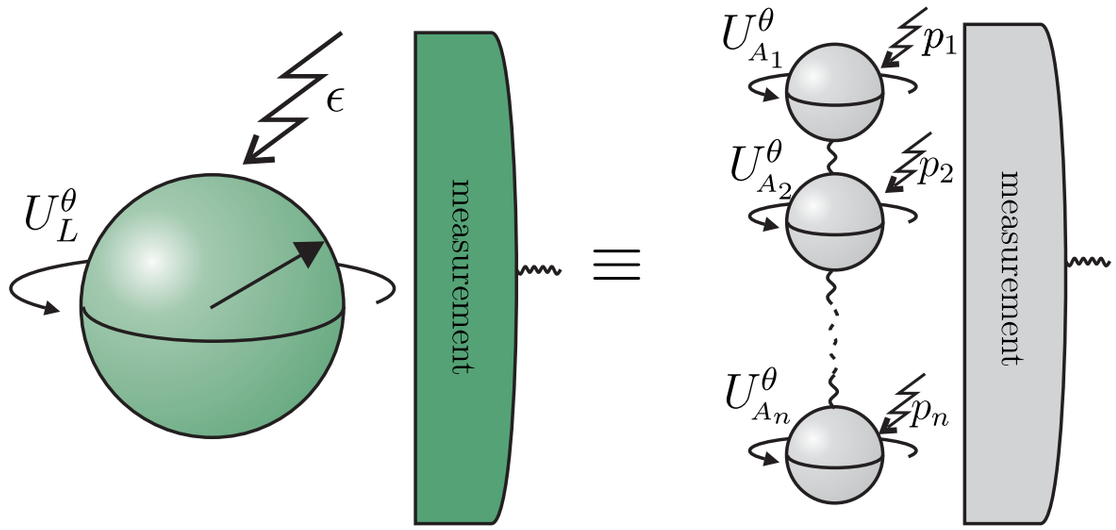
Using metrological bounds in quantum error correction

Aleksander Kubica

PERIMETER INSTITUTE

IQC Institute for Quantum Computing



work w/ Rafał Demkowicz-Dobrzański
(University of Warsaw)

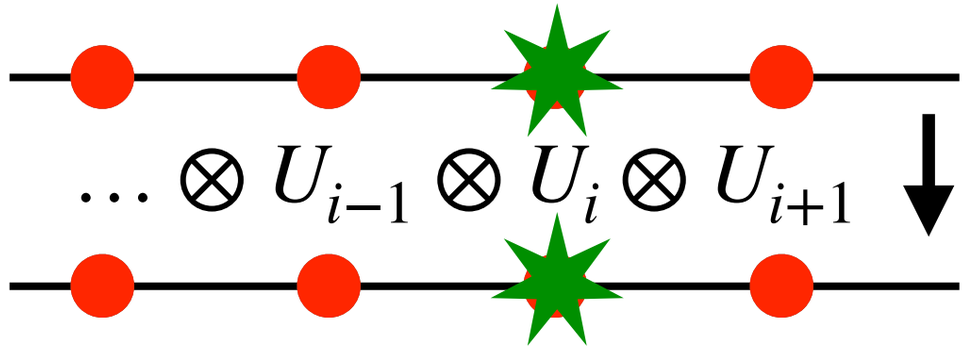
arXiv:**2004.11893**

Good protection vs. easy computation

- We want to process quantum information in a fault-tolerant way.

good protection from noisy environment ⟷ easy-to-implement logical operations

- **Transversal gates** do not spread errors!



- Can we find a non-trivial quantum error-correcting code w/ transversal gates forming a universal set of logical gates?

- **Eastin-Knill theorem** [ZCC07,EK09]: nope!

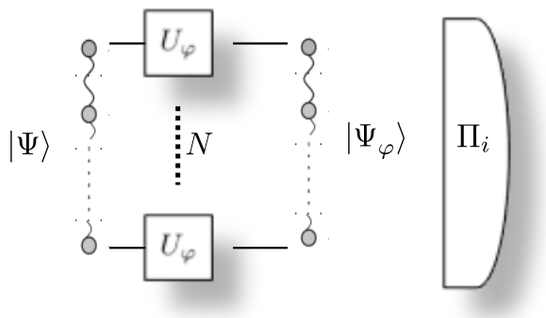
- Can we avoid this no-go result? How fundamental is E-K theorem?

