Quasi-polynomial Time Approximation of Output Probabilities of Low-depth, Geometrically-local Quantum Circuits

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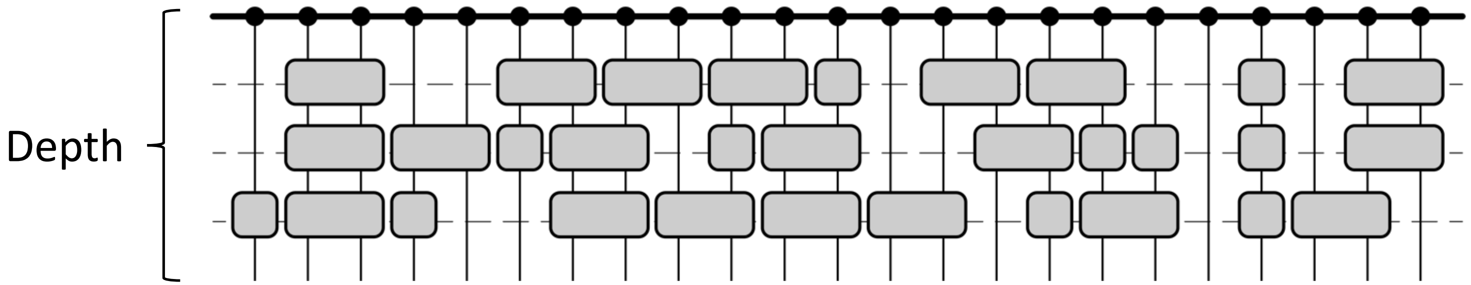
Problem Statement

Let C be a geometrically local, low-depth (logarithmic depth) quantum circuit acting on n qubits.

$\langle x|$

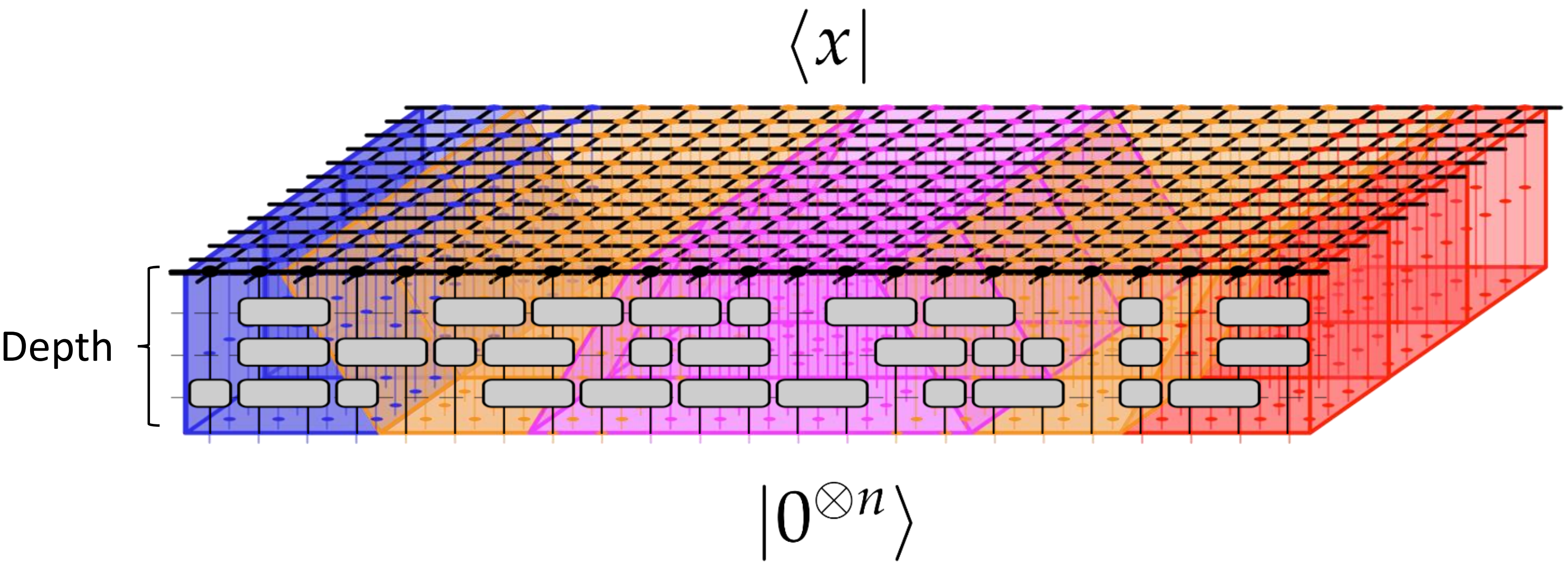


$|0^{\otimes n}\rangle$

- Goal: Compute the quantity
  $$|\langle x| C |0^{\otimes n}\rangle|^2 \pm \epsilon.$$
- What is the classical complexity of approximating $|\langle x| C |0^{\otimes n}\rangle|^2$ to *additive* error ϵ?
- 1-Dimensional Case.

Problem Statement

Let C be a 2D geometrically local, low-depth (logarithmic depth) quantum circuit acting on n qubits.



- Goal: Compute the quantity

$$|\langle x| C |0^{\otimes n}\rangle|^2 \pm \epsilon.$$

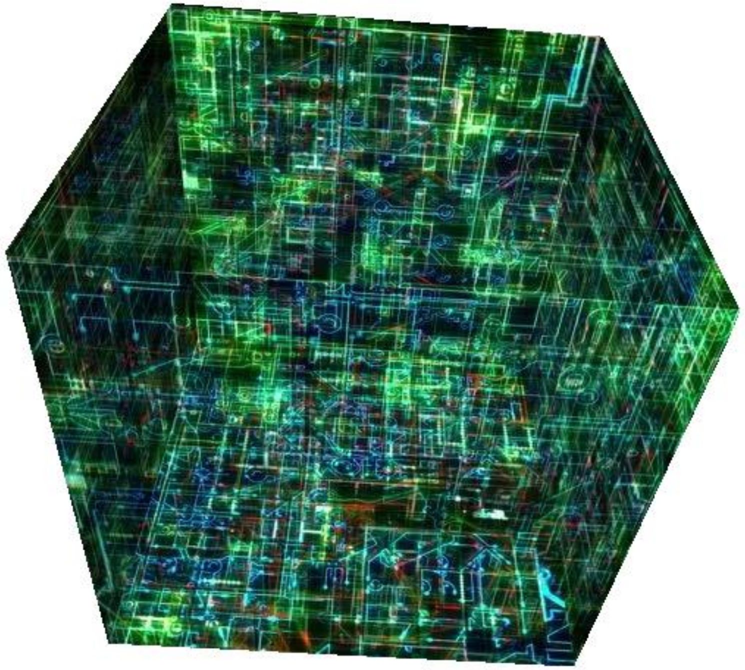
- What is the time complexity of approximating $|\langle x| C |0^{\otimes n}\rangle|^2$ to *additive* error ϵ?

- 2-Dimensional Case.
- Must solve *worst-case* over such circuits.
- Arbitrary 2-qubit gates allowed.

Problem Statement

Let C be a 3D geometrically local, low-depth (logarithmic depth) quantum circuit acting on n qubits.

$\langle x|$



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Depth $\{$ Use your imagination.

- Goal: Compute the quantity

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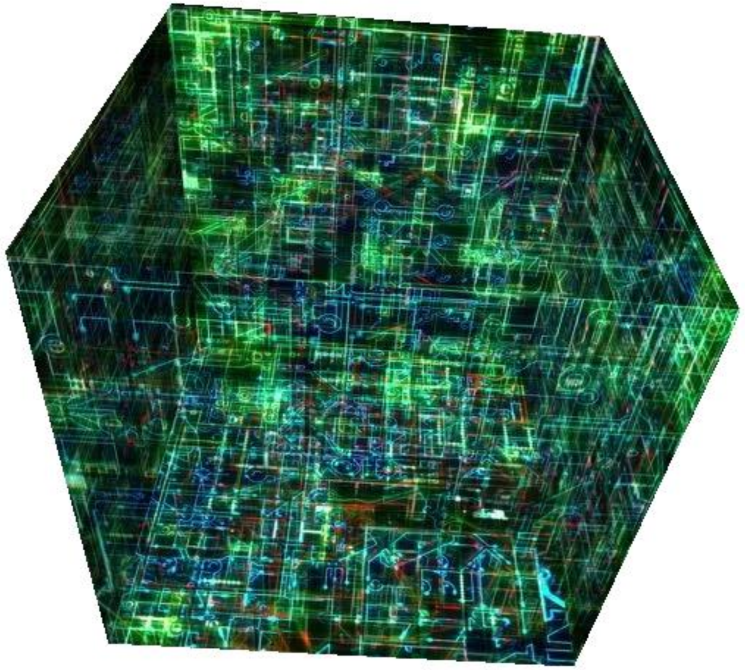
- 3-Dimensional Case.
- Must solve *worst-case* over such circuits.
- Arbitrary 2-qubit gates allowed.

Motivation

Let C be a geometrically local, low-depth (logarithmic depth) quantum circuit acting on n qubits.

$\langle x|$  $|0^{\otimes n}\rangle$

- Goal: Compute the quantity

$$|\langle x| C |0^{\otimes n}\rangle|^2 \pm \epsilon.$$

- Why should we care about this task?
- When $\epsilon \leq 2^{-n^2}$ this task is #P-Hard [Movassagh20].
- So ϵ parametrizes difficulty.
- So, can only hope to solve efficiently when $\epsilon \gg 2^{-n^2}$.

Motivation

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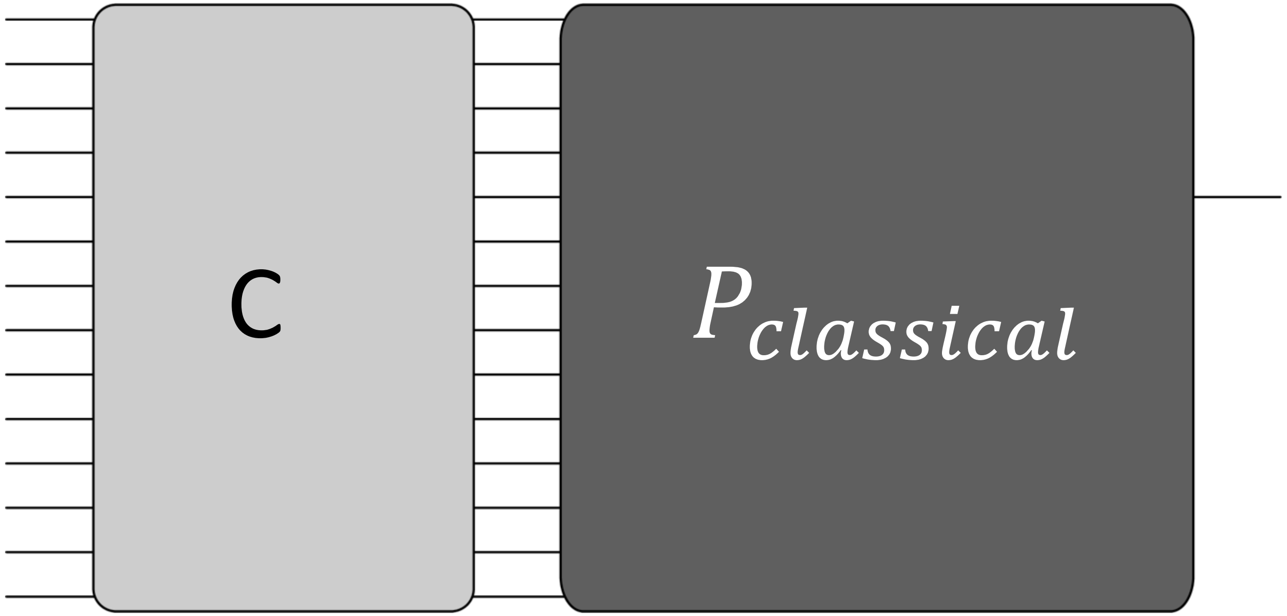

- Goal: Compute the quantity
$$|\langle x| C |0^{\otimes n}\rangle|^2 \pm \epsilon.$$
- Why should we care about this task for $\epsilon \sim \frac{1}{poly(n)}$?
- Relevant for classically simulating some hybrid quantum algorithms.
- Quantum circuit C composed with some classical post-processing.

