Circuit lower bounds for low-energy states of quantum code Hamiltonians

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The quantum PCP conjecture, a central open question in quantum complexity theory, asserts the hardness of approximating the ground state energy of a local Hamiltonian problem [1, 2]. The local Hamiltonian problem is the quantum analog of the constraint satisfaction problem which is the object of study in the classical PCP theorem. Despite the advances spanning over two decades, the quantum PCP conjecture remains wide open today. Furthermore, a necessary consequence, the No Low-Energy Trivial States (NLTS) conjecture of Freedman and Hastings, also remains open [3]. The NLTS conjecture posits that there exist local Hamiltonians such that every state of energy below a threshold linear in the system size cannot be defined by constant-depth quantum circuits.

In this work, we make progress on the NLTS conjecture by showing that for a wide class of local Hamiltonians derived from quantum error-correcting codes, every state of energy *almost* linear in the system size cannot be constructed by constant-depth quantum circuits.

Our results Defining the circuit complexity of a state ψ as the minimum depth of a fan-in/fan-out 2 quantum circuit that generates ψ starting from the all-zeros state, all classical pure states have circuit complexity 0 or 1. Furthermore, for a state ρ, the energy $\mathrm{tr}(H\rho)$ of ρ with respect to a local Hamiltonian H can be computed classically in time doubly-exponential in the circuit complexity of ρ. For this reason, in some literature, states of constant circuit complexity are referred to as "trivial" or "classical." Conversely, non-trivial pure states *i.e.* states of super-constant circuit complexity, are inherently "quantum" and possess entanglement. Our result, therefore, can also be referred to as a *circuit lower bound* on the entanglement complexity of low-energy states of quantum error-correcting codes. Our main result refers to a subclass of codes known as *stabilizer codes* where the code Hamiltonian is commuting and each Hamiltonian term is the tensor product of Pauli operators.

Theorem 1 *Let $\mathcal{C}$ be a $[[n,k,d]]$ stabilizer code of constant locality $\ell = O(1)$ and let H be the corresponding code Hamiltonian. For any state ψ on n-qubits with energy $\leq \epsilon n$, the circuit complexity of ψ is at least*

$$\Omega\left(\min\left\{\log d, \quad \log \frac{k}{n\epsilon \log \frac{1}{\epsilon}}\right\}\right). \tag{1}$$

For "reasonable" stabilizer codes of polynomial rate and polynomial distance, this theorem provides a non-trivial lower bound on the circuit complexity in the energy regime of $1/\mathsf{poly}(n)$. And in the case of linear-rate and polynomial-distance codes such as the hypergraph product code of Tillich and Zémor [4], the theorem proves a circuit lower bound of $\Omega(\delta \log n)$ for any state of energy $O(n^{1-\delta})$. So for fixed δ, say $\delta = 0.01$, it provides a circuit lower bound of $\Omega(\log n)$ for all states of energy $O(n^{0.99})$. Furthermore, it proves a circuit lower bound of $\Omega(\log \log n)$ for any state of energy $O(n/\mathsf{poly}\log n)$ and a super-constant circuit lower bound for any state of energy $o(n)$.

Likewise, for any stabilizer code of nearly-linear-rate (i.e. $n^{1-\delta}$ rate) and distance at least $n^{\Omega(\delta)}$, the theorem still proves a circuit lower bound of $\Omega(\delta \log n)$ for any state of energy $O(n^{1-2\delta})$. Codes with these properties are known to exist on constant-dimensional lattices; one example is the punctured 2D toric code with $O(n^{1-\delta})$ punctures [5, 6].

Relation to the Quantum PCP Conjecture The question of robust entanglement at high energies is strongly connected to the notion of hardness of approximation, a seminal idea of classical computer science. The probabilistically checkable proofs (PCP) theorem shows that classical constraint satisfaction problems such as 3-coloring or satisfiability are just as hard to approximate as they are to solve exactly. Since Kitaev's proof that the local Hamiltonian problem is QMA-complete, it has been evident that local Hamiltonians are the quantum analogs of constraint satisfaction problems. This connection led to the proposal of the quantum PCP conjecture which asked the analogous question about the hardness of approximating ground-states of local Hamiltonians.

A proof of the QPCP conjecture would necessarily demonstrate that complex entanglement persists at high energies, while also encoding hard computational problems (assuming $\mathsf{NP} \neq \mathsf{QMA}$). Despite numerous results both proving evidence for [3, 7, 8, 9, 10] and against [11, 12, 13] the QPCP conjecture, the problem remains wide open. For this reason, a more modest goal called the No Low-energy Trivial States (NLTS) conjecture was proposed by Freedman and Hastings [3]. This conjecture isolates the notion of robust entanglement from that of computation by positing the existence of a local Hamiltonian with a super-constant circuit lower bound on the complexity of all low-energy states.

The main result of this work is, therefore, a resolution of a weaker form of the NLTS conjecture: we show a circuit lower bound of $\Omega(\log(1/\epsilon))$ instead of a super-constant circuit lower bound. When viewed directly as a consequence of QPCP conjecture without the intermediate step of NLTS, our result in Theorem 1 is the necessary consequence of showing that for any positive constant δ, the local Hamiltonian problem is QMA-complete for promise gap $n^{-\delta}$. Currently, the problem is known to be QMA-complete for promise gap is $O(n^{-2})$ and any improvement past it would be highly non-trivial.