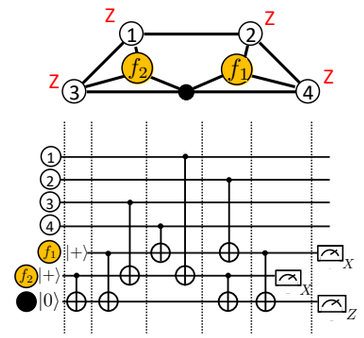
Zeng+07;Eastin&Knill'09;

Main result & outline

Main result: a simple proof of the approximate E-K theorem.

Unorthodox approach:



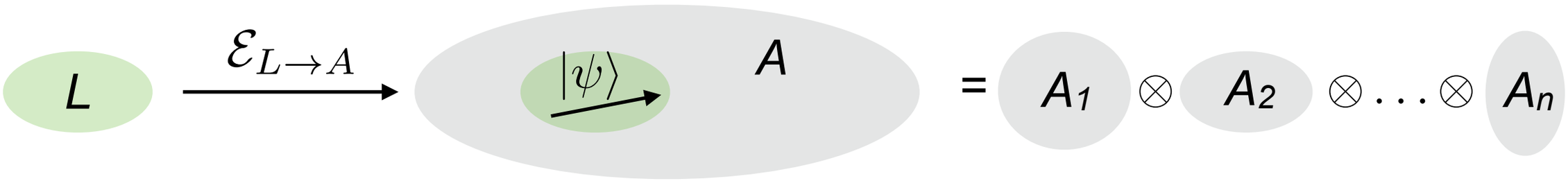


quantum metrology → quantum error correction

1. Setting the stage: quantum error correction.
2. The approximate E-K theorem.
3. Tools needed: quantum metrological bounds.
4. Our proof.

Quantum error correction

- **Quantum code** = a subspace of the Hilbert space.



$$n < \infty \quad \& \quad \dim(\mathcal{H}_*) < \infty$$

- Given a set of errors $\{K_1, K_2, \ldots\}$, can we correct them?

- **Knill-Laflamme condition** [KL97]: there exists a recovery $\mathscr{R}$ s.t.

$$\mathscr{R} \circ \left(K_i |\psi\rangle\langle\psi| K_i^\dagger \right) = |\psi\rangle\langle\psi| \iff \langle\psi_\alpha| K_i^\dagger K_j |\psi_\beta\rangle = c_{ij}\delta_{\alpha,\beta}.$$

- We correct errors **exactly**! Any linear combination of K_i's is correctable!

- Code $\mathscr{C}_{L\to A}$ has distance $2t+1$ if any error of weight $\leq t$ is correctable.

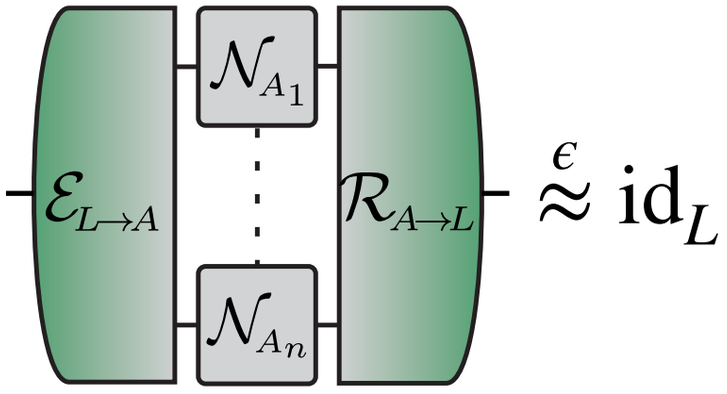
Knill&Laflamme'97;

Approximate quantum error correction

- Allowing for negligible error in recovery can lead to surprisingly better codes [LNCY97,CGS05].

- **Setting**: for given noise $\mathcal{N}_A$ can choose code $\mathscr{E}_{L\to A}$ & recovery $\mathscr{R}_{A\to L}$. Usually $\mathcal{N}_A = \bigotimes_i \mathcal{N}_{A_i}$.

