Asymptotic theory of quantum channel estimation

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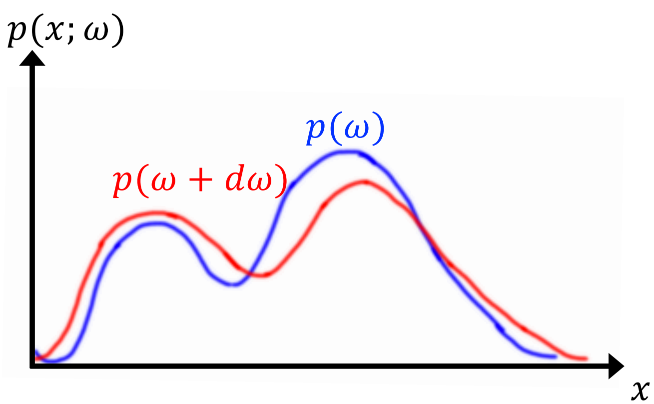
Outline

- Introduction to quantum metrology
- Asymptotic quantum channel estimation
- Examples

Classical estimation theory



Estimation precision:

$$\Delta\omega = (\mathbb{E}[(\widehat{\omega}(x)-\omega)^2])^{\frac{1}{2}}$$

For unbiased estimators $\mathbb{E}[\widehat{\omega}(x)] = \omega$, we have **the Cramér-Rao bound**:

$$\Delta\omega \geq \frac{1}{\sqrt{N_{\mathrm{expr}} \cdot F(p(x;\omega))}}$$

N_{expr}: number of experiments

$F(p(x;\omega))$: Fisher information

The bound is saturable asymptotically using the maximum likelihood estimator.

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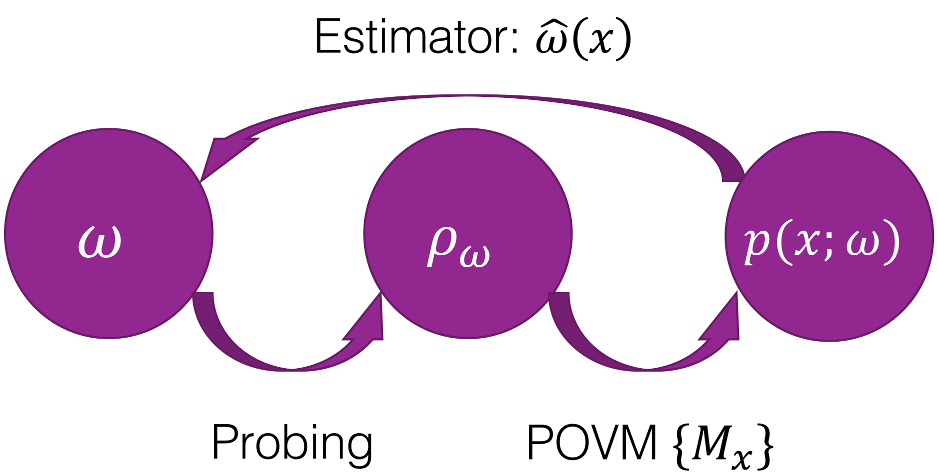
Quantum estimation theory



The quantum Cramér-Rao bound:

$$\Delta\omega \geq \frac{1}{\sqrt{N_{\mathrm{expr}} \cdot F(\rho_\omega)}}$$

N_{expr}: number of experiments

$F(\rho_\omega)$: quantum Fisher information

$F(\rho_\omega) = \max_{\{M_x\}} F(p(x:\omega))$

The quantum Cramér-Rao bound is also saturable asymptotically.

