Asymptotic theory of quantum channel estimation

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This submission is based on the paper "Asymptotic theory of quantum channel estimation" [1].

Quantum metrology is the study of parameter estimation in quantum systems, which provides a theoretical guideline for building quantum-enhanced sensors in real-world experiments. Usually, the experimentalist prepares a probe state of his/her choice, let it evolve under the system dynamics governed by an unknown parameter θ and perform measurements in the end to estimate the unknown parameter. Given the final quantum state ρ_θ, the quantum Cramér-Rao bound [2] states that $\delta\theta \geq 1/\sqrt{MF(\rho_\theta)}$, where $\delta\theta$ is the standard deviation of the estimation error, M is the number of repeated measurements and $F(\rho_\theta)$ is the quantum Fisher information (QFI). The bound is saturable asymptotically when $M \to \infty$. The QFI is a good information measure quantifies the amount of information a quantum state carries about an unknown parameter. Taking the optimization of the probe states into consideration, we can define the QFI of quantum channels [3] $F(\mathcal{N}_\theta) := \max_\rho F((\mathcal{N}_\theta \otimes \mathbb{1})(\rho))$ where $\mathcal{N}_\theta$ is a quantum channel describing the system dynamics and ρ is a probe state which could be entangled over the input system of $\mathcal{N}_\theta$ and an unbounded ancillary system.

While the state QFI $F(\rho_\theta)$ is always additive, i.e. $F(\rho_\theta^{\otimes N}) = NF(\rho_\theta)$, the channel QFI is not. It means that given N copies of parametrized quantum channels, it is possible to choose an entangled probe state which achieves a better estimation precision than product probe states, which is the underlying rationale behind practical quantum-enhanced sensors. One major breakthrough in quantum metrology was the discovery that $F(\mathcal{U}_\theta^{\otimes N}) = N^2 F(\mathcal{U}_\theta)$ where $\mathcal{U}_\theta$ is unitary [4]. Such a quadratic scaling is called the Heisenberg limit and marks the ultimate advantage of quantum-enhanced sensors. On the other hand, the $O(N)$ scaling, which is called the standard quantum limit (SQL), usually appears in noisy systems [5, 6] and has at most a constant factor improvement from product probe states.

One fundamental open problem in quantum channel estimation is to solve the channel QFI when $N \to \infty$. Here we show that there is *a simple criterion determining whether $\mathcal{N}_\theta$ follows the HL or the SQL* (all other scalings are impossible), which we call the HNKS ("Hamiltonian-not-in-Kraus-span") condition. Intuitively, the Hamiltonian (a Hermitian operator H) of a quantum channel describes the evolution of the channel with respect to θ, which reduces to the traditional definition of Hamiltonian when the channel is unitary, and the Kraus span (a matrix subspace $\mathcal{S}$) of a quantum channel contains all types of noises that could happen in the system. The HL is achievable if and only if $H \notin \mathcal{S}$. Furthermore, we show that the *"regularized" channel QFIs*

$$F_{\mathrm{HL}}(\mathcal{N}_\theta) = \lim_{N\to\infty} \frac{F(\mathcal{N}_\theta^{\otimes N})}{N^2}, \qquad F_{\mathrm{SQL}}(\mathcal{N}_\theta) = \lim_{N\to\infty} \frac{F(\mathcal{N}_\theta^{\otimes N})}{N} \tag{1}$$

are both computable via semidefinite programs and $F_{\mathrm{SQL}}(\mathcal{N}_\theta)$ is additive. According to the HNKS criterion, when $H \notin \mathcal{S}$, $F_{\mathrm{HL}}(\mathcal{N}_\theta) > 0$ and $F_{\mathrm{SQL}}(\mathcal{N}_\theta) = \infty$; when $H \in \mathcal{S}$, $F_{\mathrm{HL}}(\mathcal{N}_\theta) = 0$ and $F_{\mathrm{SQL}}(\mathcal{N}_\theta) < \infty$. Note that it was known previously that there are some efficiently computable upper bounds of $F_{\text{HL,SQL}}(\mathcal{N}_\theta)$ [3, 6, 7] and our contribution is to prove the attainablity of the upper bounds.

We provide a constructive proof where we show explicit quantum error correction (QEC) protocols asymptotically achieving the regularized QFIs in both cases, which paves the way for practical implementation of the optimal QEC protocols in real-world experiments. The proof have three steps, including (1) showing the regularized channel QFIs of single-qubit dephasing channels are achievable using spin-squeezed states [8] (when $H \in S$) or GHZ states [4] (when $H \notin S$); (2) devising two-dimensional QEC protocols which reduce all channels to single-qubit dephasing channels; (3) formulating the code optimization as semidefinite programs. The optimized QFIs in the two-dimensional logical space match the QFI upper bounds, which proves the attainability. Our proof techniques are innovative and similar results were only known in the case of Hamiltonian estimation in Markovian dynamics [9, 10].

In addition to computing the regularized channel QFIs for non-adaptive channel estimation strategies where N identical quantum channels act in parallel on a quantum state, our result (when $H \in \mathcal{S}$) also applies to adaptive channel estimation strategies where we allow where we allow quantum controls (arbitrary quantum operations) between each channel. Namely, the regularized channel QFI $F_{\text{SQL}}(\mathcal{N}_\theta)$ is the same for both non-adaptive channel estimation strategies and adaptive ones, which means that adaptive strategies provide no asymptotic advantages over non-adaptive ones for generic noisy quantum channels. This was an intriguing open question in quantum metrology [7] and is closely connected to similar problems in quantum channel discrimination where the distinction between the asymptotic powers of adaptive and non-adaptive strategies was studied extensively [11–15].

To summarize, in this talk we will present the asymptotic theory of quantum channel estimation, including the HNKS condition and the regularized channel QFIs. Our results are fundementally important in quantum information theory and also potentially applicable to practical quantum sensing.

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