- **Want:**



Bures distance for channels

$$\mathrm{d}(\mathscr{C},\mathscr{D}) = \sqrt{1-\mathscr{F}(\mathscr{C},\mathscr{D})}$$

↑ worst-case entanglement fidelity

- [BO09]: code $\mathscr{E}_{L\to A}$ is ϵ**-correctable** under noise $\mathcal{N}_A$ if there exists recovery $\mathscr{R}_{A\to L}$ s.t. $\mathrm{d}(\mathscr{R}_{A\to L}\circ\mathcal{N}_A\circ\mathscr{E}_{L\to A}, \mathrm{id}_L) \leq \epsilon$.

Leung+97;Crepeau+05;Beny&Oreshkov'09;

Main result & outline

Main result: a simple proof of the approximate E-K theorem.

Unorthodox approach:

quantum metrology

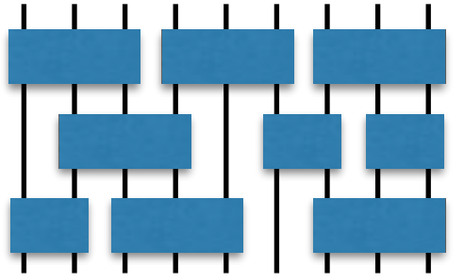
quantum error correction

1. Setting the stage: quantum error correction.
2. The approximate E-K theorem.
3. Tools needed: quantum metrological bounds.
4. Our proof.

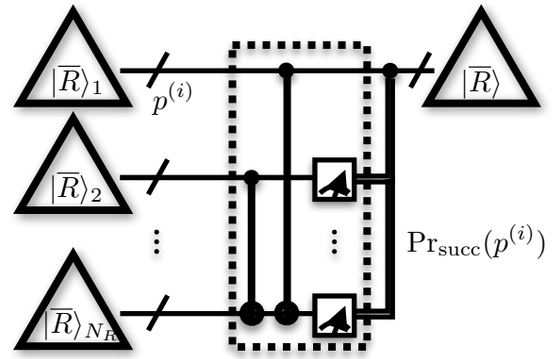
Avoiding E-K theorem

- **E-K theorem:** no quantum code w/ a universal set of transversal gates.

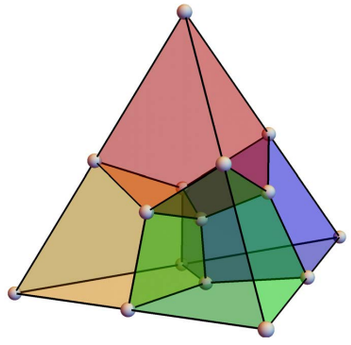
- Loosen transversality: const-depth, different partitions [JL14], …



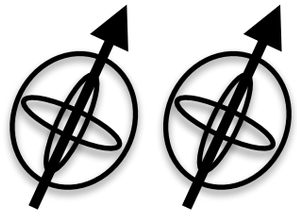
- Employ non-unitary operators: state distillation [BK05], …



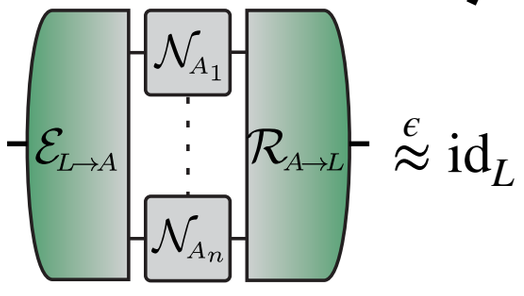
- Use different codes: code switching, gauge fixing [PR13,B15], …



- Infinite-dim codes: reference frame [HNPS17], …



- Allow for errors: approximate quantum error correction [FNASPHP19,WA19], …



Jochym-O'Connor&Laflamme'14;Bravyi&Kitaev'05;Paetznick&Reichardt'13;Bombin'15;Hayden+'17;Faist+'19; Woods&Alhambra'19

What if recovery is approximate?