Helstrom 1976, Holevo 1982
Braunstein & Caves 1994

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Quantum Fisher information

Quantum Fisher information (QFI): $F(\rho_\omega) = \max_{\{M_x\}} F(p(x{:}\,\omega))$

$$F(\rho_\omega) = \mathrm{Tr}(\rho_\omega L_\omega^2),\ \frac{L_\omega \rho_\omega + \rho_\omega L_\omega}{2} = \partial_\omega \rho_\omega$$

- Faithfulness: $F(\rho_\omega) \geq 0$ and $F(\rho_\omega) = 0$ if and only if $\partial_\omega \rho_\omega = 0$.
- Monotonicity: $F(\mathcal{N}(\rho_\omega)) \leq F(\rho_\omega)$, $\mathcal{N}$ is an arbitrary CPTP map.
- Additivity: $F(\rho_\omega \otimes \sigma_\omega) = F(\rho_\omega) + F(\sigma_\omega)$.
- Convexity: $F(p_1 \rho_\omega + p_2 \sigma_\omega) \leq p_1 F(\rho_\omega) + p_2 F(\sigma_\omega)$.
- Connection to Fidelity: $\frac{1}{4} F(\rho_\omega) d\omega^2 = d_{\mathrm{Bures}}^2(\rho_\omega, \rho_{\omega+d\omega}) = 2 - 2Fid(\rho_\omega, \rho_{\omega+d\omega})$.

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Quantum Fisher information

Example 1 (pure state):

$\rho_\omega = |\psi_\omega\rangle\langle\psi_\omega|, |\psi_\omega\rangle = e^{-i\omega Ht}|\psi_0\rangle,$

$$F(\rho_\omega) = 4t^2\langle\Delta^2 H\rangle = \Theta(t^2),$$

where $\langle\Delta^2 H\rangle = (\langle\psi_0|H^2|\psi_0\rangle - \langle\psi_0|H|\psi_0\rangle^2)$.

- For a single qubit state, when $H = Z/2$, the optimal initial state is

$$|\psi_0\rangle = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}.$$

Quantum Fisher information

Example 2 (single-qubit dephasing):

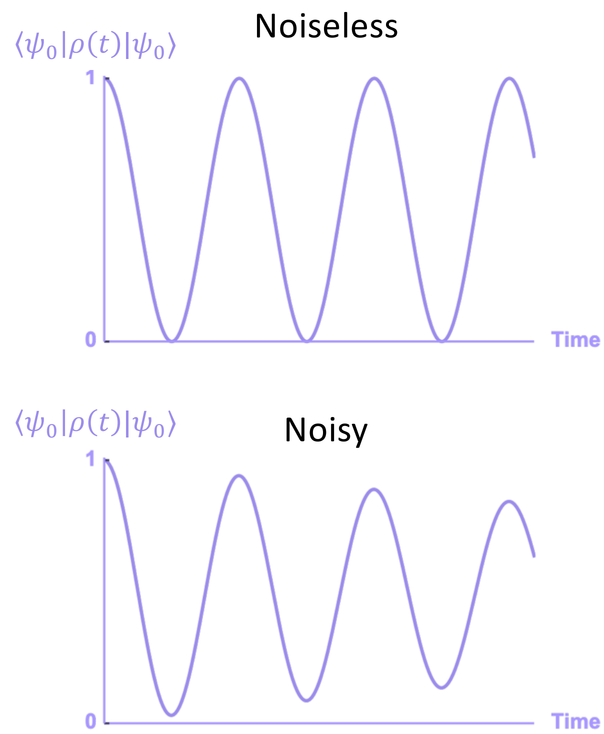
$$\frac{d\rho}{dt} = -i\left[\frac{\omega Z}{2}, \rho\right] + \frac{\gamma}{2}(Z\rho Z - \rho)$$

Input state: $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

Noiseless case ($\gamma = 0$): $F\big(\rho_\omega(t)\big) = t^2 = \Theta(t^2)$.

Noisy case ($\gamma > 0$): $F\big(\rho_\omega(t)\big) = t^2 e^{-2\gamma t}$.

If we measure and renew the qubit every constant time, we get only QFI $= \Theta(t)$.