A key property of an NLTS Hamiltonian is that it cannot live on a lattice of dimension D for a fixed constant D [2]. This is because of a "cutting" argument: Let H be a local Hamiltonian in D dimensions and Ψ a ground-state of H. For a fixed constant ϵ, partition the space of qubits into D dimensional rectangular chunks so that the side length of each rectangular chunk is of size $O((D\epsilon)^{-1})$. Let ρ_i be the reduced state of Ψ on chunk i, and $\rho = \bigotimes \rho_i$ be a state over all the qubits. It's not hard to check that ρ violates at most a ϵ-fraction of the terms of H (only the boundary terms of the rectangular division) and yet has circuit complexity at most $\exp((D\epsilon)^{-D}) = O(1)$; so it is not NLTS. Because an NLTS Hamiltonian cannot be "cuttable", it might be simpler to consider a further simplification of NLTS called Combinatorial NLTS (CNLTS) [7]. CNLTS posits that there exist a universal constant $\epsilon > 0$ and a family of local Hamiltonians H such that for any Hamiltonian H' obtained from H by removing an ϵ-fraction of the HAmiltonian terms, the circuit complexity of any ground-state of H' is super-constant.

So, crucially CNLTS Hamiltonians cannot live on a lattice of constant dimension. However, as we previously noted the Hamiltonians in Theorem 1 can live on lattices of dimension as low as 2. Therefore, almost linear NLTS Hamiltonians are not necessarily CNLTS. This result is surprising; the CNLTS weakening mimics classical constraint satisfaction problems since every term either is or isn't satisfied and ignores the possibility of partially satisfying constraints. A priori, one might assume that handling the partially satisfied constraints would be the hard step. Our result gives a technique for handling the partially satisfied constraints but cannot handle the combinatorial aspect of the NLTS conjecture. This suggests that the "hard" step in the NLTS conjecture may be captured in the CNLTS conjecture.

The physics perspective The crucial role of entanglement in the theory of quantum many-body systems is widely known, with some seminal examples including topological phases of matter

[14] and quantum computation with physically realistic systems [15, 16]. But entanglement also brings new challenges, as the classical simulation of realistic many-body systems faces serious computational overheads.

Estimating the ground energy of such systems is one of the major problems in condensed matter physics [17], quantum chemistry [18], and quantum annealing [19, 20]. One of the key methods to address this problem is to construct *ansatz quantum states* that achieve as low energy as possible and are also suitable for numerical simulations. A leading ansatz, used in Variational Quantum Eigensolvers [21, 22, 18] or Quantum Adiabatic Optimization Algorithm [23], is precisely the class of quantum states that can be generated by low-depth quantum circuits.

Theorem 1 shows that there are Hamiltonians for which any constant-depth ansatz cannot estimate their ground energies beyond a fairly large threshold. As discussed earlier, we provide examples even in the physically realistic two-dimensional setting. The 2D punctured toric code Hamiltonians on n qubits with distance d (which is a free parameter) require a circuit of depth $\Omega(\log d)$, if we want an approximation to ground energy better than $O(n/d^7)$.

Overview of proof techniques Error-correcting codes are a fruitful source of lower bound techniques because of the way information is spread across the set of qubits. Error-correcting codes of distance d have a local-indistinguishability property for all subsets of size $< d$. This means that for any size $< d$ subset S of the qubits, the reduced density matrix ρ_S for any $\rho \in \mathcal{C}$ is an invariant of the code-space. Consider then a state ψ near a code $\mathcal{C}$. The reduced density matrices $\{\psi_S\}$ approximate the reduced density matrices of the closest state of $\mathcal{C}$. By local-indistinguishability, the $\{\psi_S\}$ in turn approximate the reduced density matrices for all code-states.

In particular, they approximate the reduced density matrices of the entangled maximally-mixed state Θ of the code. This state has entropy $S(\Theta)$ equal to the rate of the code, k. We now show that if ψ has low circuit complexity, then the entropy $S(\Theta)$ is bounded. Assume that ψ is the output of a low-depth circuit W, then for any qubit i,

$$\mathrm{tr}_{-\{i\}}(W^\dagger \psi W) \approx \mathrm{tr}_{-\{i\}}(W^\dagger \Theta W). \tag{2}$$

This is because $\mathrm{tr}_{-L_i}(\psi) \approx \mathrm{tr}_{-L_i}(\Theta)$ where L_i is the support of the lightcone of qubit i with respect to W. However, the left-hand side of (2) equals the pure state $|0\rangle\langle 0|$ and so the entropy of $\mathrm{tr}_{-\{i\}}(W^\dagger \Theta W)$, the ith qubit of $W^\dagger \Theta W$, is small. This gives us an overall bound on the entropy of Θ and also the rate of the code. Assuming that $\mathcal{C}$ is a high-rate code, this yields a lower bound on the circuit complexity of ψ.

If the state ψ has low-energy with respect to the code Hamiltonian, we cannot guarantee that it is close in trace-distance to a code-state. A general strategy in earlier works [7, 10] was to build a low-depth decoding circuit to bring the state closer to the ground-space. But this required a local-testability property, which is not available to us. We appeal to the observation that every eigenspace of a stabilizer code Hamiltonian possesses the local-indistinguishability property. Instead of attempting to construct a decoding circuit, we simply measure the syndrome using a constant-depth circuit (which uses the LDPC nature of the code Hamiltonian). This allows us to decohere ψ to a mixture of states that live within the eigenspaces. The aforementioned local-indistinguishability property and the low energy of ψ help us to prove results similar to (2). We crucially use the low-energy property: it implies that for most L_i, the energy of the Hamiltonian terms within L_i is small.

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