- Rephrasing E-K theorem:

approximate ϵ-correctable under $\mathcal{N}_A = \bigotimes_i \mathcal{N}_{A_i}$

quantum code $\mathscr{E}_{L\to A}$

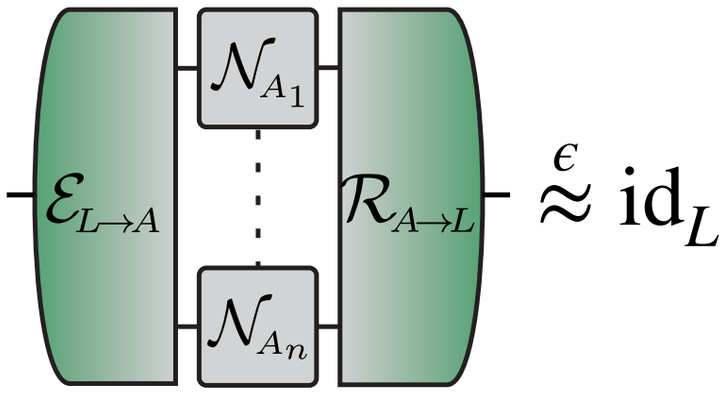
w/ transversal universal gates

$U(1)$-rotations

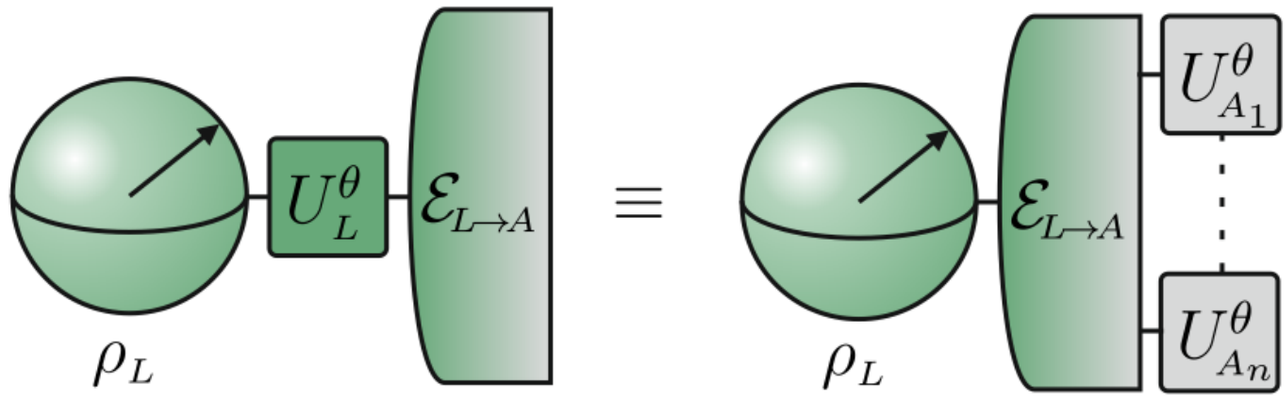
$\longrightarrow$ code distance = 1

$\epsilon \geq ???$

ϵ-correctability



$U(1)$-covariance & transversality



$U_L^\theta = e^{-i\theta T_L}$ $\qquad$ $\bigotimes_i U_{A_i}^\theta = e^{-i\theta \sum_i T_{A_i}}$

- $U(1)$-covariance is a weaker assumption than a universal gate set: rotation along the *Z* axis vs. rotations along the *X*, *Y* and *Z* axes.

The approximate E-K theorem

- **Theorem:** let $\mathscr{E}_{L\to A}$ be a $U(1)$-covariant code w/ transversal logical gates $U_L^\theta = e^{-i\theta T_L}$. If it is ϵ-correctable under noise $\mathcal{N}_A = \bigotimes_i \mathcal{N}_{A_i}$, then

$$\epsilon \geq (\Delta T_L)^2/(3\sqrt{6}F^{\uparrow}).$$

eigenvalue spread of T_L

bound on the quantum Fisher info of any metrological protocol

- **Price to pay:** code's quality is limited!

- Easy to evaluate for relevant noise models (erasure or depolarizing): $F^{\uparrow} = \sum_i (\Delta T_{A_i})^2 g(\mathcal{N}_{A_i})$ with $g(\mathcal{N}_{A_i}) = (1-p_i)/p_i$.

- **A different version**: if $\mathscr{E}_{L\to A}$ encodes one qubit into n qudits & has a transversal universal gate set, then $\epsilon \geq [3\sqrt{6}\sum_i (d_{A_i}-1)^2 g(\mathcal{N}_{A_i})]^{-1}$.

Main result & outline

Main result: a simple proof of the approximate E-K theorem.