Quantum Fisher information

Example 3 (N-qubit dephasing):

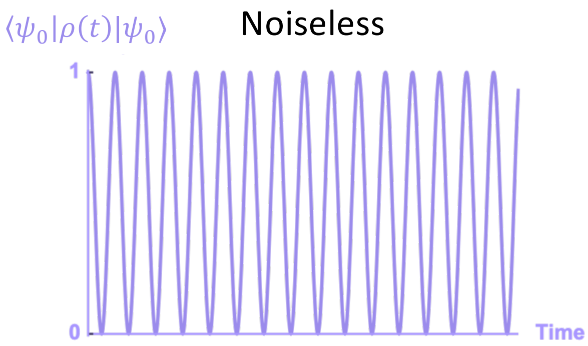
Input state (GHZ state):

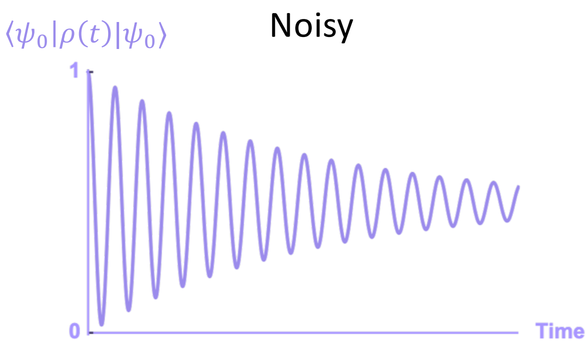
$$|\psi_0\rangle = \frac{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}}{\sqrt{2}}$$

Noiseless case ($\gamma = 0$): $F\big(\rho_\omega(t)\big) = N^2 t^2 = \Theta(N^2)$.

Noisy case ($\gamma > 0$): $F\big(\rho_\omega(t)\big) = N^2 t^2 e^{-2N\gamma t}$.

Average QFI over time: $\max_{t>0} F\big(\rho_\omega(t)\big)/t = N/2e\gamma = \Theta(N)$.





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Quantum Fisher information

In quantum metrology, the resource we care about is the number of channels used N or the probing time t. There are two types of estimation precision limits:

- The Heisenberg limit (HL): QFI $= \Theta(N^2)$ or $\Theta(t^2)$

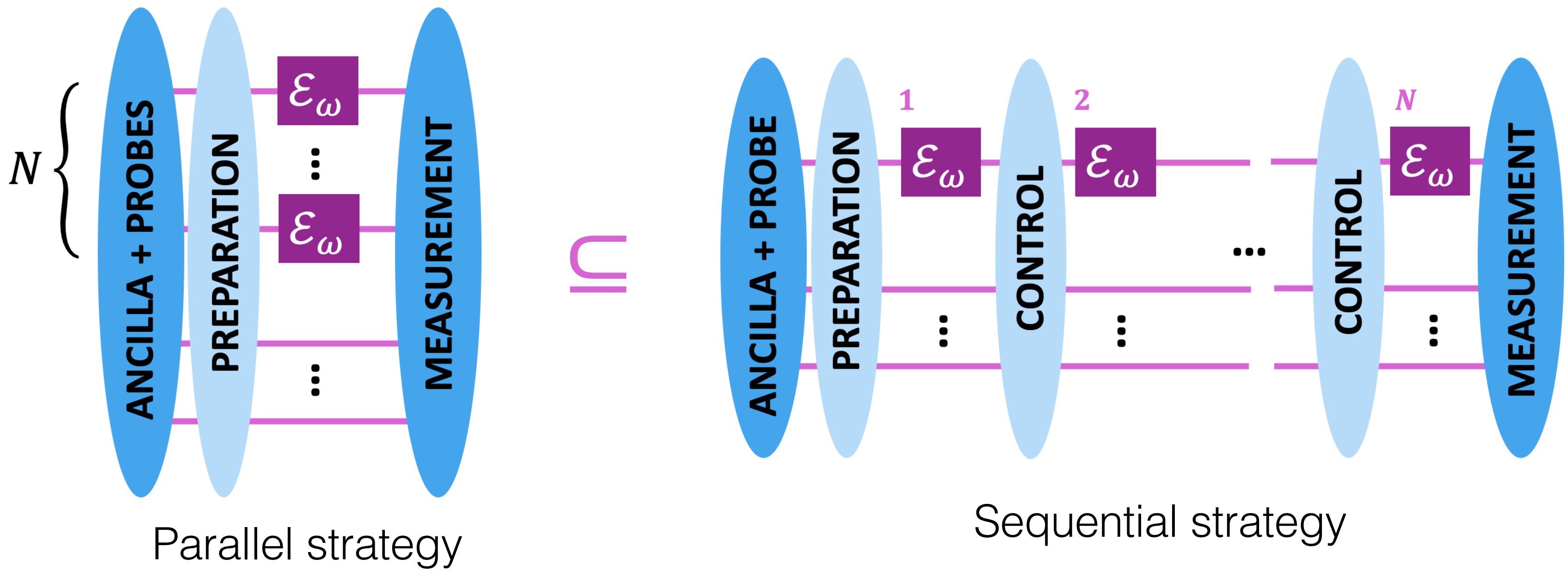
— the ultimate estimation precision limit allowed by quantum mechanics.