Motivation

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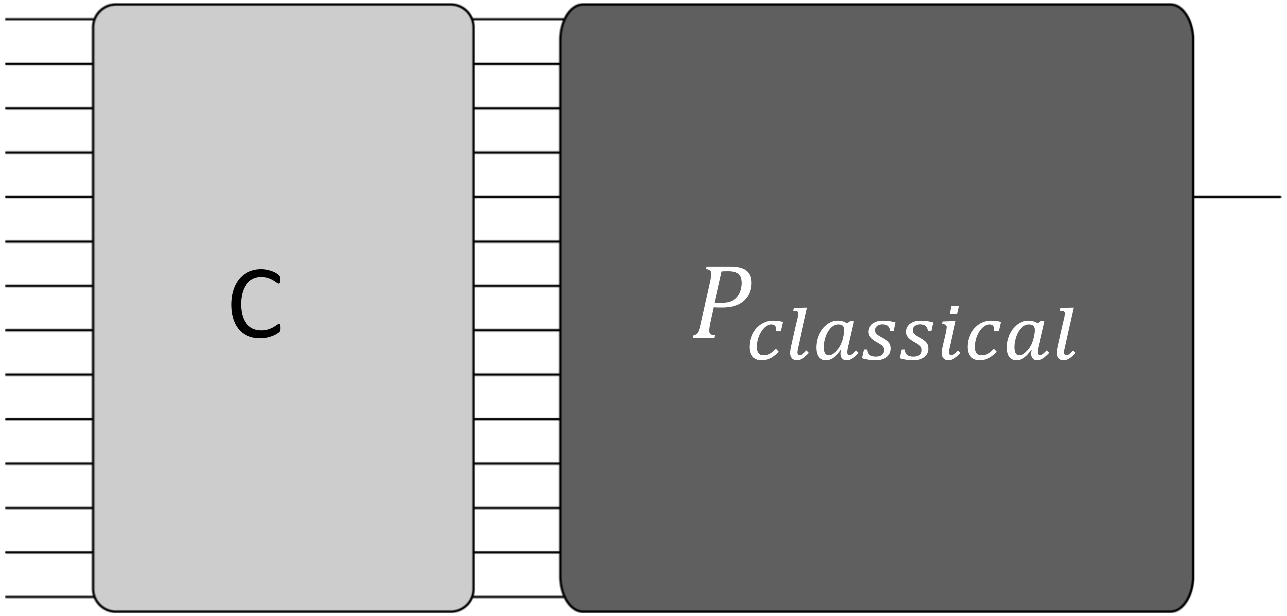

- Goal: Compute the quantity

$$|\langle x| C |0^{\otimes n}\rangle|^2 \pm \epsilon.$$

- Why should we care about this task for $\epsilon \sim \frac{1}{poly(n)}$?
- Our algorithm can estimate this output bit when
  - $P_{classical}$ = AND
    $$|\langle 0^{\otimes n}| X^{\otimes n} C |0^{\otimes n}\rangle|^2 \pm \epsilon.$$
  - $P_{classical}$ = OR
    $$1 - |\langle 0^{\otimes n}| C |0^{\otimes n}\rangle|^2 \pm \epsilon.$$
  - $P_{classical}$ = XOR
    $$|\langle 0^{\otimes n}| C Z^{\otimes n} C^{\dagger} |0^{\otimes n}\rangle|^2 \pm \epsilon.$$

WLOG can focus on $|\langle 0^{\otimes n}| C |0^{\otimes n}\rangle|^2 \pm \epsilon$, since $|\langle x| C |0^{\otimes n}\rangle|^2 = |\langle 0^{\otimes n}| \prod_i X^{x_i} C |0^{\otimes n}\rangle|^2$.

Motivation

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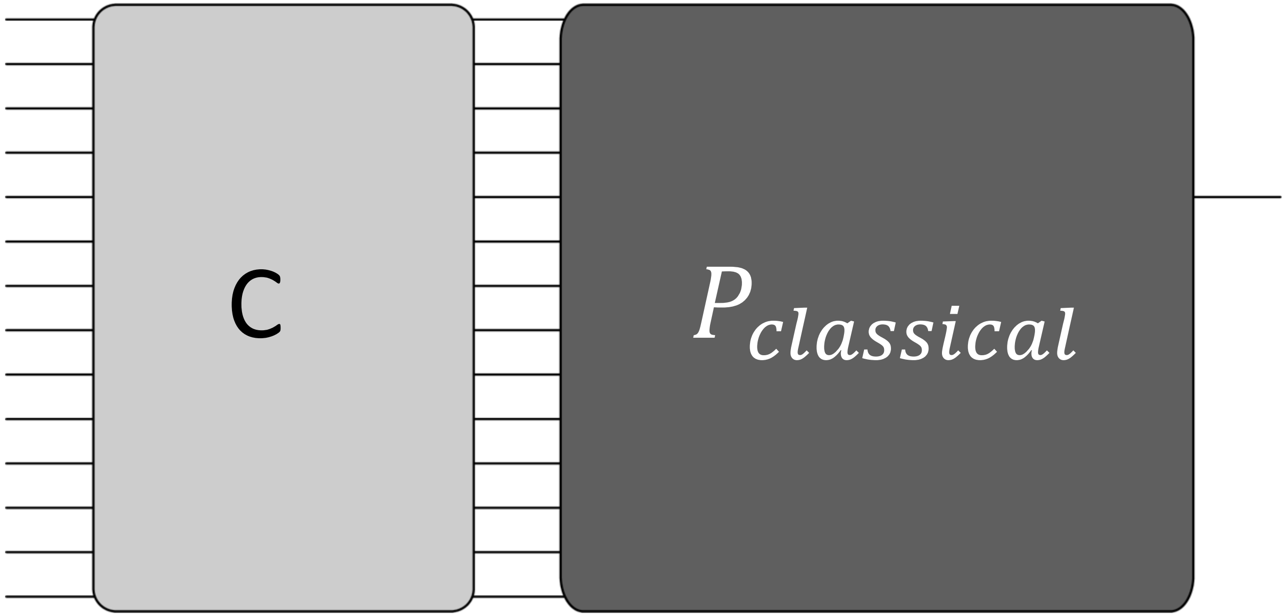


- Goal: Compute the quantity

$$|\langle 0^{\otimes n}| C |0^{\otimes n}\rangle|^2 \pm \epsilon.$$

- Why should we care about this task for $\epsilon \sim \frac{1}{poly(n)}$?
- Can also estimate:

$$|\langle 0^{\otimes n}| C(\prod_i P_i)C^\dagger |0^{\otimes n}\rangle|^2 \pm \epsilon.$$

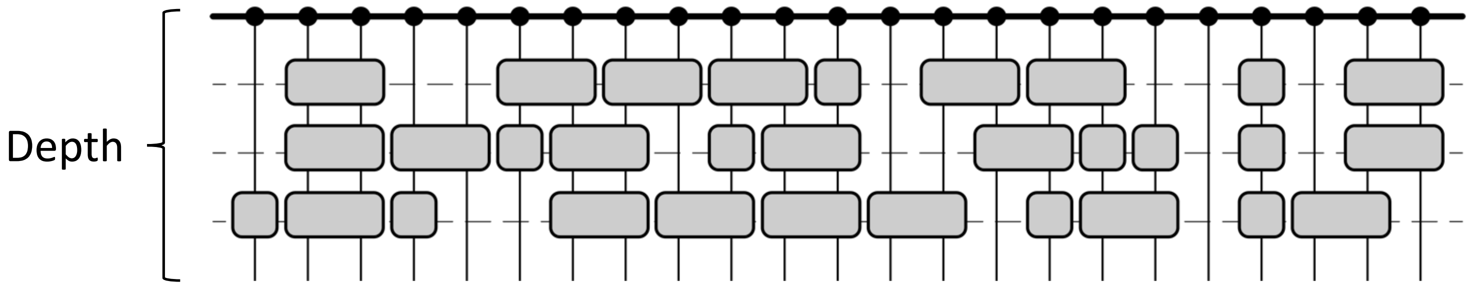
- Techniques: Controlling global correlations even though lightcones overlap!

WLOG can focus on $|\langle 0^{\otimes n}| C |0^{\otimes n}\rangle|^2 \pm \epsilon$, since $|\langle x| C |0^{\otimes n}\rangle|^2 = |\langle 0^{\otimes n}| \prod_i X^{x_i} C |0^{\otimes n}\rangle|^2$.

Background

Let C be a geometrically local, low-depth (logarithmic depth) quantum circuit acting on n qubits.

$\langle x|$



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- Goal: Compute the quantity

$$|\langle 0^{\otimes n}| C |0^{\otimes n}\rangle|^2 \pm \epsilon.$$

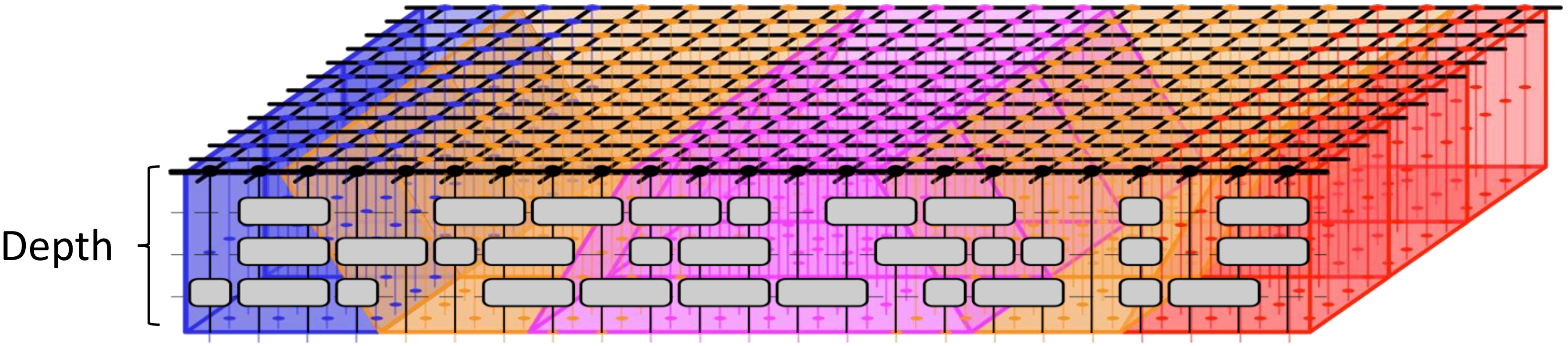
- What is the classical complexity of approximating $|\langle x| C |0^{\otimes n}\rangle|^2$ to *additive* error ϵ?
- Well known solutions to 1D case in poly-time:

e.g. Matrix Product States.

Background

Let C be a 2D geometrically local, low-depth (logarithmic depth) quantum circuit acting on n qubits.

$\langle x|$



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- Goal: Compute the quantity

$$|\langle 0^{\otimes n}| C |0^{\otimes n}\rangle|^2 \pm \epsilon.$$

- What is the time complexity of approximating $|\langle x| C |0^{\otimes n}\rangle|^2$ to *additive* error ϵ?
- Polynomial time solution to the 2D problem is non-trivial:
- Elegant classical algorithm of [Bravyi, Gosset, Movassagh '20].
- ϵ-approximation in poly(n, 1/ ϵ) time.

Background

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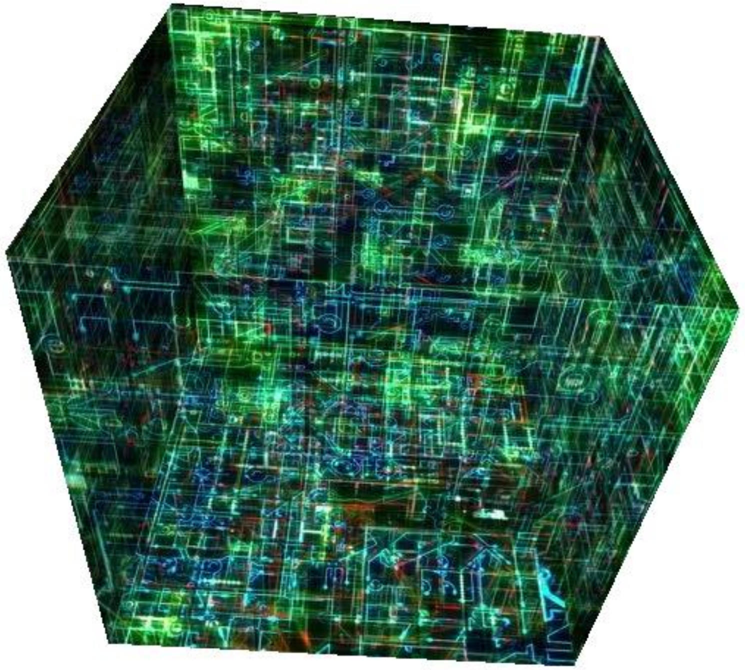

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Depth { Use your imagination.

- Goal: Compute the quantity

$$|\langle 0^{\otimes n}| C |0^{\otimes n}\rangle|^2 \pm \epsilon.$$

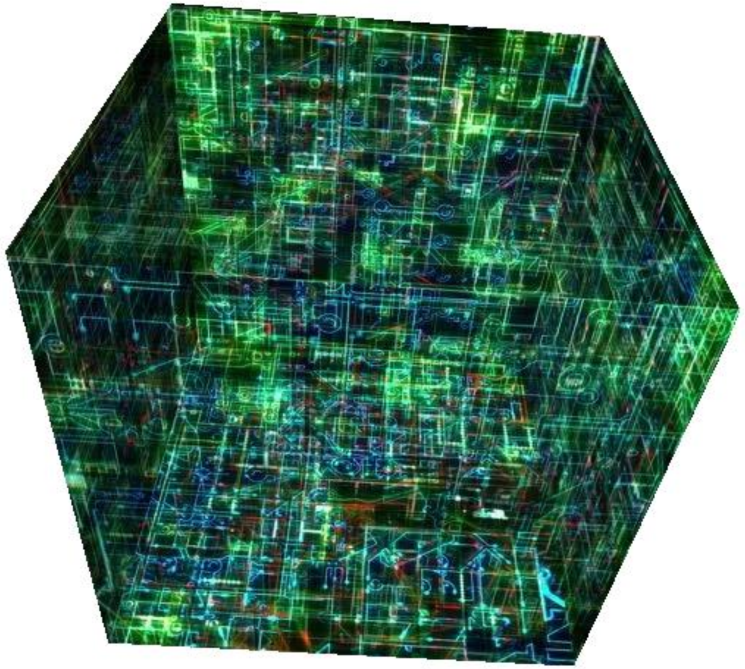
- What is the time complexity of approximating $|\langle x| C |0^{\otimes n}\rangle|^2$ to *additive* error ϵ?
- Techniques of [BGM20] may be limited to sub-exponential time in the 3D case.
- [Bravyi, Gosset, Movassagh ‘20], 3D case:

ϵ-approximation in poly($2^{n^{1/3}}$, $1/\epsilon$) time.