Unorthodox approach:

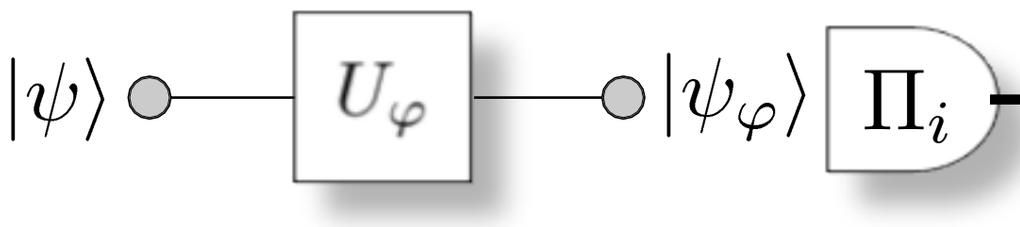
quantum metrology

quantum error correction

1. Setting the stage: quantum error correction.
2. The approximate E-K theorem.
3. Tools needed: quantum metrological bounds.
4. Our proof.

Quantum metrology

- Channel estimation problem:



unbiased estimator $\tilde{\varphi}(i)$ ⟶ $\min_{\Pi_i,\tilde{\varphi},\psi} \Delta^2\tilde{\varphi}$

- **Quantum Fisher information** useful for bounding the variance via the quantum Cramer-Rao inequality.

$$\Delta^2\tilde{\varphi} \geq \frac{1}{F}$$

- Quantum Fisher info for pure states is given by

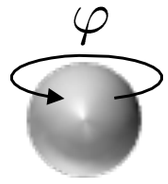
$$F(|\psi_\varphi\rangle) = 4\left(\langle\dot{\psi}_\varphi|\dot{\psi}_\varphi\rangle - \left|\langle\dot{\psi}_\varphi|\psi_\varphi\rangle\right|^2\right)$$

- For a unitary evolution $|\psi_t\rangle = e^{-iHt}|\psi\rangle$, time-energy uncertainty relation!

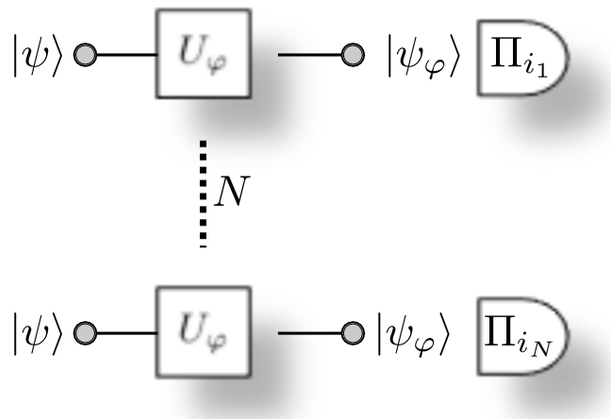
$$F(|\psi_t\rangle) = 4(\langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2) \longrightarrow \Delta^2\tilde{t} \geq \frac{1}{4\Delta^2 H}$$

Metrological bounds

- Parameter estimation w/ N uses of the channel, e.g. $\boxed{U_\varphi} = e^{i\frac{\sigma_z}{2}\varphi}$

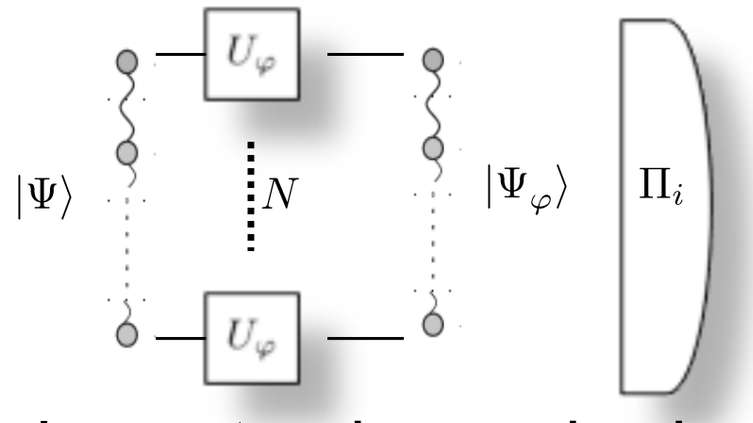


- Scaling of quantum Fisher info [GLM06,FI08,DKG12,DM14,…]:



uncorrelated scheme

$F(|\psi_\varphi\rangle^{\otimes N}) \propto N$



entanglement-enhanced scheme

$F(|\Psi_\varphi\rangle) \propto N^2$

- In the presence of noise $\mathcal{N} = \bigotimes_i \mathcal{N}_i$, quantum Fisher info may scale only linearly in the number N of uses of the parameter-encoding unitary!