- The standard quantum limit (SQL): QFI $= \Theta(N)$ or $\Theta(t)$

— achievable using "classical" strategies (no need to maintain the coherence in & between probes for a long time).

Quantum channel estimation

$$\mathcal{E}_\omega(\rho) = \sum_{i=1}^{r} K_{\omega,i}\rho K_{\omega,i}^\dagger$$



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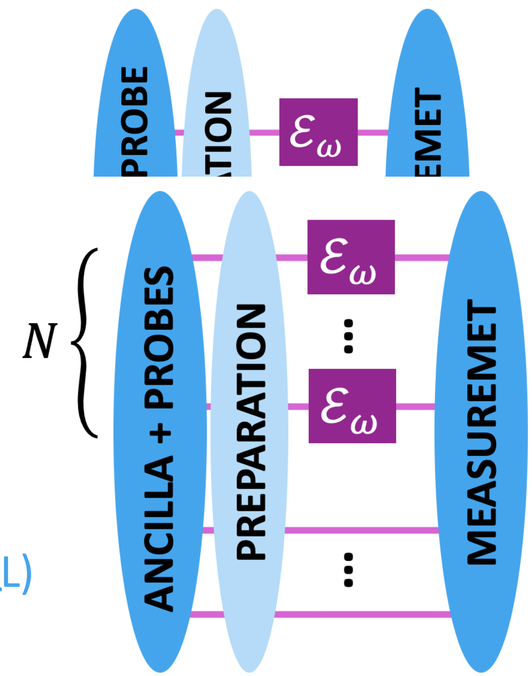
Quantum channel estimation



The channel QFI:

$$F(\mathcal{E}_\omega) = \max_\rho F((\mathcal{E}_\omega \otimes \mathbb{I})(\rho))$$

The asymptotic channel QFI:

$F\left(\mathcal{E}_\omega^{\otimes N}\right) \propto N^2$: the Heisenberg limit (HL)

$F\left(\mathcal{E}_\omega^{\otimes N}\right) \propto N$: the standard quantum limit (SQL)

The asymptotic channel QFI either follows the HL or the SQL.

Examples: unitary channels, depolarizing noise.

The same is true for $F^{(\mathrm{seq})}(\mathcal{E}_\omega, N)$----the QFI optimized over all sequential strategies.

Fujiwara & Imai 2008, Demkowicz-Dobrzanski *et al.* 2012
Demkowicz-Dobrzanski & Maccone 2014, SZ & Jiang 2020

Necessary and sufficient condition

Theorem 1: The HL is achievable using parallel strategies (or sequential strategies) if and only if **the Hamiltonian-not-in-Kraus-Span (HNKS) condition** is satisfied:

$$H(\mathcal{E}_\omega) \notin \mathcal{S}(\mathcal{E}_\omega),$$

where

$$\mathbf{K} = \begin{pmatrix} K_{\omega,1} \\ \vdots \\ K_{\omega,r} \end{pmatrix}$$

Hamiltonian: $H(\mathcal{E}_\omega) \coloneqq i \sum_i K_{\omega,i}^\dagger \partial_\omega K_{\omega,i} = i\mathbf{K}^\dagger \dot{\mathbf{K}}$,

Kraus span: $\mathcal{S}(\mathcal{E}_\omega) = \mathrm{span}\{K_{\omega,i}^\dagger K_{\omega,j}, \forall i,j\}$.

