Eliminating Intermediate Measurements in Space-Bounded Quantum Computation

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Quantum Logspace Algorithm for Powering Matrices with Bounded Norm

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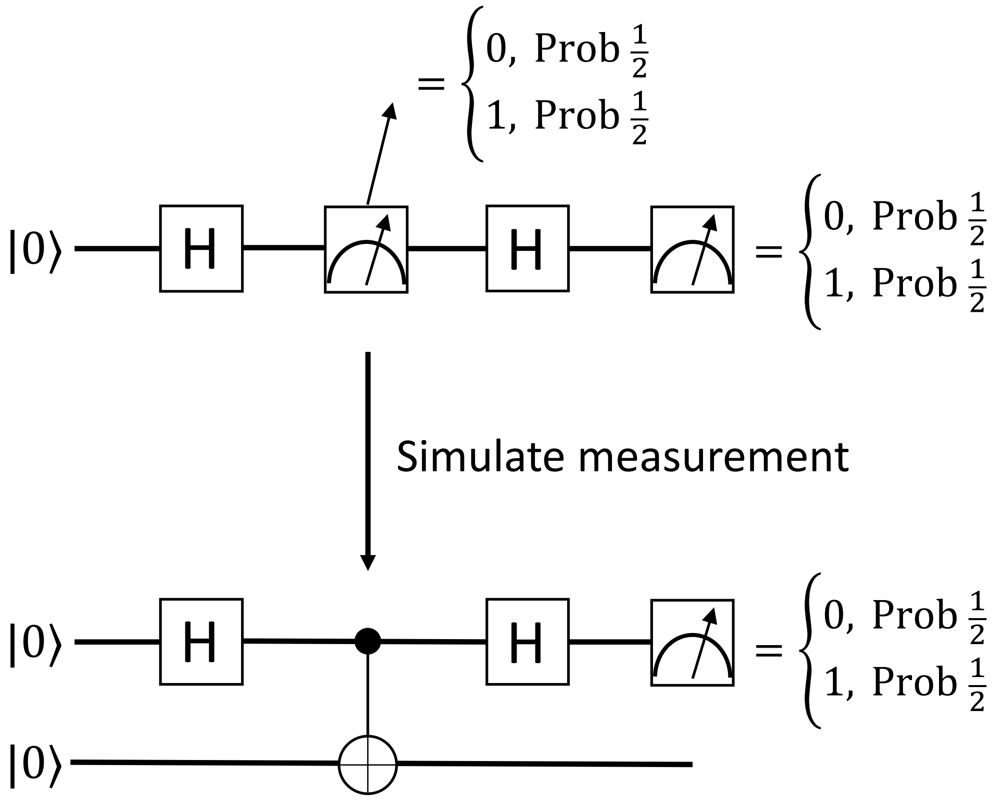
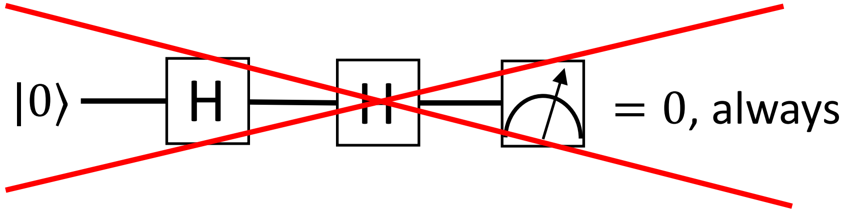
Principle of Deferred Measurement





Okay if only care about time

Not okay if also care about space

Cost = 1 ancilla (space) per measurement

Ex: $\log(n)$-space and $poly(n)$-time

May have $poly(n)$ measurements

⇒ exponential blowup to $poly(n)$-space

Possible without space blowup? [Watrous’01, Watrous’03, van Melkebeek-Watson’12, Ta-Shma’13, Fefferman-Lin’16, etc.]