$$\max F(|\Psi_\varphi\rangle) \leq F^{\uparrow} = 4\sum_i \min_{K_{i,k}^\varphi} \| \sum_k \dot{K}_{i,k}^{\varphi\dagger} \dot{K}_{i,k}^{\varphi} \| .$$

Kraus representation of $\mathcal{N}_i \circ \mathcal{U}_\varphi$ s.t. $\sum_k \dot{K}_{i,k}^{\varphi\dagger} K_{i,k}^{\varphi} = 0$

Giovannetti+'06;Fujiwara&Imai'08;Demkowicz-Dobrzański+'12;Demkowicz-Dobrzański&Maconne'14;

Main result & outline

Main result: a simple proof of the approximate E-K theorem.

Unorthodox approach:

quantum metrology

quantum error correction

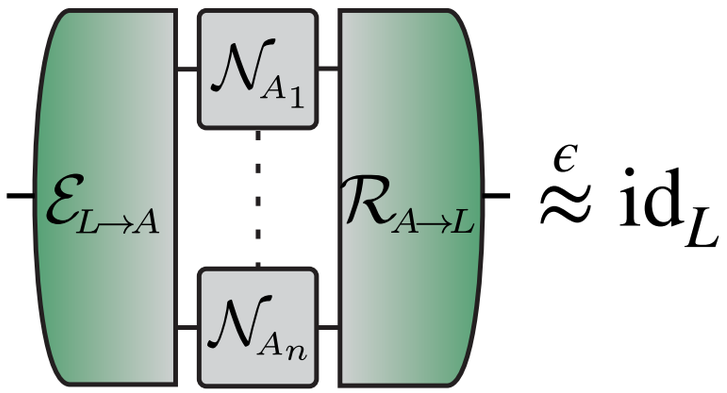
1. Setting the stage: quantum error correction.
2. The approximate E-K theorem.
3. Tools needed: quantum metrological bounds.
4. Our proof.

The approximate E-K theorem

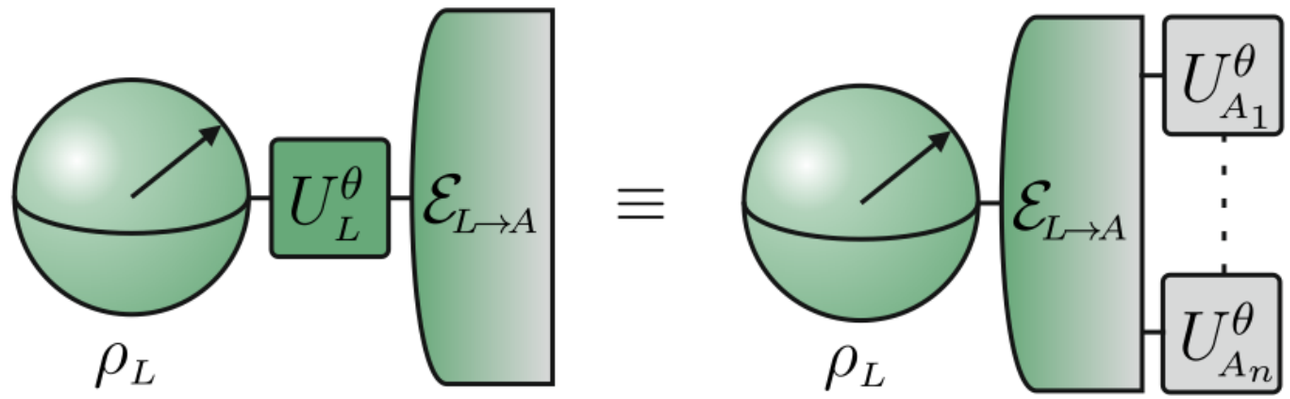
- **Theorem:** let $\mathscr{E}_{L\to A}$ be a $U(1)$-covariant code w/ transversal logical gates $U_L^\theta = e^{-i\theta T_L}$. If it is ϵ-correctable under noise $\mathscr{N}_A = \bigotimes_i \mathscr{N}_{A_i}$, then