Remark: For unitary channel $U_\omega = e^{-iH\omega}$, or $\mathcal{E}_\omega = \mathcal{N} \circ \mathcal{U}_\omega$, $H(\mathcal{E}_\omega) = H$.

Fujiwara & Imai 2008, Demkowicz-Dobrzanski *et al.* 2012
Demkowicz-Dobrzanski & Maccone 2014, SZ & Jiang 2020

The asymptotic QFI

$$F_{\mathrm{HL}}(\mathcal{E}_\omega) = \lim_{N\to\infty} \frac{F\left(\mathcal{E}_\omega^{\otimes N}\right)}{N^2}, \qquad F_{\mathrm{SQL}}(\mathcal{E}_\omega) = \lim_{N\to\infty} \frac{F\left(\mathcal{E}_\omega^{\otimes N}\right)}{N}$$

Theorem 2: When $H \notin \mathcal{S}$,

$\min_h \|\beta\|$ is a distance between H and $\mathcal{S}$

$$F_{\mathrm{HL}}(\mathcal{E}_\omega) = 4 \min_h \|\beta\|^2, \quad \beta = H - \mathbf{K}^\dagger h \mathbf{K}.$$

Theorem 3: When $H \in \mathcal{S}$,

$$F_{\mathrm{SQL}}(\mathcal{E}_\omega) = 4 \min_{h:\beta=0} \|\alpha\|, \quad \alpha = \left(\dot{\mathbf{K}} - ih\mathbf{K}\right)^\dagger (\dot{\mathbf{K}} - ih\mathbf{K}).$$

In particular, $F_{\mathrm{SQL}}(\mathcal{E}_\omega) = F_{\mathrm{SQL}}^{(\mathrm{seq})}(\mathcal{E}_\omega)$.

$\mathbf{K} = \begin{pmatrix} K_{\omega,1} \\ \vdots \\ K_{\omega,r} \end{pmatrix}$, Hermitian $h \in \mathbb{C}^{r\times r}$

Fujiwara & Imai 2008, Demkowicz-Dobrzanski *et al.* 2012
Demkowicz-Dobrzanski & Maccone 2014, SZ & Jiang 2020

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Proof of Theorem 2 and 3

$$F_{\mathrm{HL}}(\mathcal{E}_\omega) = 4 \min_h \|\beta\|^2, \qquad F_{\mathrm{SQL}}(\mathcal{E}_\omega) = 4 \min_{h:\beta=0} \|\alpha\|.$$

$$\beta = H - \mathbf{K}^\dagger h \mathbf{K}, \quad \alpha = \left(\dot{\mathbf{K}} - ih\mathbf{K}\right)^\dagger (\dot{\mathbf{K}} - ih\mathbf{K}).$$

- **Upper bounds:**

Fujiwara & Imai 2008, Demkowicz-Dobrzanski *et al.* 2012
Demkowicz-Dobrzanski & Maccone 2014

$$F\left(\mathcal{E}_\omega^{\otimes N}\right) \le 4(N\|\alpha\| + N(N-1)\|\beta\|^2)$$

$$F^{(\mathrm{seq})}(\mathcal{E}_\omega, N) \le 4(N\|\alpha\| + N(N-1)\|\beta\|(\|\beta\| + \|\alpha\| + 1))$$