- Could 3D be drastically more complex than 2D?

Main Result

Let C be a 3D geometrically local, depth-d quantum circuit acting on n qubits.

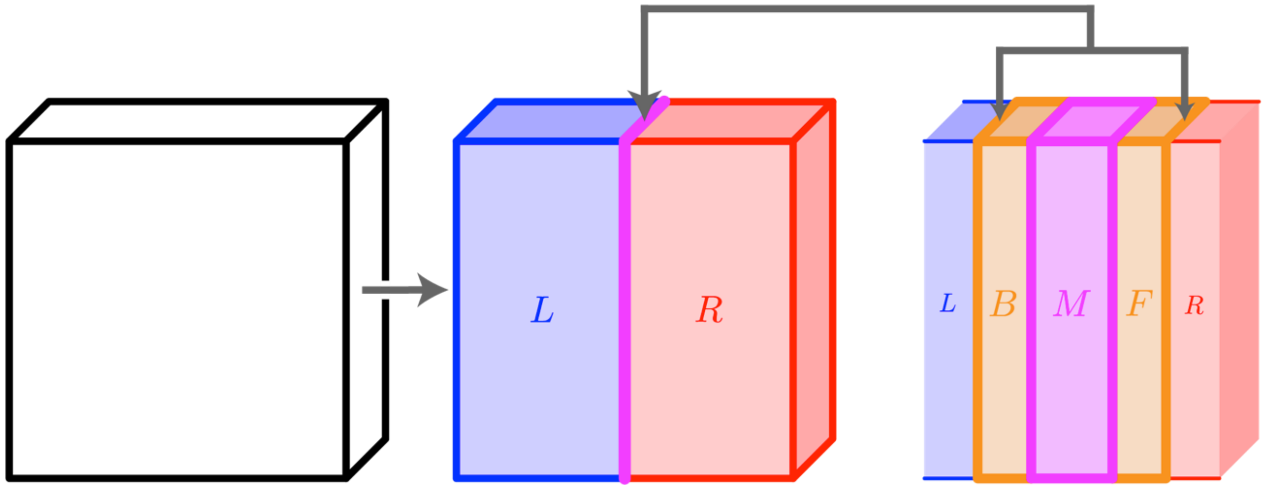
$\langle x|$  $|0^{\otimes n}\rangle$

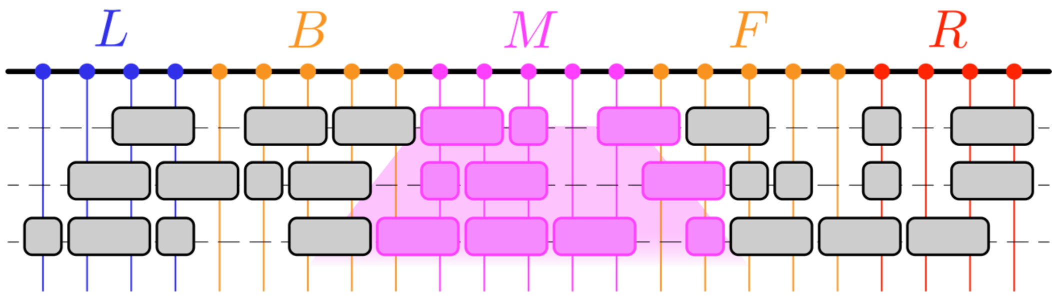
- Gives an inverse-polynomial additive approximation for any polynomial (asymptotically).
- Solves worst-case circuits.
- Allows arbitrary 2-qubit unitary gates.
- We believe our result will generalize to constant dimension D > 3, but we only prove the 3D case.

Theorem. *There exists a classical algorithm that approximates the quantity* $|\langle x|\, C\, |0^{\otimes n}\rangle|^2$ *to additive error* $1/n^{\log(n)}$ *for any* $x \in \{0,1\}^n$ *in time* $n^{\mathrm{polylog}(n)} 2^{d^3}$.

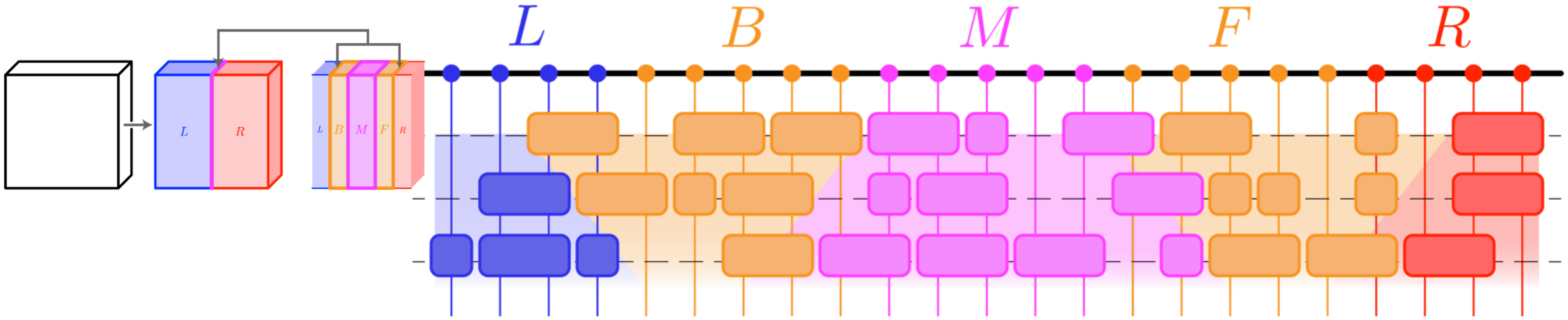
3D Circuits – A New Approach

- Unable to extend the [BGM20] algorithm, we are forced to pursue a new approach.
- Idea:
  - Lightcones in low-depth geometrically local circuits are local.
  - So, it is natural to consider a Divide-and-Conquer approach to estimating $\langle 0^{\otimes n} | C | 0^{\otimes n} \rangle$.
  - But, how should the division step work?





3D Circuits – Divide-and-Conquer



$$|\psi\rangle_{B\cup F} \equiv \langle 0|_M \, C_{B\cup M\cup F} \, |0\rangle_{B\cup M\cup F} = \sum_{i=1} \lambda_i \, |v_i\rangle_B \otimes |w_i\rangle_F$$

Imagine that this state has most of it's mass in it's top few Schmidt vectors across the division M.

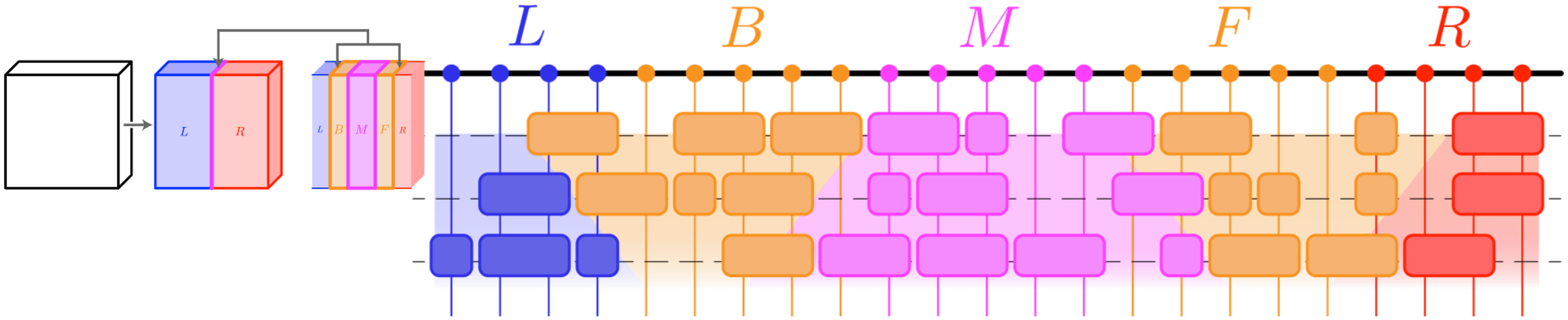
$$\langle 0_{\mathrm{ALL}}| \, C \, |0_{\mathrm{ALL}}\rangle = \langle 0_{\mathrm{ALL}}| \, C_{L\cup R} \, |0_{L\cup R}\rangle \otimes |\psi_{B\cup F}\rangle \approx \sum_{i=1}^{p(n)} \lambda_i \, \langle 0_{\mathrm{ALL}}| \, C_{L\cup R} \, |0_{L\cup R}\rangle \otimes |v_i\rangle_B \otimes |w_i\rangle_F$$

$$= \sum_{i=1}^{p(n)} \lambda_i \, \langle 0|_{L\cup B} \, C_L \, |0\rangle_L \otimes |v_i\rangle_B \cdot \langle 0|_{F\cup R} \, C_R \, |0\rangle_R \otimes |w_i\rangle_F$$

The original quantity is close to a sum of a few 3D problems of about half the size!

This looks like the beginnings of Divide-and-Conquer.

3D Circuits – Divide-and-Conquer



$$|\psi\rangle_{B\cup F} \equiv \langle 0|_M \, C_{B\cup M\cup F} \, |0\rangle_{B\cup M\cup F} = \textstyle\sum_{i=1} \lambda_i \, |v_i\rangle_B \otimes |w_i\rangle_F$$

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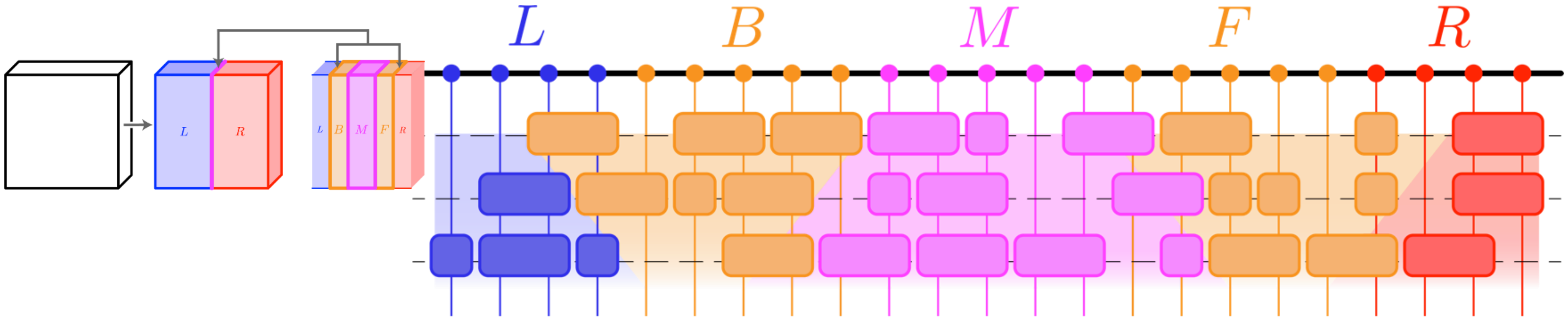
Two Problems:

1) Why should the state have most of it’s mass on a few Schmidt coefficients?
2) We would need to construct the corresponding Schmidt vectors via low-depth, geometrically local quantum circuits. Not clear how to do this.

3D Circuits – Divide-and-Conquer



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Problem #2: How to construct the corresponding Schmidt vectors via low-depth, geometrically local quantum circuits?

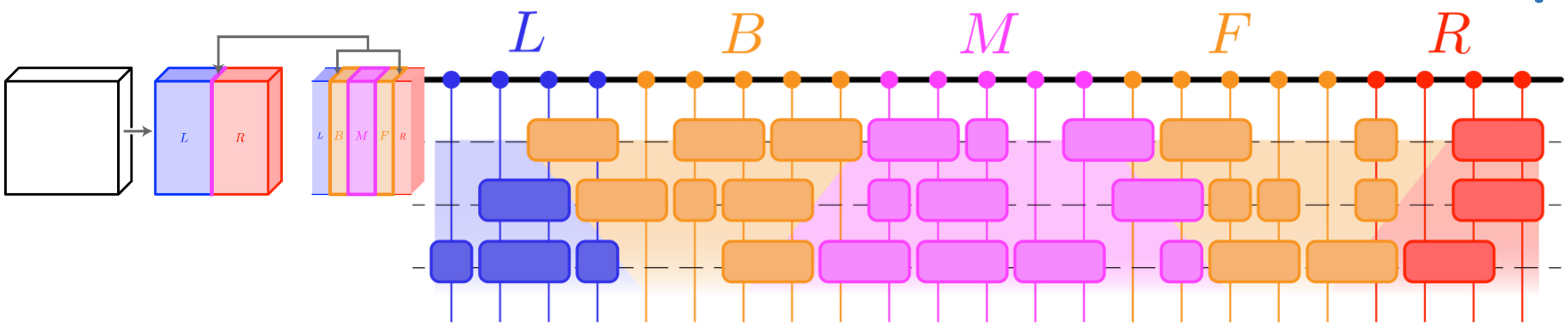
Idea: Use Block-Encodings,

Lemma (Lemma 45 of [GSLW19]). *The following is a 2D-local constant-depth circuit which gives a block encoding for* $\rho_F \equiv \mathrm{tr}_B(|\psi\rangle\langle\psi|_{B\cup F})$:

$$(C^\dagger_{B\cup M\cup F} \otimes I_{F'})(I_{B\cup M} \otimes SWAP_{FF'})(C_{B\cup M\cup F} \otimes I_{F'})$$

Traditionally used in quantum algorithms. Here we use them as a subroutine of a classical algorithm!

