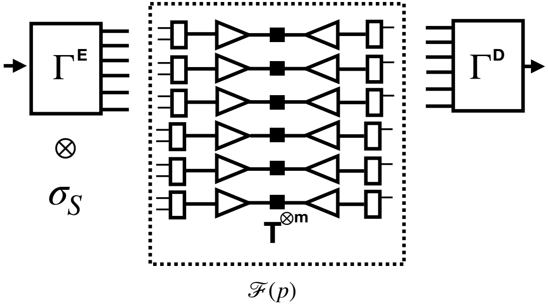

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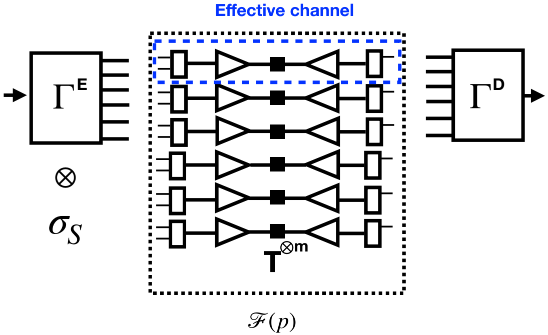

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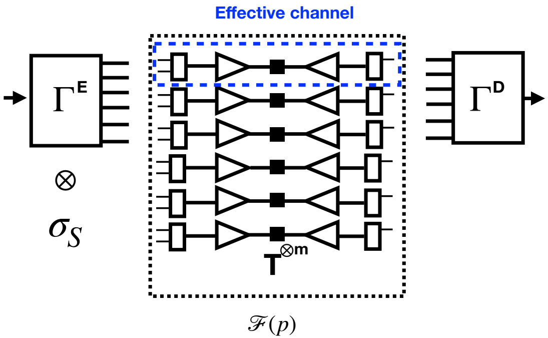

Effective channel: $T_{p,l} : \mathcal{M}_2 \otimes \mathcal{M}_2^{\otimes(7^l-1)} \to \mathcal{M}_2$

Effective channel

Have to find coding scheme (E_m, D_m) such that

$$\epsilon_{cl}\left(D_m \circ T_{p,l}^{\otimes m}(\cdot \otimes \sigma_S^m) \circ E_m\right) \to 0 \quad \text{as } m \to \infty.$$

$\rightsquigarrow$ **Fully-quantum arbitrarily varying channel[3]!**

Luckily, there is more structure:

Lemma (Christandl, AMH 2020)

There exists interface circuits Enc_l *and* Dec_l *for the QECC* $\mathcal{C}_l$ *such that the effective channel* $T_{p,l} : \mathcal{M}_2 \otimes \mathcal{M}_2^{\otimes(7^l-1)} \to \mathcal{M}_2$ *is of the form*

$$T_{p,l} = (1 - 2cp)T \otimes Tr_S + 2cpN_l,$$

for some quantum channel $N_l : \mathcal{M}_2 \otimes \mathcal{M}_2^{\otimes(7^l-1)} \to \mathcal{M}_2$ *and some constant* c.

See also: "Long-distance quantum communication over noisy networks without long-time quantum memory", Mazurek et al. Phys. Rev. A 90, 062311 (2014)

[3] "Fully Quantum Arbitrarily Varying Channels: Random Coding Capacity and Capacity Dichotomy", Boche, Deppe, Nötzel, Winter, Proc. ISIT (2018)

Step 3: Construct a coding scheme.

Construction of coding scheme

Exploit the special structure of

$$T_{p,l} = (1-2cp)T \otimes \mathrm{Tr}_S + 2cpN_l.$$

Lemma (Simple postselection)

For any syndrome state σ, any $m \in \mathbb{N}$ and any $\delta > 0$ we have

$$T_{p,l}^{\otimes m}(\cdot \otimes \sigma) \leq 2^{m(2cp+\delta)}\,\tilde{T}_p^{\otimes m} + \exp(-m\frac{2cp\delta^2}{3})K$$

for a quantum channel $K : \mathcal{M}_2^{\otimes m} \to \mathcal{M}_2^{\otimes m}$ and where

$$\tilde{T}_p = (1-2cp)T + 2cp\frac{\mathbb{1}_2}{2}Tr.$$

Then, we can show that:

$$\epsilon_{cl}\left(\left[\Gamma^{D_m}_{\mathcal{C}_{l_m}} \circ \mathrm{Enc}_{l_m} \circ T^{\otimes m} \circ \mathrm{Dec}_{l_m} \circ \Gamma^{E_m}_{\mathcal{C}_{l_m}}\right]_{\mathcal{F}(p)}\right) \leq 2^{m(2cp+\delta)}\epsilon_{cl}\left(D_m \circ \tilde{T}_p^{\otimes m} \circ E_m\right) + \epsilon_m.$$

where $\epsilon_m \to 0$ as $m \to \infty$.

Construction of coding scheme

Have to find coding scheme (E_m, D_m) such that

$$2^{m(2cp+\delta)}\epsilon_{cl}\left(D_m \circ \tilde{T}_p^{\otimes m} \circ E_m\right) \to 0.$$

as $m \to \infty$, for quantum channel

$$\tilde{T}_p = (1-2cp)T + 2cp\frac{\mathbb{1}_2}{2}\text{Tr}.$$

1. Random coding: This is possible for rates

$$R < \frac{1}{k}\chi\left(\tilde{T}_p^{\otimes k}\right) - g'(p),$$

where $g'(p) \to 0$ as $p \to 0$.

2. Continuity bound: Since $\tilde{T}_p \approx T$ any rate

$$R < \frac{1}{k}\chi\left(T^{\otimes k}\right) - g(p),$$

where $g(p) \to 0$ as $p \to 0$, is fault-tolerantly achievable.

Quantitative bound

Theorem (Christandl, AMH 2020)

There exists a $p_0 \in [0,1]$ such that for every $p \leq p_0$ we have

$$C_{\mathcal{F}(p)}(T) \geq \frac{1}{k}\chi\left(T^{\otimes k}\right) - 2\sqrt{2kcp} - 12cp - (1+4cp)\, h_2\left(\frac{4cp}{1+4cp}\right),$$

for any $k \in \mathbb{N}$ and where h_2 denotes the binary entropy.

Corollary (Christandl, AMH 2020)

For every quantum channel $T : \mathcal{M}_{d_A} \to \mathcal{M}_{d_B}$ and every $\delta > 0$, there exists a threshold $p(\delta, T) > 0$ such that

$$C_{\mathcal{F}(p)}(T) \geq C(T) - \delta.$$

for all $0 \leq p \leq p(\delta, T)$. In particular, we have

$$\lim_{p \searrow 0} C_{\mathcal{F}(p)}(T) = C(T),$$

for all quantum channels $T : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$.

There is more!

We have similar results for:

- Fault-tolerant classical capacity of classical-quantum channels.
- Fault-tolerant quantum capacity.

Check out our article: arXiv:2009.07161

Thank you for your attention.