Main Results

Can eliminate intermediate measurement without space blowup or time blowup

Quantum logspace algorithms and matching hardness results for many natural linear algebraic problems

For well-conditioned matrix A, approximate

$\det(A)$

A^{-1}

A^m

etc.

$\Rightarrow$ can eliminate intermediate measurements

Why is Space Important?

Current experimental quantum computers: Noisy Intermediate Scale era [Preskill'18]

Intermediate Scale = few qubits

IBM Q System
53 Qubits
Cost: $\gg$ \$400

Apple iPhone 7
128 GB
Cost: \$400

(somewhat unfair comparison)

Near-term quantum computers have few qubits $\Rightarrow$ space is important

Why Eliminate Intermediate Measurements?

Measurements are a natural resource, just like time and space

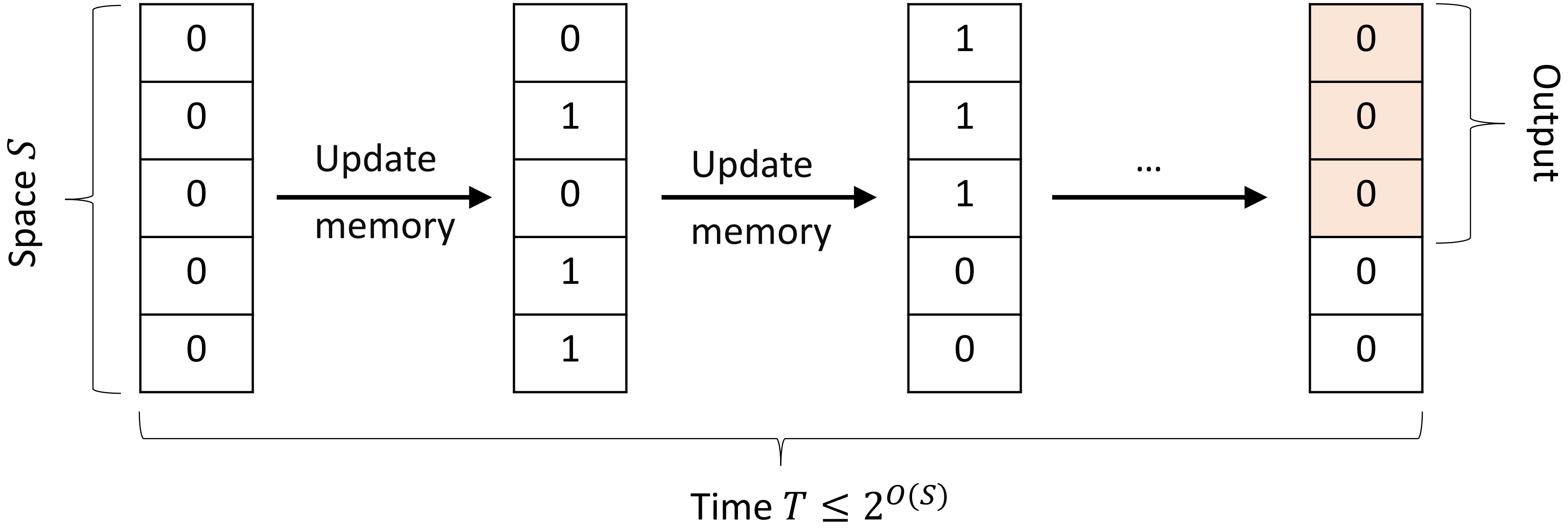
Unitary computations (i.e., without intermediate measurements) are reversible

- Undo computation: useful in design/analysis of algorithms
  - "Tidy" subroutine calling [Bennet-Bernstein-Brassard-Vazirani'97]
  - Quantum rewinding [Watrous'09]
- No heat generation (Landauer's Principle)

Shows definition of "quantum space $s(n)$" is robust

- general $=$ unitary
- For probabilistic space
  - $\subseteq$ general quantum (easy)
  - $\subseteq$ unitary quantum (previously unknown)

Our Model of Space Bounded Computation



We require that there is a classical **deterministic space S** Turing Machine which on input $x \in \{0,1\}^*$, outputs the description of the update operations.