3D Circuits – Divide-and-Conquer



$$|\psi\rangle_{B\cup F} \equiv \langle 0|_M \, C_{B\cup M\cup F} \, |0\rangle_{B\cup M\cup F} = \sum_{i=1} \lambda_i \, |v_i\rangle_B \otimes |w_i\rangle_F$$

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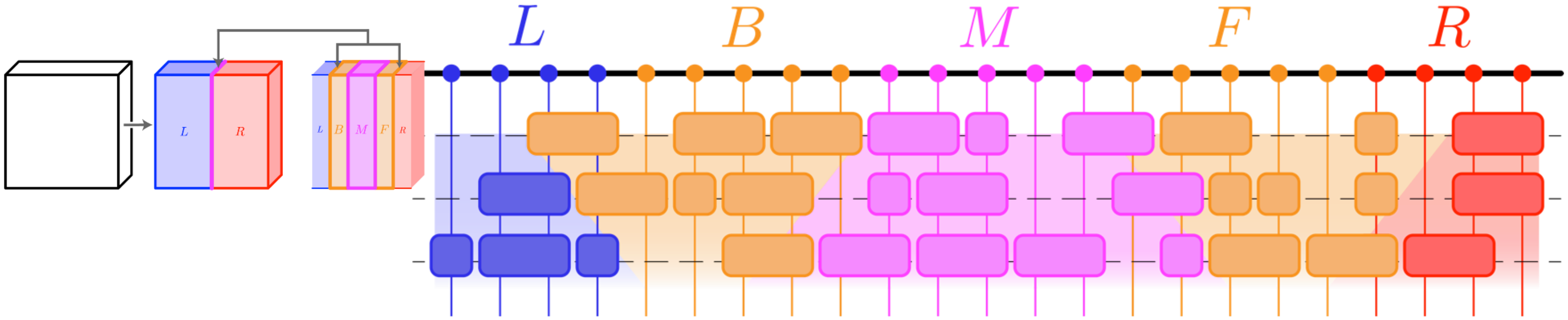
Idea: Use Block-Encodings,

That is,

$$\rho_F \equiv \mathrm{tr}_B(|\psi\rangle\langle\psi|_{B\cup F}) = \langle 0_{ALL/F}| \, (C^\dagger_{B\cup M\cup F} \otimes I_{F'})(I_{B\cup M} \otimes \mathrm{SWAP}_{FF'})(C_{B\cup M\cup F} \otimes I_{F'}) \, |0_{ALL/F}\rangle$$

3D Circuits – Divide-and-Conquer



$$|\psi\rangle_{B\cup F} \equiv \langle 0|_M C_{B\cup M\cup F} |0\rangle_{B\cup M\cup F} = \sum_{i=1} \lambda_i |v_i\rangle_B \otimes |w_i\rangle_F$$

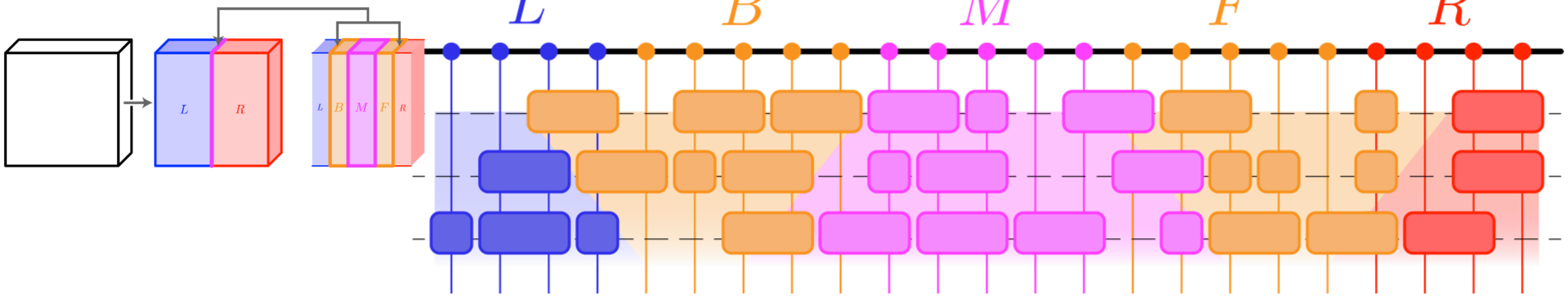
Problem #2: How to construct the corresponding Schmidt vectors via low-depth, geometrically local quantum circuits?

Idea: Use Block-Encodings,

Furthermore,

$$\rho_F^K \equiv (\mathrm{tr}_B(|\psi\rangle \langle\psi|_{B\cup F}))^K$$
$$= \langle 0_{ALL/F}| \prod_{i=1}^{K} (C^\dagger_{B\cup M\cup F} \otimes I_{F_1,\ldots,F_K})(I_{B\cup M} \otimes \mathbf{SWAP}_{FF_i})(C_{B\cup M\cup F} \otimes I_{F_1,\ldots,F_K}) |0_{ALL/F}\rangle$$

3D Circuits – Divide-and-Conquer



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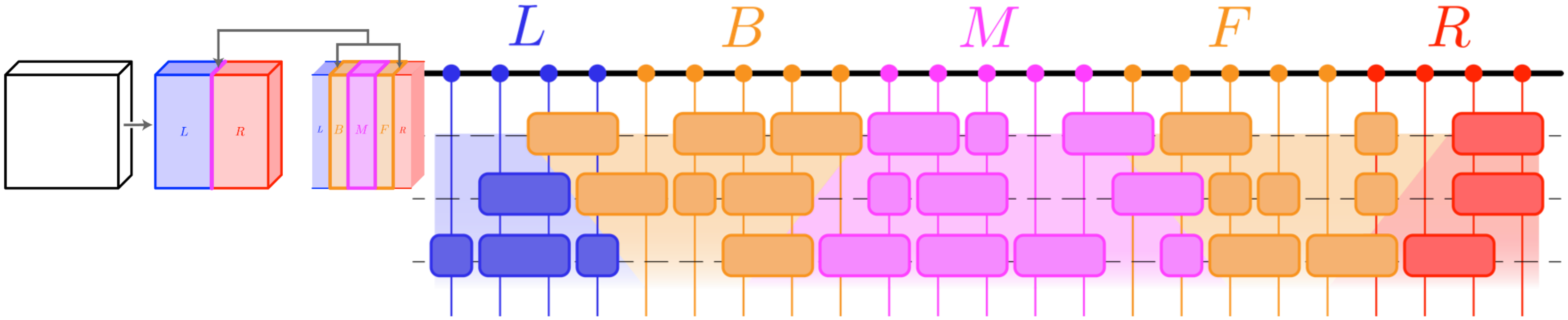
Problem #2: How to construct the corresponding Schmidt vectors via low-depth, geometrically local quantum circuits?

$$\rho_F^K \equiv (\mathrm{tr}_B(|\psi\rangle\langle\psi|_{B\cup F}))^K$$

$$= \langle 0_{ALL/F}| \prod_{i=1}^{K} (C^\dagger_{B\cup M\cup F} \otimes I_{F_1,\ldots,F_K})(I_{B\cup M} \otimes \mathbf{SWAP}_{FF_i})(C_{B\cup M\cup F} \otimes I_{F_1,\ldots,F_K}) |0_{ALL/F}\rangle$$

$$\approx \gamma |w_1\rangle\langle w_1|_F \text{ when } |\lambda_1 - \lambda_2| \text{ is sufficiently large.}$$

3D Circuits – Divide-and-Conquer



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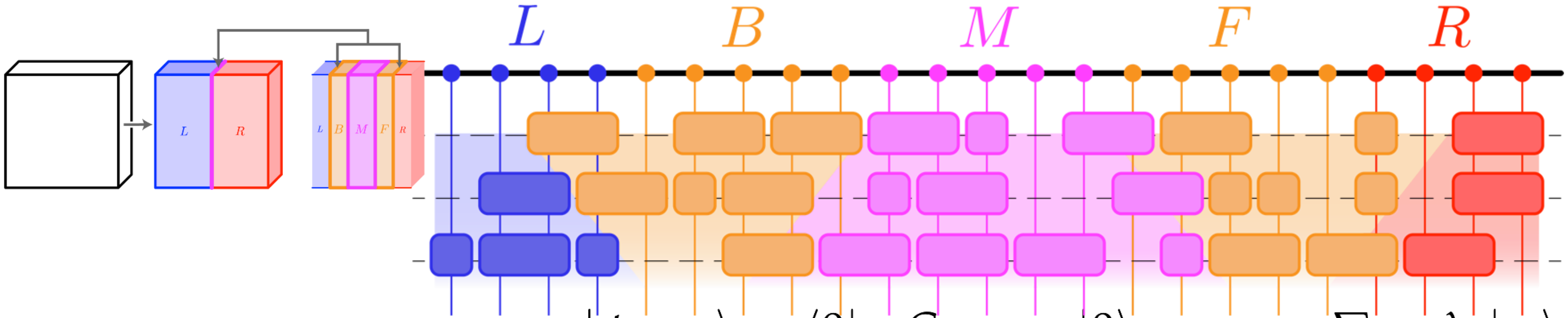
Problem #1: Why should the state have most of it's mass on a few Schmidt coefficients? In particular, why should $|\lambda_1 - \lambda_2|$ even be non-zero?

3D Circuits – Divide-and-Conquer



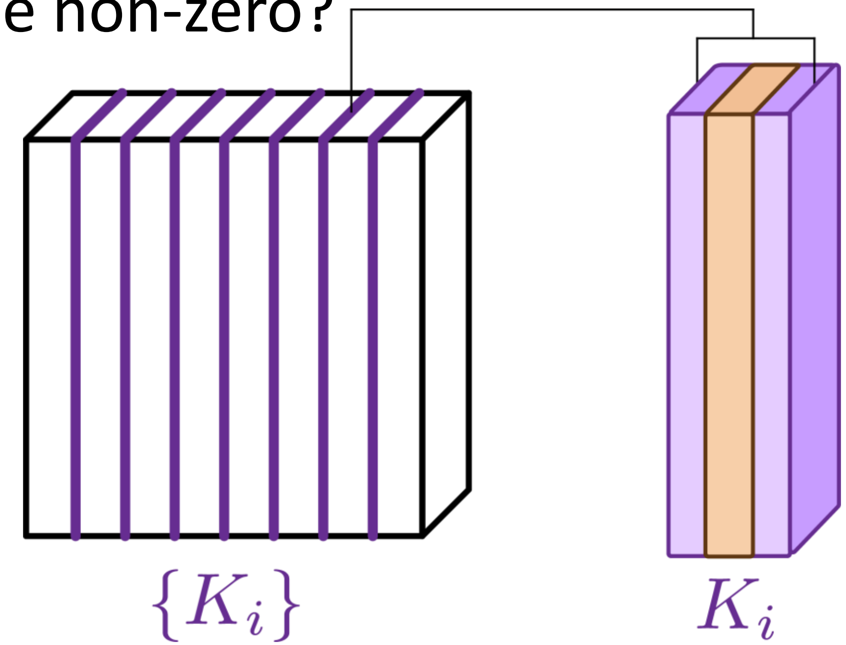
$$|\psi_{B_i \cup F_i}\rangle \equiv \langle 0|_M \, C_{B\cup M\cup F} \, |0\rangle_{B\cup M\cup F} = \sum_{i=1} \lambda_i \, |v_i\rangle_B \otimes |w_i\rangle_F$$

Problem #1: Why should the state have most of it's mass on a few Schmidt coefficients? In particular, why should $|\lambda_1 - \lambda_2|$ even be non-zero?

Assume that $\langle 0^{\otimes n}| \, C \, |0^{\otimes n}\rangle \geq 1/n^{\log(n)}$.

Lemma. *In every interval of length* $\log^5(n)$ *there are at least* $\log(n)$ *cuts* K_i *satisfying* $\lambda_1^i \geq 3/4$.

$$So, |\lambda_1^i - \lambda_2^i| \geq \frac{3-\sqrt{7}}{4} = const$$



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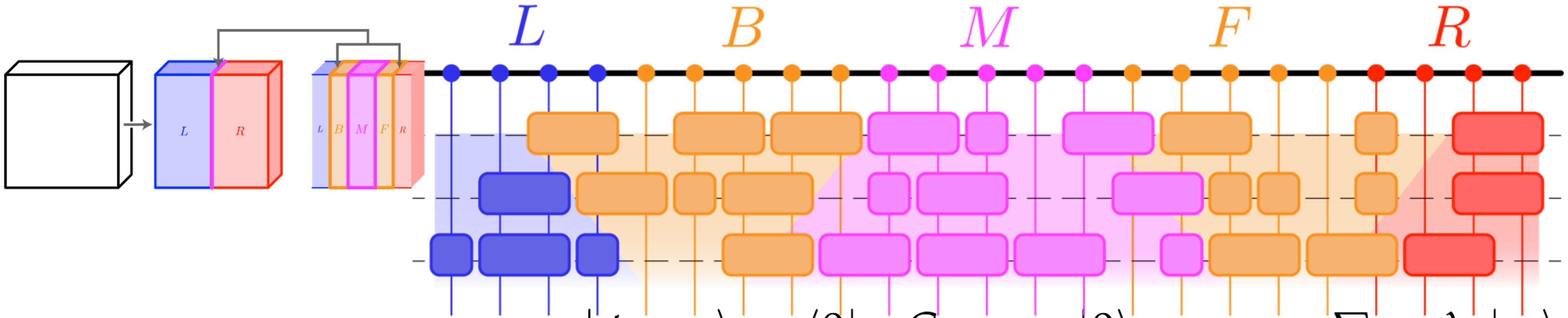

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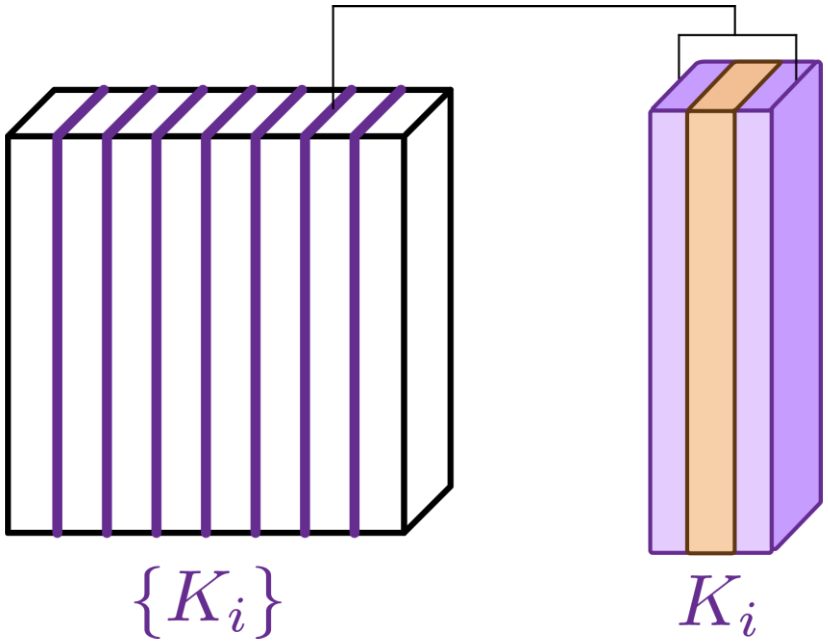
Assume that $\langle 0^{\otimes n}| \, C \, |0^{\otimes n}\rangle \geq 1/n^{\log(n)}$.