$$\epsilon \geq (\Delta T_L)^2/(3\sqrt{6}F^{\uparrow}).$$

ϵ-correctability

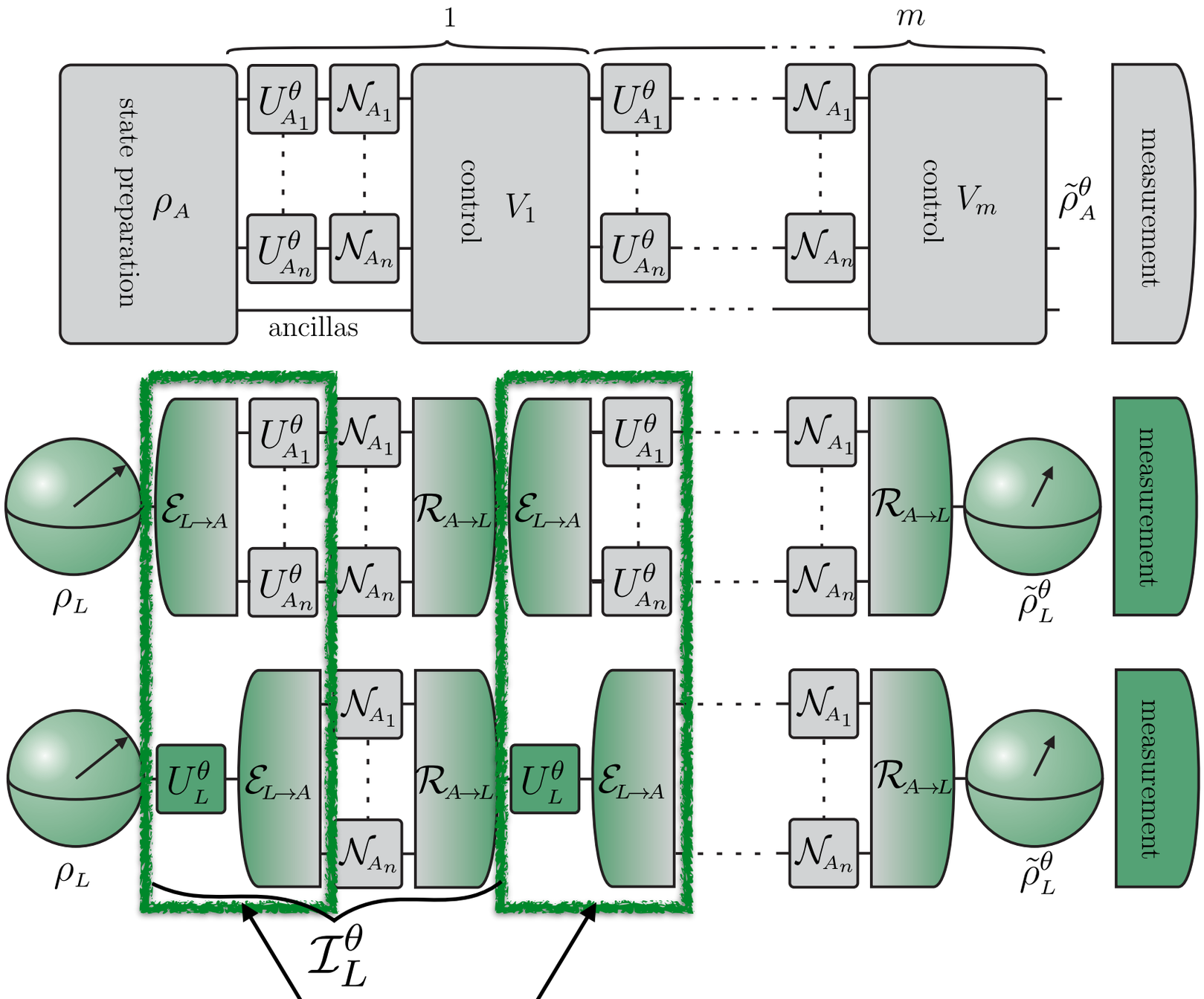


$U(1)$-covariance & transversality



- **Intuition:** if $\mathscr{E}_{L\to A}$ performed too well under $\mathscr{N}_A$, then we would be able to construct a protocol based on $\mathscr{E}_{L\to A}$ and $\mathscr{R}_{A\to L}$ violating quantum metrological bounds on the precision of parameter estimation!