An algorithm **computes** a function $f: \{0,1\}^n \to \{0,1\}^m$ if the output of the algorithm is $f(x)$ (with probability at least $\frac{2}{3}$ if the algorithm is randomized/quantum).

Logspace Algorithms: Space $O(\log n)$ and Time $\mathbf{poly}(n)$

Classical Logspace Algorithms:

$\boldsymbol{L}$: Classical algorithms where the updates are deterministic transition matrices.

$\boldsymbol{BPL}$: Classical randomized algorithms where the updates are stochastic matrices.

Quantum Logspace Algorithms:

$\boldsymbol{BQ_U L}$: (Unitary Quantum algorithms)

- Store qubits.
- Updates are by unitary quantum channels.
- The output is the outcome of measuring some qubits at the end of the computation.

$\boldsymbol{BQ_Q L}$: (Pure Quantum algorithms)

- Updates are by unital quantum channels.

$\boldsymbol{BQL}$: (Quantum algorithms)

- Updates are by general quantum channels.

Logspace Algorithms:
Space $O(\log n)$ and Time $\mathrm{poly}(n)$

$\boldsymbol{BQ_UL}$ Unitary Logspace	$\boldsymbol{BQ_QL}$ Pure Quantum Logspace	$\boldsymbol{BQL}$ General Quantum Logspace
Unitary quantum channels	Unital quantum channels	General quantum channels
✓ Unitary operations.	✓ Unitary operations. ✓ Intermediate Measurements.	✓ Unitary operations. ✓ Intermediate Measurements. ✓ Reset qubits.
Purely quantum memory, Unitary Operations	Purely quantum memory, Unitary Operations + Intermediate Measurements	Quantum Memory, Unitary Operations, Intermediate Measurements + Classical Memory + Copy measurement outcome to classical memory

Our Result:
Contraction Matrix Powering is in $\boldsymbol{BQ_UL}$.

Given: An $n \times n$ real contraction matrix A (i.e., $\|A\| \leq 1$),
$T \leq \text{poly}(n), \epsilon \geq \frac{1}{\text{poly}(n)}$ and indices $s, t \in \{1, \dots, n\}$
Estimate $\mathrm{A}^{\mathrm{T}}[s, t]$ up to ϵ additive error.

We [Girish-Raz-Zhan'20] show that this problem can be solved in $\boldsymbol{BQ_UL}$.
Equivalently, Iterated Contraction Matrix Multiplication is in $\boldsymbol{BQ_UL}$.

It is not known if this problem can be solved in $\boldsymbol{BPL}$.

Applications:
Eliminating Intermediate Measurements

We [Girish-Raz-Zhan'20] show that $\boldsymbol{BQ_U L = BQ_Q L}$.