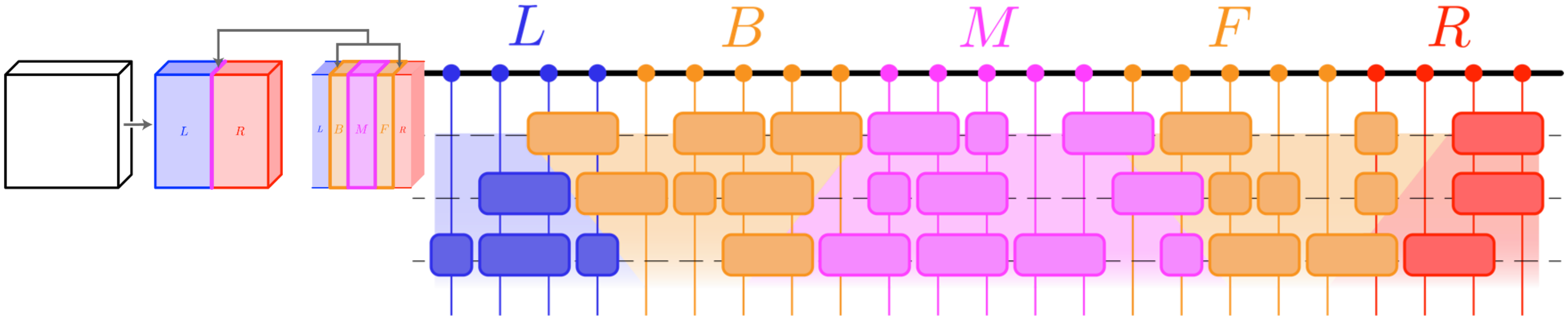
Lemma. *In every interval of length* $\log^5(n)$ *there are at least* $\log(n)$ *cuts* K_i *satisfying:* $|\lambda_1^i - \lambda_2^i| \geq \frac{3-\sqrt{7}}{4} = const.$

$$\rho_F^K = \langle 0_{ALL/F}| \prod_{i=1}^{K} (C^{\dagger}_{B\cup M\cup F} \otimes I_{F_1,\ldots,F_K})(I_{B\cup M} \otimes \mathrm{SWAP}_{FF_i})(C_{B\cup M\cup F} \otimes I_{F_1,\ldots,F_K}) \, |0_{ALL/F}\rangle$$
$$\approx |w_1\rangle \langle w_1|_F + O((3/4)^K)$$

So, we can find many cuts for which we can construct the largest Schmidt vector with a depth-K, 2D local quantum circuit!



3D Circuits – Divide-and-Conquer



Recall:

$$|\psi\rangle_{B\cup F} \equiv \langle 0|_M C_{B\cup M\cup F} |0\rangle_{B\cup M\cup F} = \sum_{i=1} \lambda_i |v_i\rangle_B \otimes |w_i\rangle_F$$

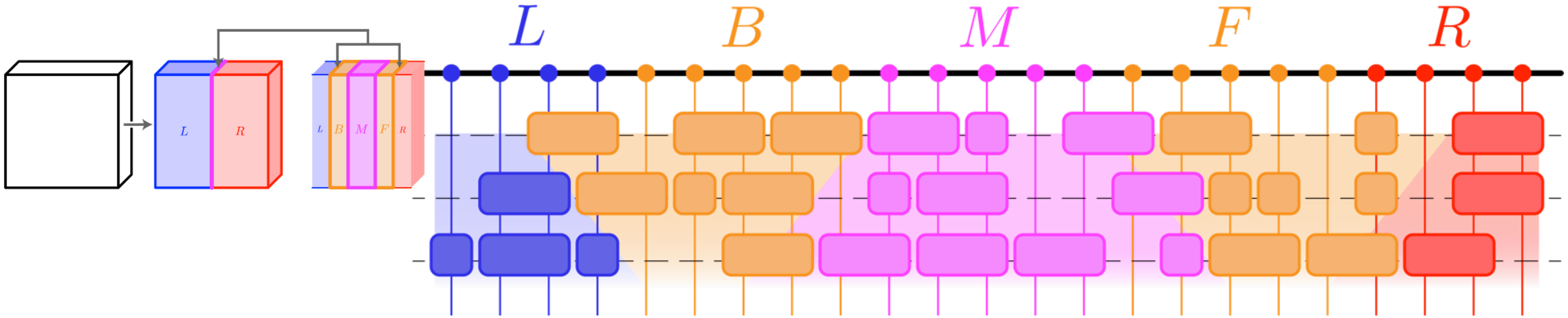
Imagine that this state has most of it's mass in it's top few Schmidt vectors across the division M.

$$\langle 0_{\text{ALL}}| C |0_{\text{ALL}}\rangle = \langle 0_{\text{ALL}}| C_{L\cup R} |0_{L\cup R}\rangle \otimes |\psi_{B\cup F}\rangle$$

$$\approx \sum_{i=1}^{p(n)} \lambda_i \langle 0_{\text{ALL}}| C_{L\cup R} |0_{L\cup R}\rangle \otimes |v_i\rangle_B \otimes |w_i\rangle_F = \sum_{i=1}^{p(n)} \lambda_i \langle 0|_{L\cup B} C_L |0\rangle_L \otimes |v_i\rangle_B \cdot \langle 0|_{F\cup R} C_R |0\rangle_R \otimes |w_i\rangle_F$$

Problem: We can only construct the top Schmidt vector with a geometrically local constant depth quantum circuit!

3D Circuits – Divide-and-Conquer



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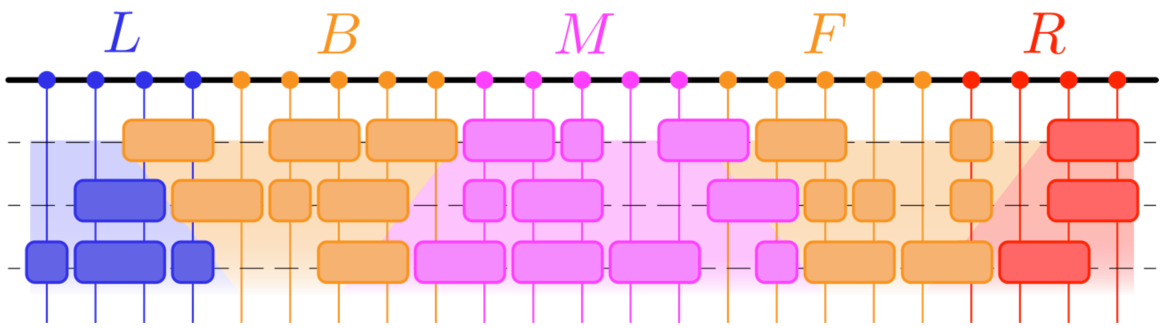
$$\approx \sum_{i=1}^{p(n)} \lambda_i \langle 0_{\mathrm{ALL}}| C_{L\cup R} |0_{L\cup R}\rangle \otimes |v_i\rangle_B \otimes |w_i\rangle_F \approx \lambda_1 \langle 0|_{L\cup B} C_L |0\rangle_L \otimes |v_1\rangle_B \cdot \langle 0|_{F\cup R} C_R |0\rangle_R \otimes |w_1\rangle_F \pm 1/4$$

Problem: We can only construct the top Schmidt vector with a geometrically local constant depth quantum circuit!

But this results in a constant sized error term since $\lambda_1 \geq 3/4$.

3D Circuits – Divide-and-Conquer



$$|\psi\rangle_{B\cup F} \equiv \langle 0|_M \, C_{B\cup M\cup F} \, |0\rangle_{B\cup M\cup F} = \sum_{i=1} \lambda_i \, |v_i\rangle_B \otimes |w_i\rangle_F$$

$$\langle 0_{\mathrm{ALL}}| \, C \, |0_{\mathrm{ALL}}\rangle = \lambda_1 \, \langle 0|_{L\cup B} \, C_L \, |0\rangle_L \otimes |v_1\rangle_B \cdot \langle 0|_{F\cup R} \, C_R \, |0\rangle_R \otimes |w_1\rangle_F \pm 1/4$$

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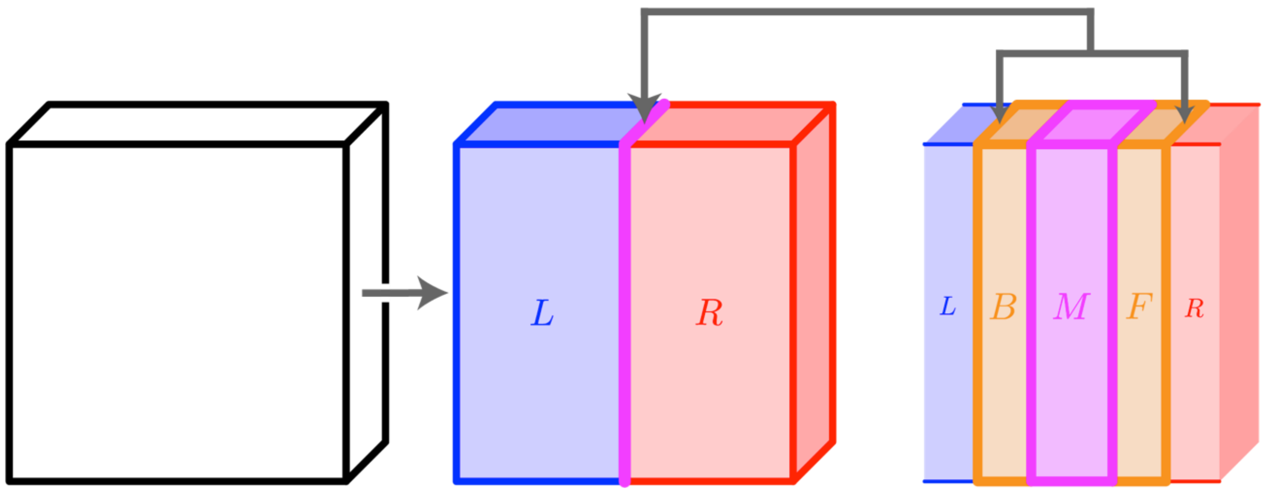

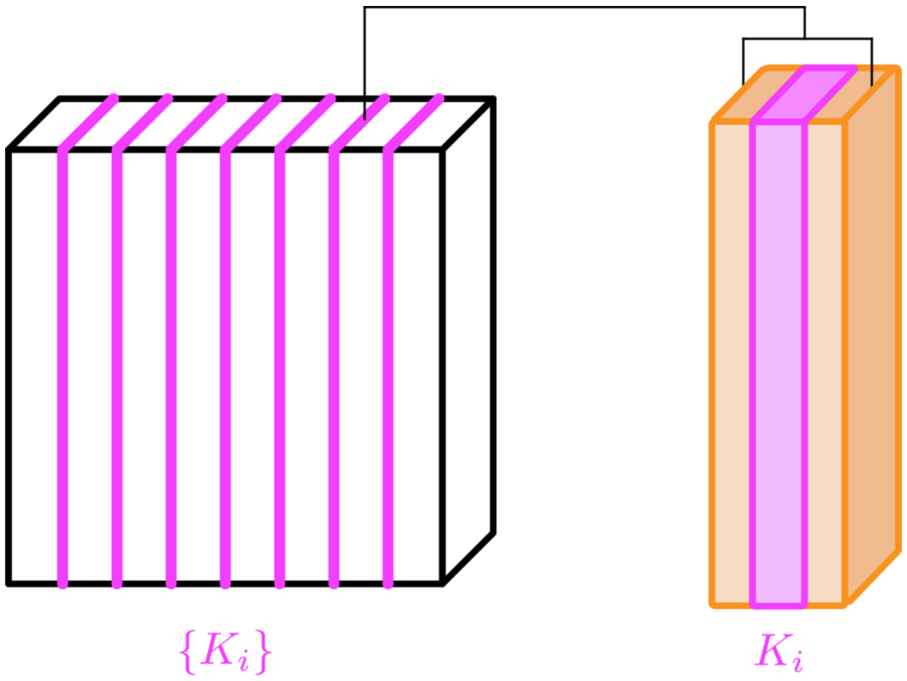

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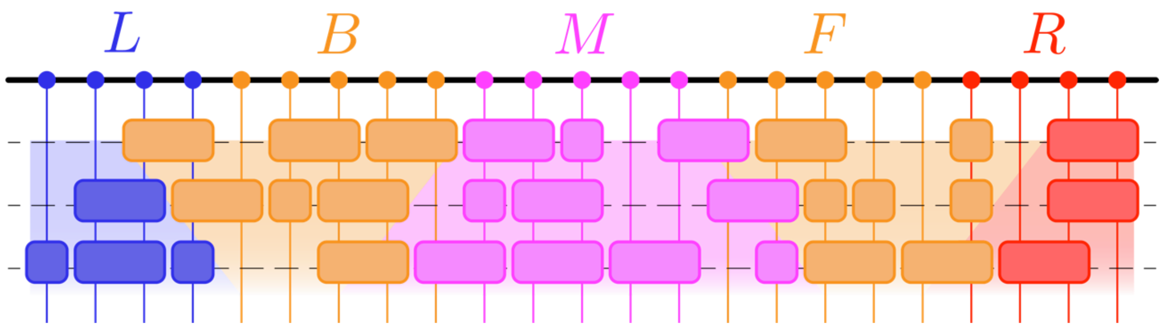