Sketch of our proof



metrological bound

$$\max_{\rho_A,\theta,\{V_i\}_i} F(\tilde{\rho}^\theta_A) \le mF^\uparrow$$

$\bigvee|$

protocol based on $\mathscr{E}_{L\to A}$ and $\mathscr{R}_{A\to L}$

$$\max_{\rho_L,\theta} F[(\mathcal{I}^\theta_L)^m(\rho_L)]$$

$\bigvee|$

technical lemma

$$\min_\theta \left\{1 - 8\mathrm{d}[(\mathcal{I}^\theta_L)^m, \mathcal{U}^{m\theta}_L]^2\right\} \times (m\Delta T_L)^2$$

$\bigvee|$

ϵ-correctability, contractivity,…

$$(1-8m^2\epsilon^2)(m\Delta T_L)^2$$

- The tightest bound on ϵ for $m = 3F^\uparrow/(2(\Delta T_L)^2)$.

A couple of comments

- Compared w/ previous works [FNASPHY19,WA19]: **simple** proof, same scaling *1/n*, applicable **beyond the erasure noise**!

- **Corollary (Eastin-Knill)**: there does not exist a finite-dimensional $SU(d_L)$-covariant code w/ transversal gates and distance $D > 1.$
  Proof by contradiction: for the erasure noise

$$\epsilon \propto p^D \qquad < \qquad \frac{(\Delta T_L)^2}{3\sqrt{6}F^{\uparrow}} \propto p$$

for sufficiently small p

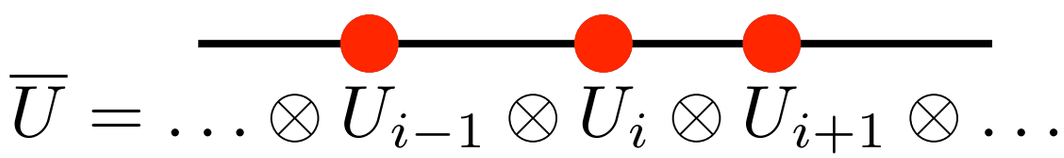
- If quantum Fisher info scales quadratically, then we get a trivial bound! Consistent with the 3-qubit repetition code: corrects any bit-flip error and has logical gates $U_L^\theta = e^{-i\theta\sigma_L^Z}$ implemented via $U_A^\theta = e^{-i\theta(\sigma_1^Z+\sigma_2^Z+\sigma_3^Z)}$.

Faist+'19;Woods&Alhambra'19

Summary

- Tension between error correction and fault-tolerant computation.

vs.

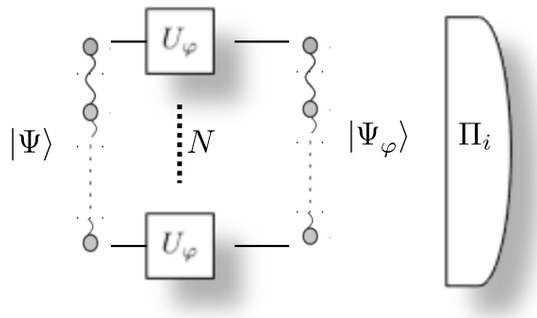


$$\overline{U} = \ldots \otimes U_{i-1} \otimes U_i \otimes U_{i+1} \otimes \ldots$$

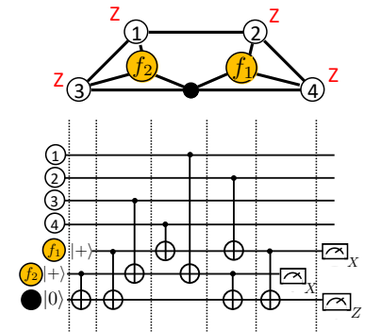
- The approximate Eastin-Knill theorem:

$$\epsilon \geq \frac{(\Delta T_L)^2}{3\sqrt{6}F^{\uparrow}}, \qquad F^{\uparrow} = 4\sum_{i=1}^{n} \min_{\{K_{i,k}^{\theta}\}_k, \beta_i = 0} \|\alpha_i\|.$$

- Novel approach!



quantum metrology



quantum error correction

THANKS FOR LISTENING!
arXiv:**2004.11893**