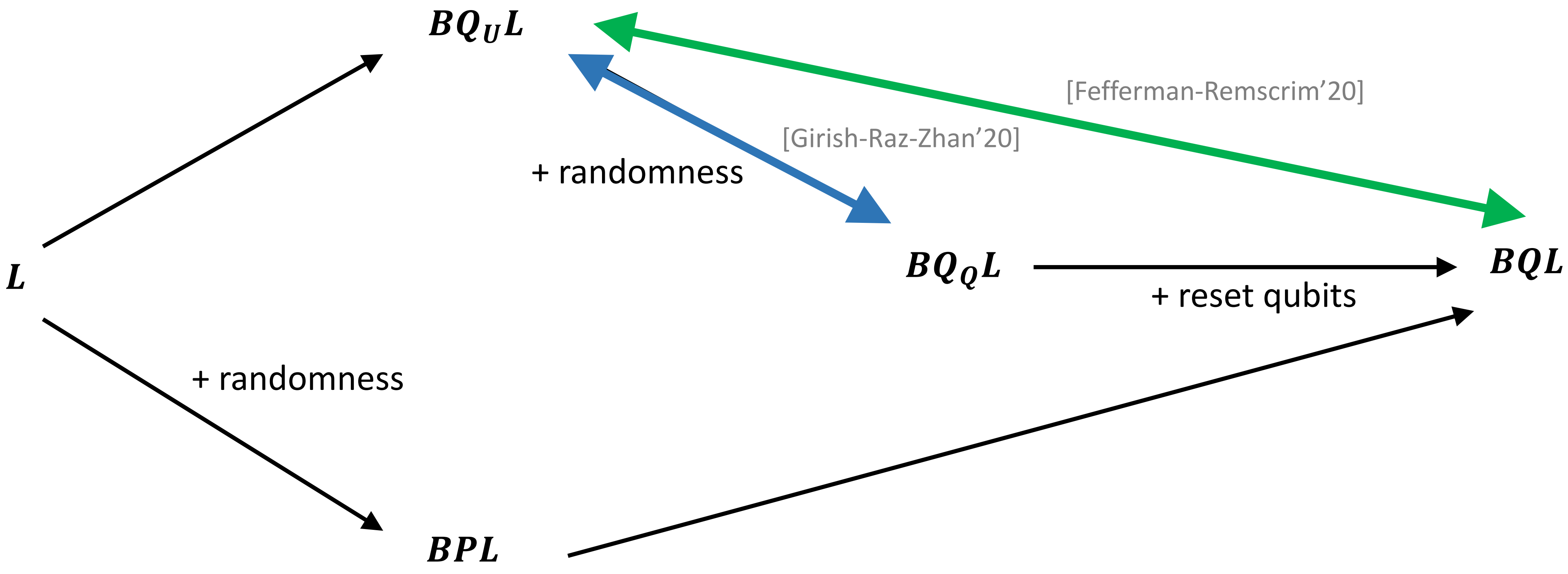
Measurements during computation don't give more power to quantum logspace algorithms with only quantum memory.

In general, we show that a pure quantum algorithm of space S and time T *with intermediate measurements,*
can be simulated in space $O(S + \log T)$ and time $\mathrm{poly}(T, 2^{O(S)})$ *without intermediate measurements.*
$\Rightarrow \boldsymbol{BQ_U SPACE}(s(n)) = \boldsymbol{BQ_Q SPACE}(s(n))$ for any space-constructible $s(n) = \Omega(\log(n))$.

[Fefferman-Remscrim'20] show that $\boldsymbol{BQ_U L = BQ_Q L = BQL}$.
$\Rightarrow \boldsymbol{BQ_U SPACE}(s(n)) = \boldsymbol{BQSPACE}(s(n))$ for any space-constructible $s(n) = \Omega(\log(n))$.

The ability to do intermediate measurements or reset qubits doesn't give more power to quantum logspace algorithms.