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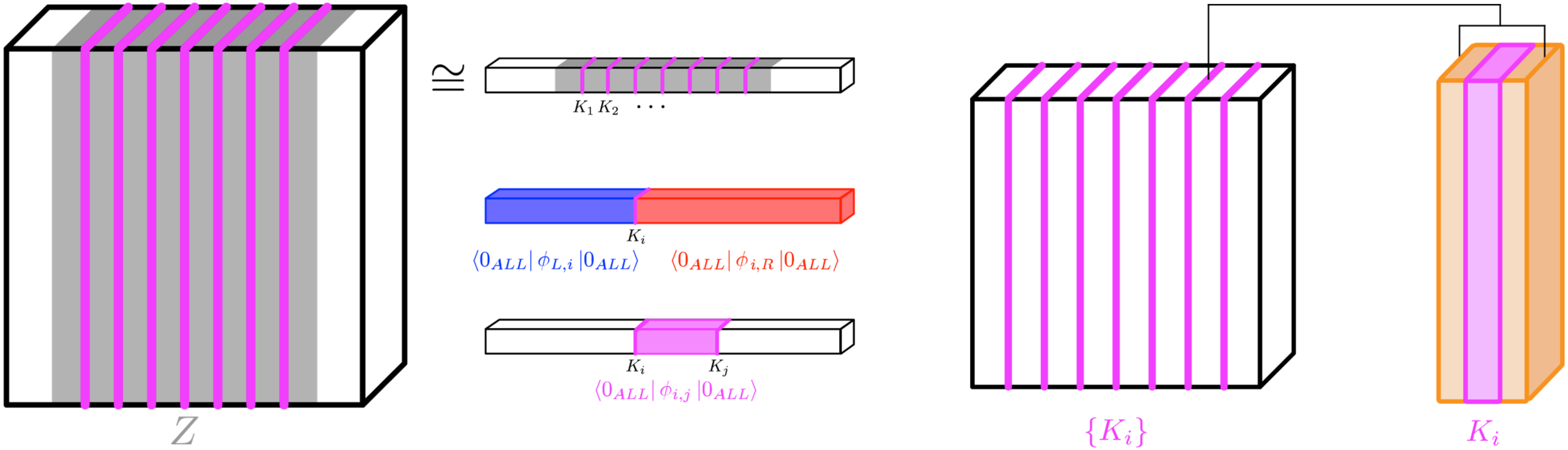

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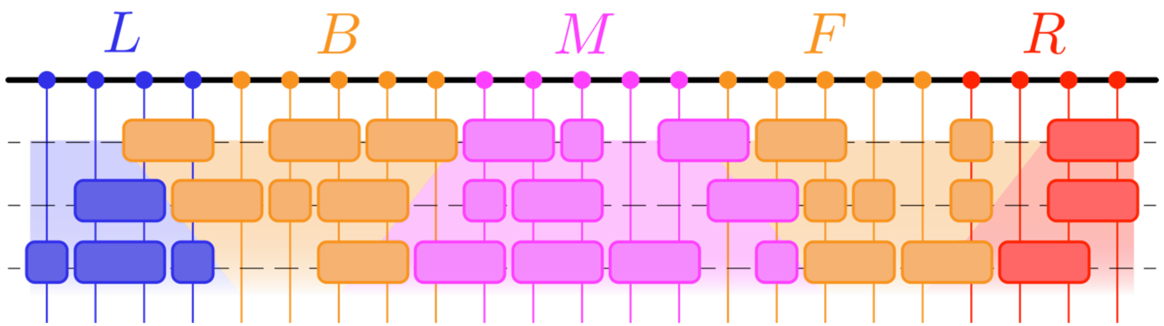

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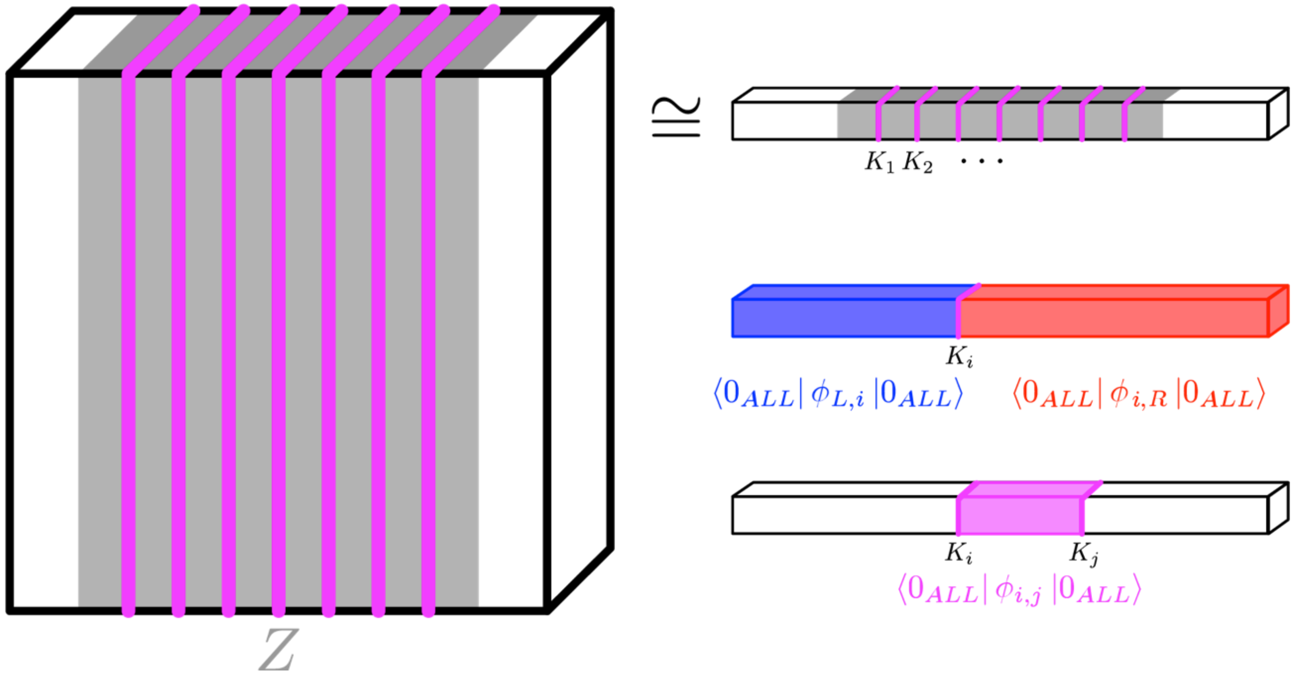

Theorem.

$$\left\| C |0^{\otimes n}\rangle\langle 0^{\otimes n}| C^\dagger - \sum_{\sigma\in\mathcal{P}([\Delta])} (-1)^{|\sigma|+1}\Psi_\sigma \right\| = \left(\frac{1-\lambda}{\lambda}\right)^\Delta \le \frac{1}{2^\Delta}$$

Where $\lambda = \min_i \lambda_1^i \ge 3/4$

$$\Psi_{\{i\}} = \phi_{L,i} \otimes \phi_{i,R}$$

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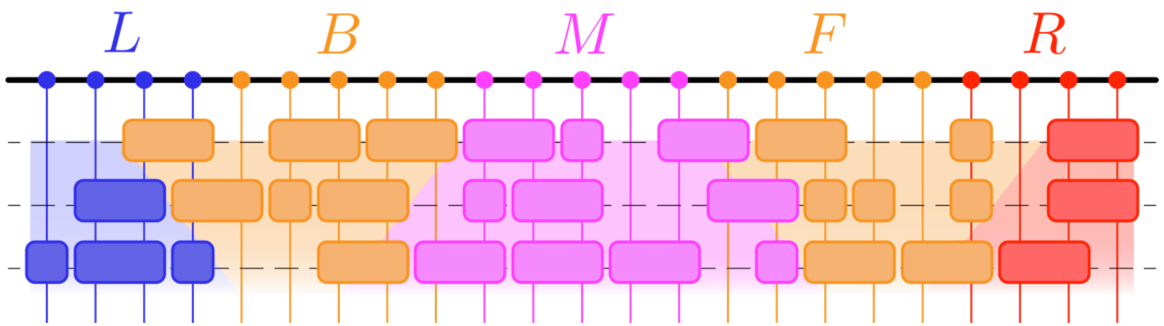

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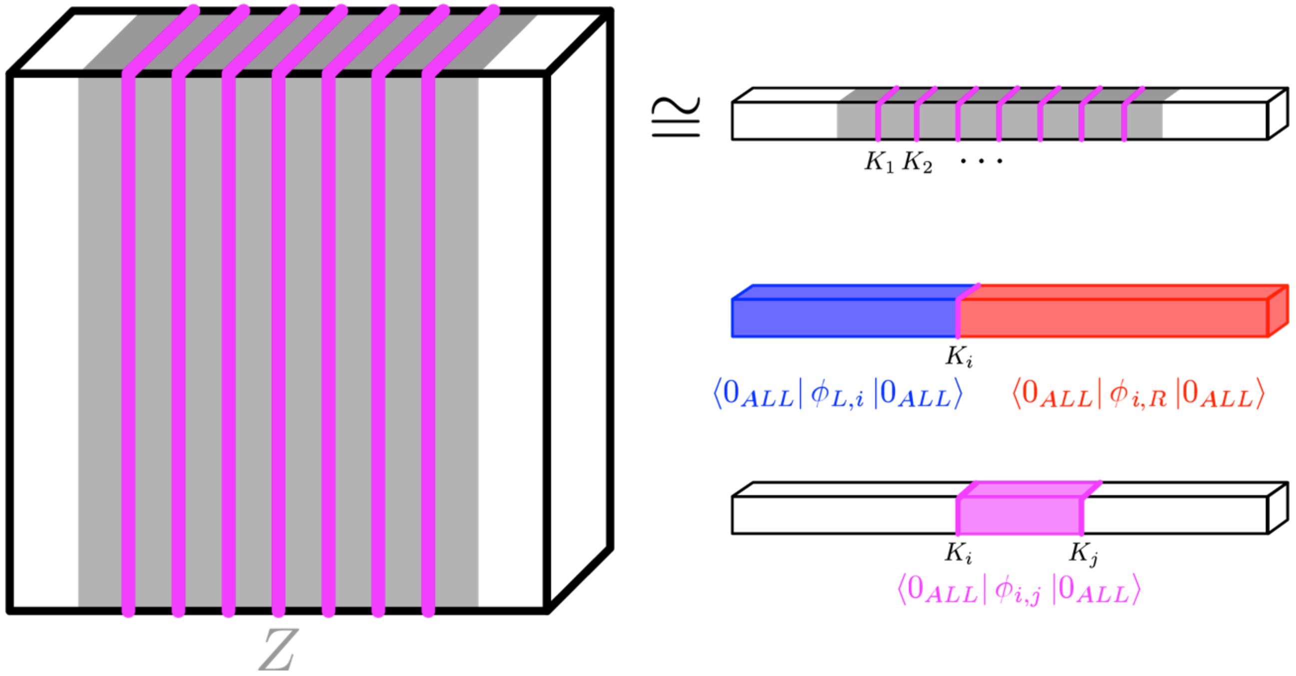

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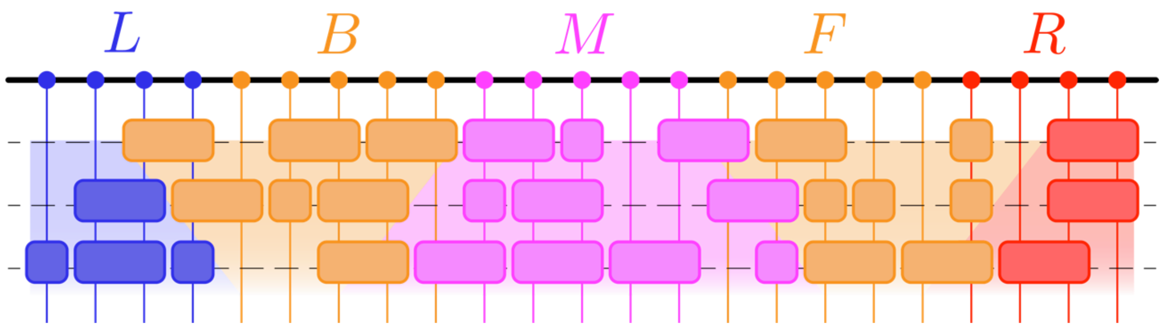

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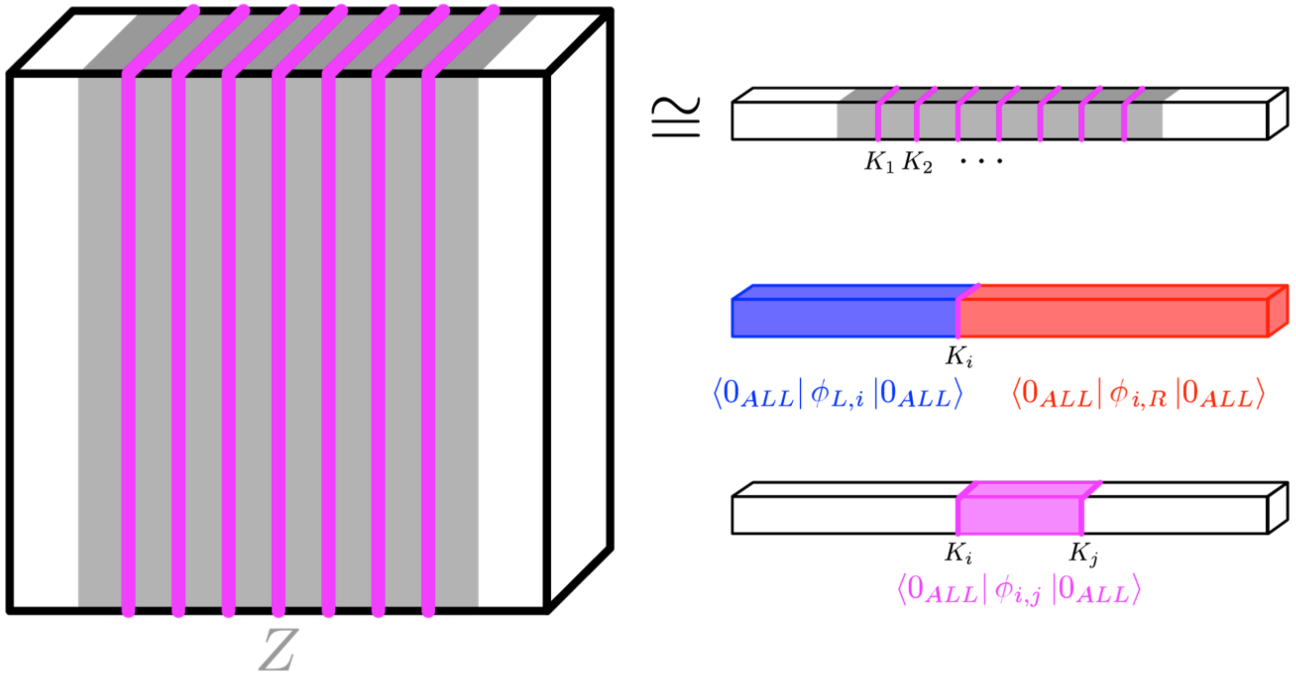

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$\phi_{L,i}, \phi_{i,R},$ and $\phi_{i,j}$ "synthesized" using:

$$\langle 0_{ALL/F}| \prod_{i=1}^{K} (C_{B\cup M\cup F}^\dagger \otimes I_{F_1,\ldots,F_K})(I_{B\cup M} \otimes \mathrm{SWAP}_{FF_i})(C_{B\cup M\cup F} \otimes I_{F_1,\ldots,F_K}) \, |0_{ALL/F}\rangle$$

$$\approx |w_1\rangle \langle w_1|_F + O((3/4)^K)$$

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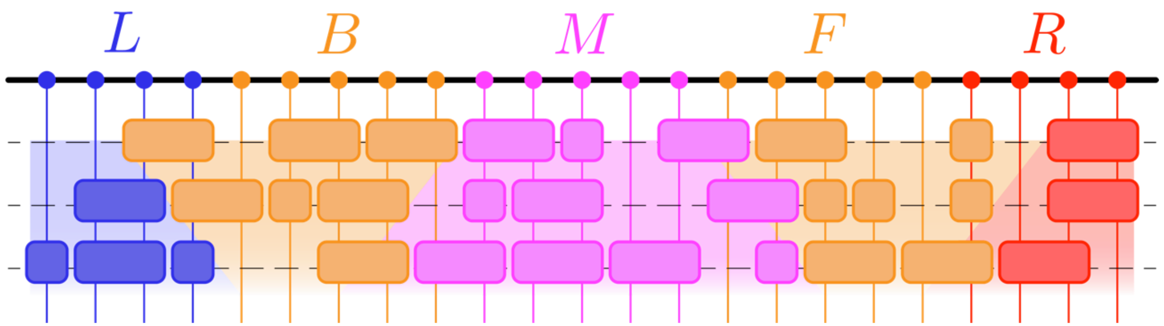

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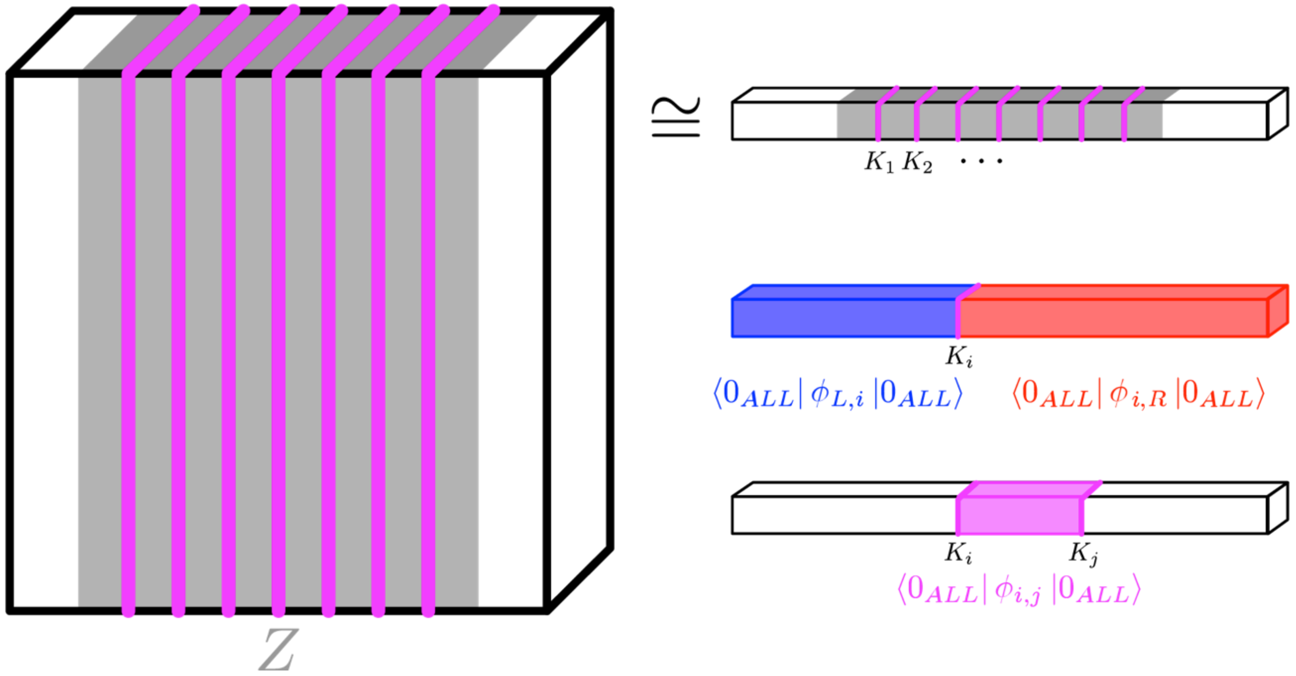

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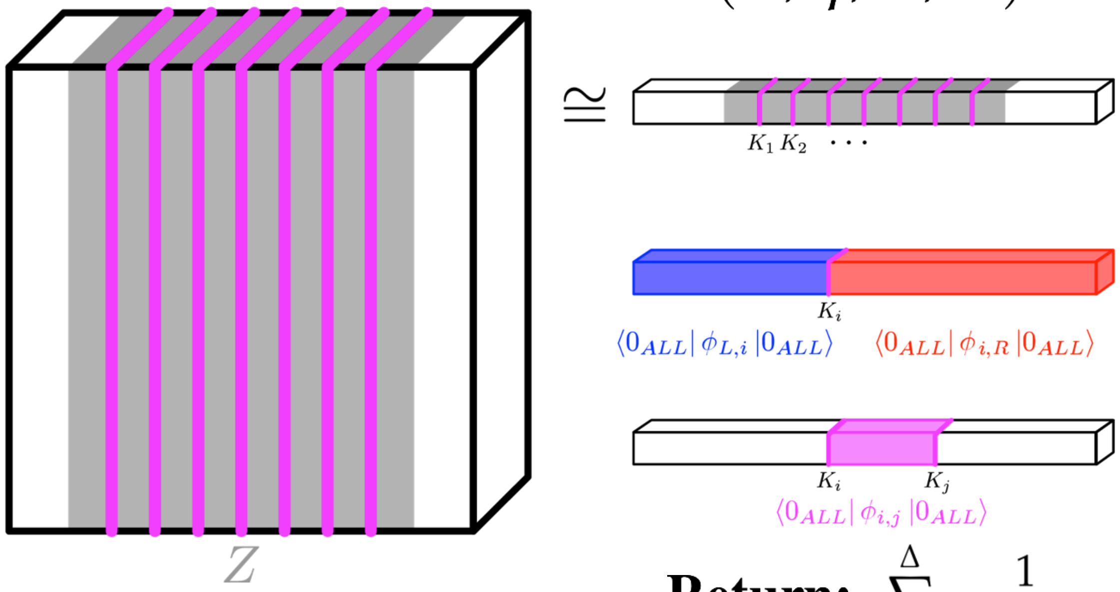
$$\mathcal{A}(C,\eta,\Delta,\mathcal{B})$$



- To compute the quantity

$$|\langle 0^{\otimes n}|\,C\,|0^{\otimes n}\rangle|^2 \pm O(1/n^{\log(n)}).$$

  - Find $\Delta \sim \log(n)$ light-cone-separated, "heavy" cuts $\{K_i : i \in I\}$ within $\log^5(n)$ of the center.
  - For each $i \in I$, approximate λ_1^i, and circuit diagrams for $\phi_{L,i}, \phi_{i,j}, \phi_{j,R}$.

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Return:

$$\sum_{i=1}^{\Delta} \frac{1}{(\lambda_1^i)^{4K+1}} \mathcal{A}(\phi_{L,i},\eta-1)\cdot\mathcal{A}(\phi_{i,R},\eta-1)$$

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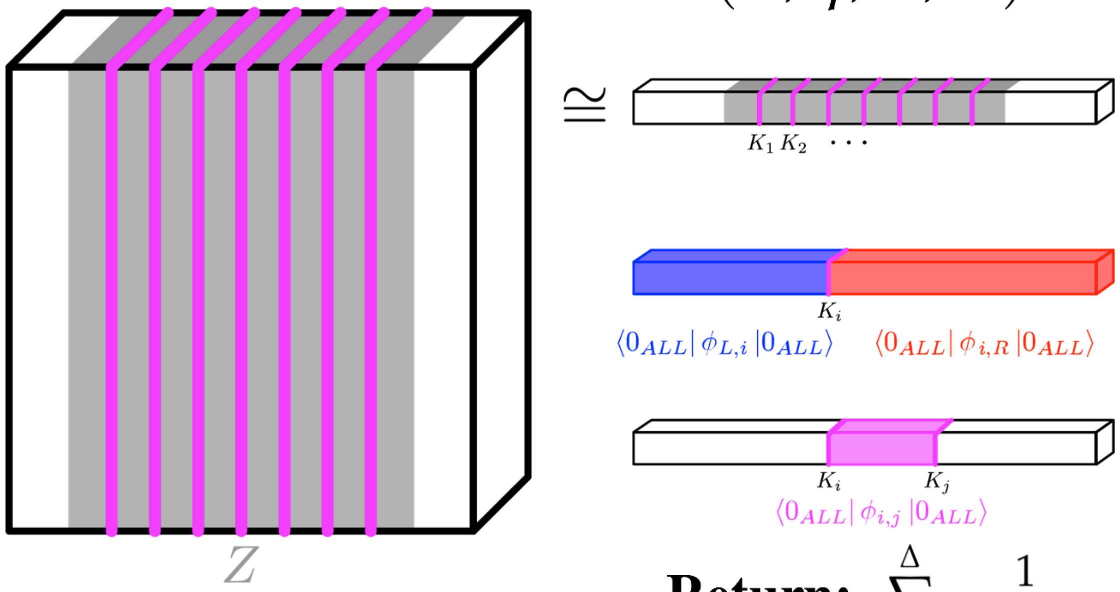

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Here $\mathcal{A}(\phi,\eta-1)$ is shorthand for $\mathcal{A}(\phi,\eta-1,\Delta,\mathcal{B})$, and represents the algorithm $\mathcal{A}$ making a recursive call to itself to approximate a smaller 3D circuit.

Here $\mathcal{B}(\phi,\epsilon)$ represents a call to the 2D algorithm of [BGM20] to ϵ-approximate a circuit that is “almost” 2D.