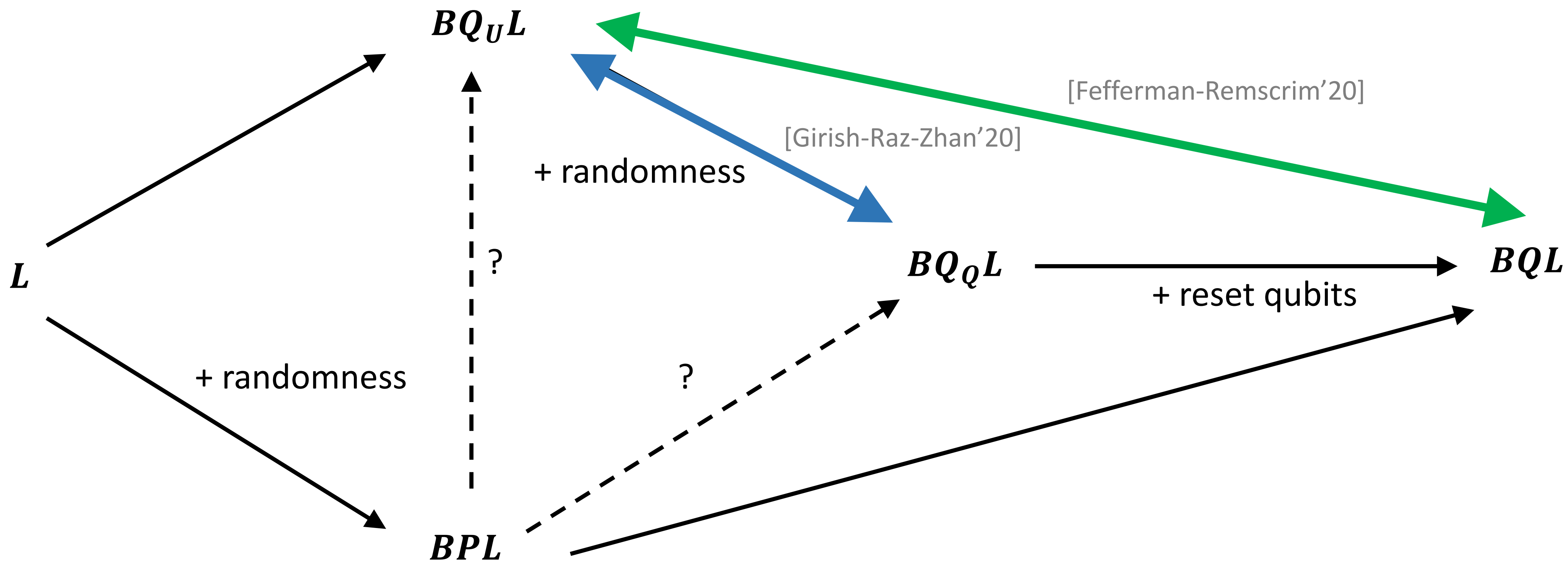
Applications: Derandomizing $\boldsymbol{BQ_QL}$



$\boldsymbol{A} \rightarrow \boldsymbol{B}$ denotes that $\boldsymbol{B}$ is at least as powerful as $\boldsymbol{A}$.

We show that $\boldsymbol{BQ_QL}$ algorithms don’t require randomness.

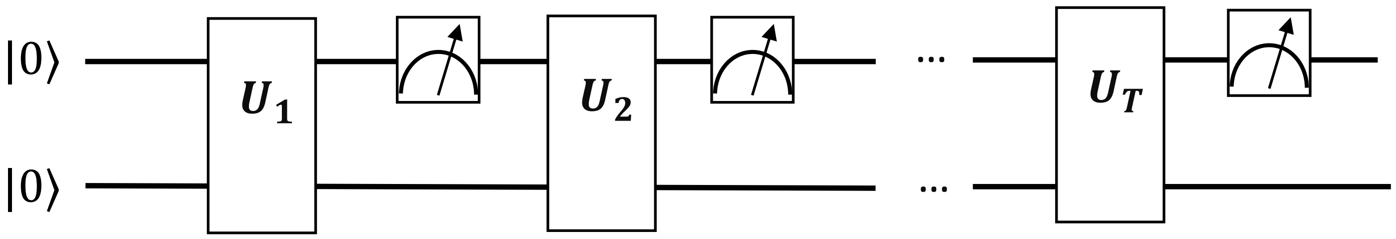
Applications:
$\boldsymbol{BPL}$ versus $\boldsymbol{BQ_UL}$



$\boldsymbol{A}$ → $\boldsymbol{B}$ denotes that $\boldsymbol{B}$ is at least as powerful as $\boldsymbol{A}$.

Connection between Contraction Matrix Multiplication and $\boldsymbol{BQ_QL}$

A generic $\boldsymbol{BQ_QL}$ algorithm



The probability that this circuit outputs 1 is precisely

Unitary matrices

$$\mathrm{vec}(\Pi_1\Pi_1^\dagger)^\dagger(U_T \otimes \overline{U_T})\, M \ldots (U_2 \otimes \overline{U_2})\, M\, (U_1 \otimes \overline{U_1})(v_0 \otimes v_0) \quad = a^\dagger(A_{\mathrm{poly}(n)} \times \cdots \times A_1)b$$

Vector† Contraction matrices Unit vector

$\Pi_1 =$ projection matrix onto $\{\, |i\rangle : i_1 = 1 \,\}$
$v_0 = |0^S\rangle$
$M =$ a diagonal matrix with diagonal entries in $\{0,1\}$.