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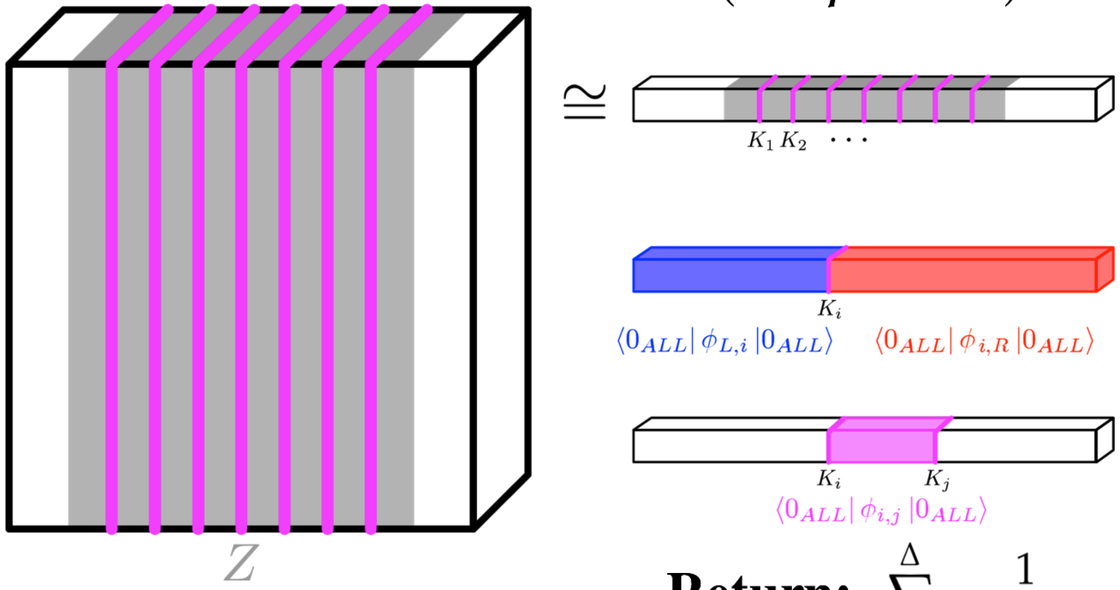

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Theorem. *This algorithm approximates the quantity* $\left|\left\langle 0^{\otimes n}\right| C \left|0^{\otimes n}\right\rangle\right|^2$ *to additive error* $1/n^{\log(n)}$ *in time* $n^{\mathrm{polylog}(n)}2^{d^3}$.

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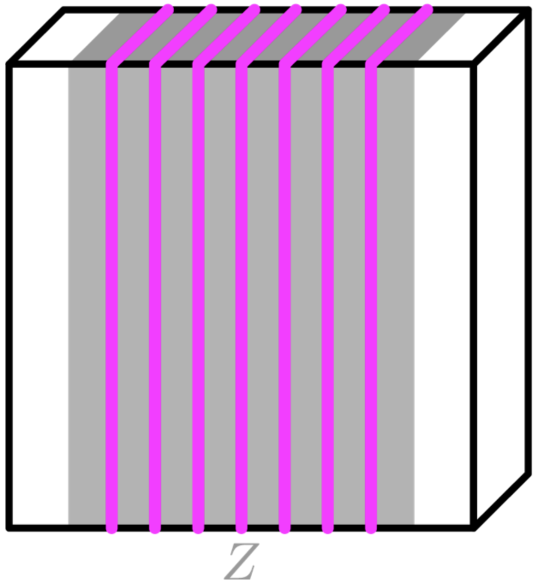

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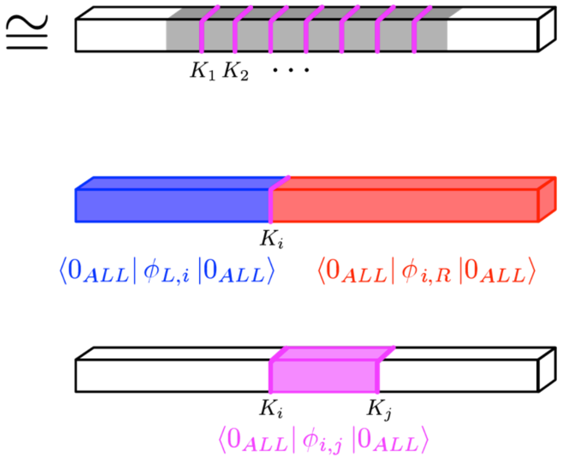

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Runtime Analysis:

- Logarithmically many recursive steps.
- Instance size ``halves" at each step.
- Quasipolynomial additive time cost at each step.
- Standard recursive time analysis gives Quasipolynomial runtime bound.

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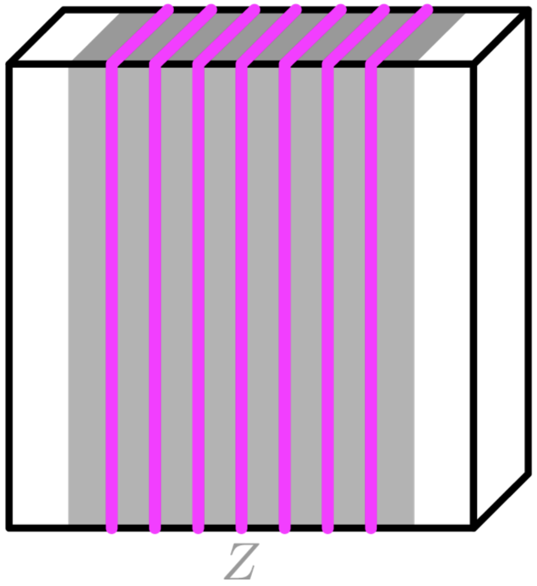

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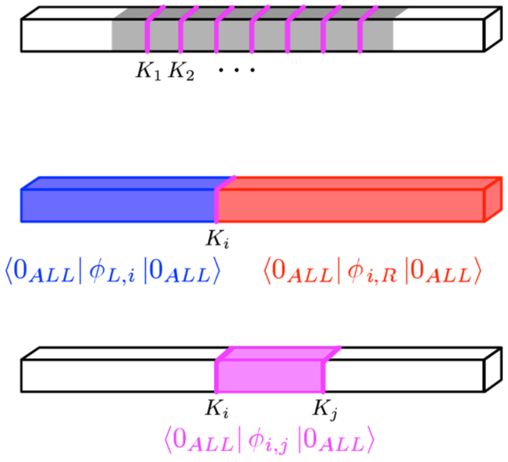

$\mathcal{A}(C,\eta,\Delta,\mathcal{B})$

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Error Analysis:

- Somewhat involved.
- Intuition follows from our two approximation theorems.

Theorem. $\langle 0_{ALL/F}|\prod_{i=1}^{K}(C^\dagger_{B\cup M\cup F}\otimes I_{F_1,\dots,F_K})(I_{B\cup M}\otimes \mathrm{SWAP}_{FF_i})(C_{B\cup M\cup F}\otimes I_{F_1,\dots,F_K})\,|0_{ALL/F}\rangle \approx |w_1\rangle\langle w_1|_F + O((3/4)^K)$

Theorem. *This algorithm approximates the quantity* $|\langle 0^{\otimes n}|\,C\,|0^{\otimes n}\rangle|^2$ *to additive error* $1/n^{\log(n)}$ *in time* $n^{\mathrm{polylog}(n)}2^{d^3}$.

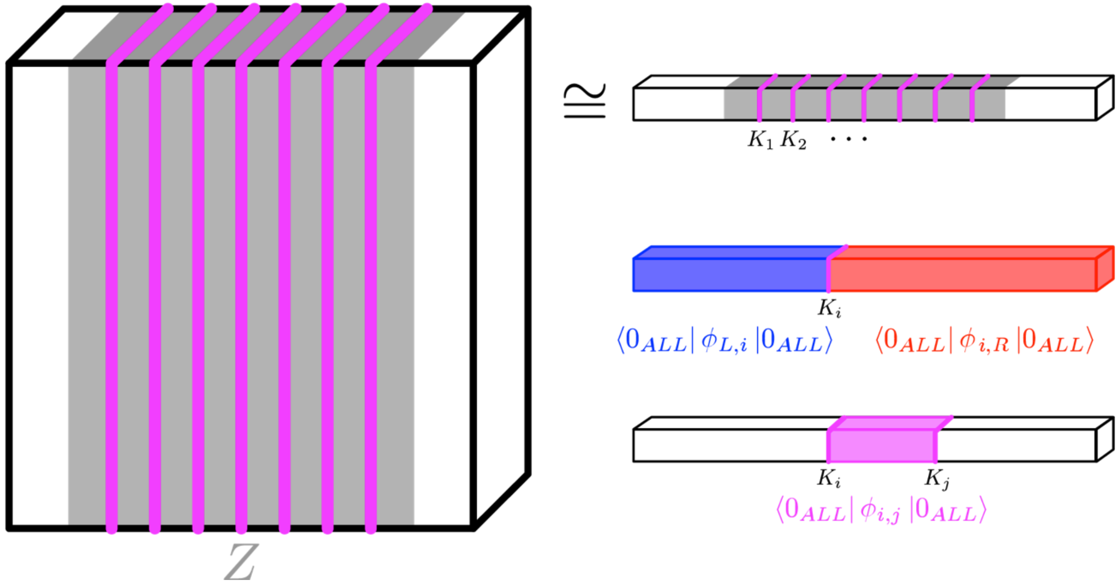
Theorem.

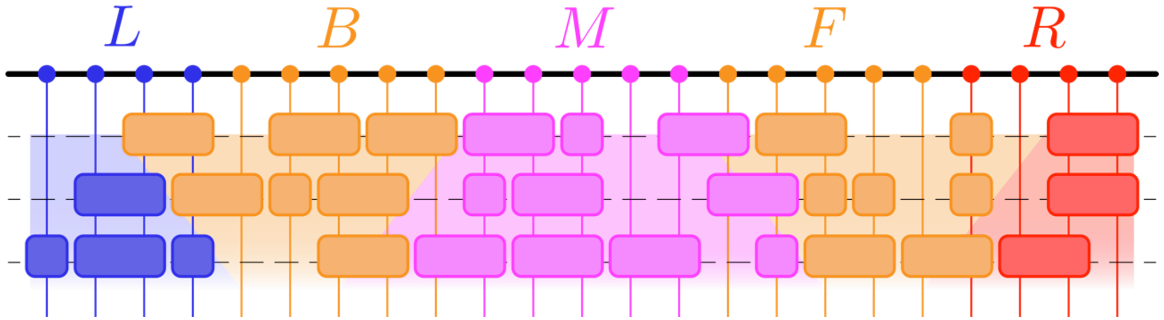
$$\left\| C \left|0^{\otimes n}\right\rangle \left\langle 0^{\otimes n}\right| C^{\dagger} - \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|+1} \Psi_{\sigma} \right\|$$

$$= \left(\frac{1-\lambda}{\lambda}\right)^{\Delta} \leq \frac{1}{2^{\Delta}}$$

$$|\psi\rangle_{B\cup F} \equiv \langle 0|_M C_{B\cup M\cup F} |0\rangle_{B\cup M\cup F} = \sum_{i=1} \lambda_i |v_i\rangle_B \otimes |w_i\rangle_F$$

Questions?





$$\phi_{L,i} = C_{L_i} \left(|0_{L_i}\rangle \langle 0_{L_i}| \otimes |v_1\rangle \langle v_1|_{B_i} \right) C_{L_i}^{\dagger}$$

$$\phi_{i,R} = C_{R_i} \left(|0_{R_i}\rangle \langle 0_{R_i}| \otimes |w_1\rangle \langle w_1|_{F_i} \right) C_{R_i}^{\dagger}$$

$$\phi_{i,j} = C_{i,j} \left(|w_1\rangle \langle w_1|_{F_i} \otimes \left|0_{[i,j]}\right\rangle \left\langle 0_{[i,j]}\right| \otimes |v_1\rangle \langle v_1|_{B_j} \right) C_{i,j}^{\dagger}$$

$$\langle 0_{ALL/F}| \prod_{i=1}^{K} (C_{B\cup M\cup F}^{\dagger} \otimes I_{F_1,\ldots,F_K})(I_{B\cup M} \otimes \mathrm{SWAP}_{FF_i})(C_{B\cup M\cup F} \otimes I_{F_1,\ldots,F_K}) |0_{ALL/F}\rangle$$

$$\approx |w_1\rangle \langle w_1|_F + O((3/4)^K)$$

Return:

$$\sum_{i=1}^{\Delta} \frac{1}{(\lambda_1^i)^{4K+1}} \mathcal{A}(\phi_{L,i}, \eta-1) \cdot \mathcal{A}(\phi_{i,R}, \eta-1)$$

$$- \sum_{i=1}^{\Delta} \sum_{j=i+1}^{\Delta} \frac{1}{(\lambda_1^i \lambda_1^j)^{4K+1}} \mathcal{A}(\phi_{L,i}, \eta-1) \cdot \mathcal{B}(\phi_{i,j}, \epsilon) \cdot \mathcal{A}(\phi_{j,R}, \eta-1)$$

$$+ \sum_{i=1}^{\Delta} \sum_{j=i+2}^{\Delta} \frac{1}{(\lambda_1^i \lambda_1^j)^{4K+1}} \mathcal{A}(\phi_{L,i}, \eta-1) \cdot \mathcal{A}(\phi_{j,R}, \eta-1)$$

$$\cdot \left[\sum_{\sigma \in \mathcal{P}(\{j-1,\ldots,i+1\})} (-1)^{|\sigma|+1} \mathcal{B}\left(\phi_{i,\sigma_1} \otimes_{k \in |\sigma|-1} \phi_{\sigma_k,\sigma_{k+1}} \otimes \phi_{\sigma_{|\sigma|},\sigma_j}, \frac{\epsilon}{2^{\Delta}} \right) \right.$$

Theorem. *This algorithm approximates the quantity* $|\langle 0^{\otimes n}| C |0^{\otimes n}\rangle|^2$ *to additive error* $1/n^{\log(n)}$ *in time* $n^{\mathrm{polylog}(n)} 2^{d^3}$.