Our Approach

Step 1. Embed each of the contraction matrices $\boldsymbol{A_1}, \ldots, \boldsymbol{A_T}$ inside *unitary* logspace quantum circuits.

$$A \hookrightarrow \begin{bmatrix} A & \sqrt{\mathbb{I} - AA^\dagger} \\ \sqrt{\mathbb{I} - A^\dagger A} & -A^\dagger \end{bmatrix}$$

Step 2. Compose the quantum circuits carefully.

$$\begin{bmatrix} A & * & 0 \\ * & * & 0 \\ 0 & 0 & \mathbb{I} \end{bmatrix} \times \begin{bmatrix} B & 0 & * \\ 0 & \mathbb{I} & 0 \\ * & 0 & * \end{bmatrix} = \begin{bmatrix} AB & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Step 3. Estimate $\boldsymbol{a}^\dagger \boldsymbol{M} \boldsymbol{b}$ using a quantum circuit that embeds $\boldsymbol{M}$.

An argument similar to Grover's search.

Well-Conditioned Matrix Inversion

Given: invertible $n \times n$ matrix A and indices $i, j \in \{1, \ldots, n\}$

Approx $A^{-1}[i, j]$ to precision $1/poly(n)$

Difficulty depends on condition number $\kappa(A) = \frac{\sigma_1(A)}{\sigma_n(A)}$

$\sigma_1(A)$ = largest singular value, $\sigma_n(A)$ = smallest singular value

Well-conditioned: $\kappa(A) = poly(n)$,

Different params (sparse $2^n \times 2^n$ matrix) in $\boldsymbol{BQP}$ [Harrow-Hassidim-Lloyd'09]

in $\boldsymbol{BQL}$ [Ta-Shma'13]

is $\boldsymbol{BQ_U L}$-complete [Fefferman-Lin'16]

Determinant

$\boldsymbol{DET} =$ problems reducible to computing $\det(A)$ [Cook'85]

$n \times n$ matrix A, entries are n-bit integers

Natural complete problems: Determinant, Matrix Inversion, Matrix Powering, Iterated Matrix Multiplication, etc.

$poly$-conditioned matrix inversion is $\boldsymbol{BQ_UL}$-complete [Fefferman-Lin'16]

We generalize: $poly$-conditioned $\boldsymbol{DET}$-complete problems are $\boldsymbol{BQ_UL}$-complete

$\Rightarrow$ can eliminate intermediate measurements

Difficulty: "standard" reductions generally do not preserve being well-conditioned

Well-Conditioned Matrix Powering

Given: $n \times n$ matrix A, $t \leq poly(n)$, and indices $i, j \in \{1, \ldots, n\}$

Approx $A^t[i,j]$ to precision $1/poly(n)$

Promise: $\|A^k\| \leq poly(n), \forall k \in \{1, \ldots, t\}$ (analogue of "poly-conditioned" for powering)

Simple reduction to $poly$-conditioned matrix inversion $\Rightarrow \in \boldsymbol{BQ_UL}$

$B =$

I	$-A$	0	0	0	0
0	I	$-A$	0	0	0
0	0	I	$-A$	0	0
0	0	0	I	$-A$	0
0	0	0	0	I	$-A$
0	0	0	0	0	I

$B^{-1} =$

I	A	A^2	A^3	A^4	A^5
0	I	A	A^2	A^3	A^4
0	0	I	A	A^2	A^3
0	0	0	I	A	A^2
0	0	0	0	I	A
0	0	0	0	0	I

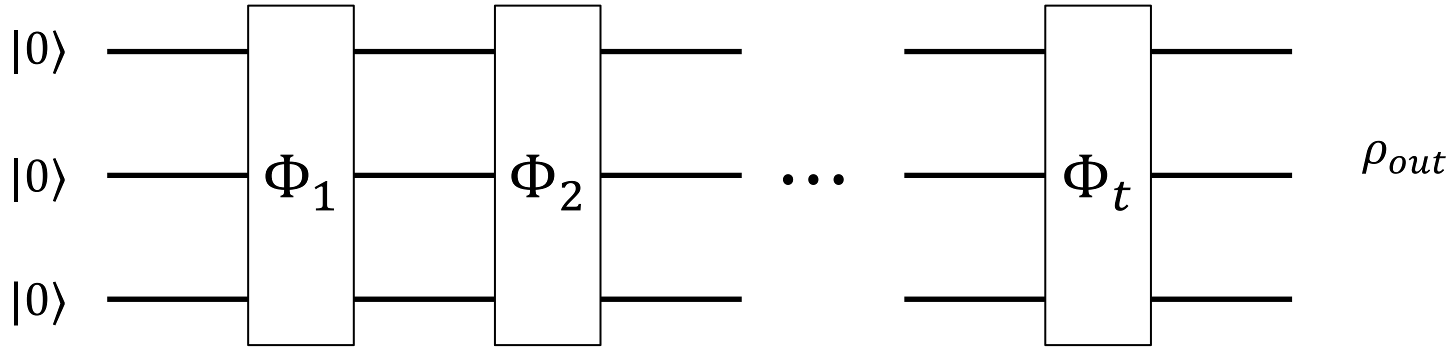
A^t appears in top-right block of B^{-1} and B is $poly$-conditioned

Well-Conditioned Iterated Matrix Product

Similar to powering: Now approx entry of $A_1 \cdots A_t$, $t \leq poly(n)$, $\|A_k \cdots A_h\| \leq poly(n), \forall\ 1 \leq k \leq h \leq t$

Can reduce to $poly$-conditioned matrix inversion $\Rightarrow \in \boldsymbol{BQ_UL}$

Is $\boldsymbol{BQL}$-hard: General quantum circuit is sequence of (arbitrary) quantum channels



$\Rightarrow \boldsymbol{BQL} = \boldsymbol{BQ_UL}$

$\Rightarrow \boldsymbol{BQSPACE}\big(s(n)\big) = \boldsymbol{BQ_USPACE}(s(n))$, any space-constructible $s(n) = \Omega(\log(n))$

Also show results for $\boldsymbol{RQL}$ and $\boldsymbol{NQL}$, and for "$\boldsymbol{QMA}$" and "$\boldsymbol{DQC1}$" versions of $\boldsymbol{BQL}$, $\boldsymbol{RQL}$, and $\boldsymbol{NQL}$

Well-Conditioned Determinant

Given: $n \times n$ matrix A, $\kappa(A) = poly(n)$

- Approx $\log(|\det(A)|)$ to precision $1/poly(n)$ (Approx $|\det(A)|$ to multiplicative factor $1 + 1/poly(n)$)

We show: is $\boldsymbol{BQL}(= \boldsymbol{BQ_U L})$-complete

- Other well-conditioned $\boldsymbol{DET}$-complete also $\boldsymbol{BQL}$-complete
- "Standard" reductions do not preserve well-conditioned (e.g. Berkowitz's algorithm)
- Our reductions: various power series approximations, quantum "instance compression"

[Boix-Adsera, Eldar, and Mehraban'19]: $\in \boldsymbol{DSPACE}(\log^2 n \, poly(\log\log n))$

- Used (different) power series approximations
  - Suggests source of quantum advantage
- We show $\in \boldsymbol{DSPACE}(\log^2 n)$
  - Recall: $\boldsymbol{BQL} \subseteq \boldsymbol{DSPACE}(\log^2 n)$ [Watrous'03]
- Improve dependence on $\kappa(A)$?
  - Would show $\boldsymbol{BQL} \subseteq \boldsymbol{DSPACE}(\log^{2-\delta} n)$