- **Attainability:**

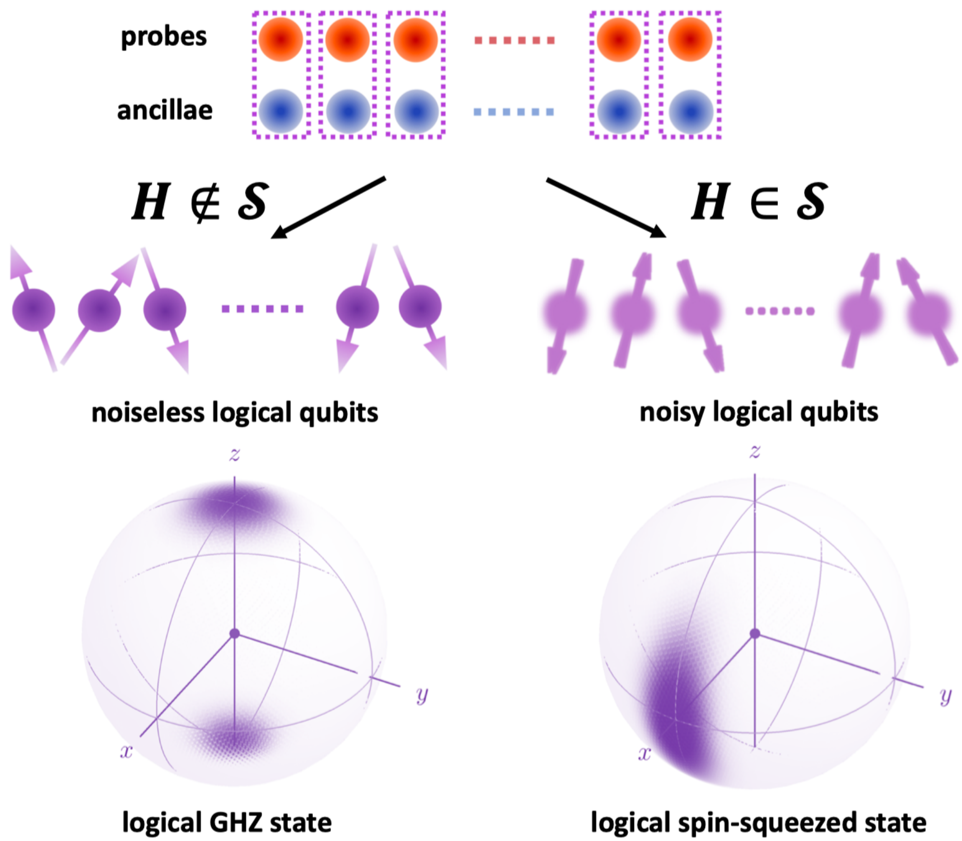
SZ & Jiang 2020

Quantum error correction (QEC) protocols

Hamiltonian estimation in master equations
Demkowicz-Dobrzanski *et al.* 2017, SZ *et al.* 2018, SZ & Jiang 2019
SZ, QIP 2018

Attainability: the QEC protocol

The Heisenberg limit is recovered.



The optimal SQL coefficient $F_{\mathrm{SQL}}(\mathcal{E}_\omega)$ is achievable.

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Attainability: the QEC protocol

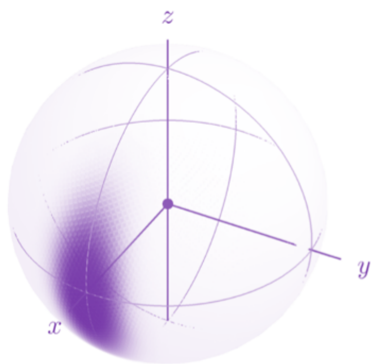
- **Step 1: Asymptotic QFIs for dephasing channels**
- Step 2: Reduction to dephasing channels
- Step 3: Code optimization

Dephasing channels $\mathcal{D}_\omega$:

$$\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \xrightarrow{\mathcal{D}_\omega} \begin{pmatrix} \rho_{00} & \xi\rho_{01} \\ \xi^*\rho_{10} & \rho_{11} \end{pmatrix}$$

When $|\xi| = 1, \quad F_{\mathrm{HL}}(\mathcal{D}_\omega) = |\dot{\xi}|^2,$

When $|\xi| < 1, \quad F_{\mathrm{SQL}}(\mathcal{D}_\omega) = \frac{|\dot{\xi}|^2}{1-|\xi|^2},$



logical spin-squeezed state

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Attainability: the QEC protocol

- Step 1: Asymptotic QFIs for dephasing channels
- **Step 2: Reduction to dephasing channels**
- Step 3: Code optimization

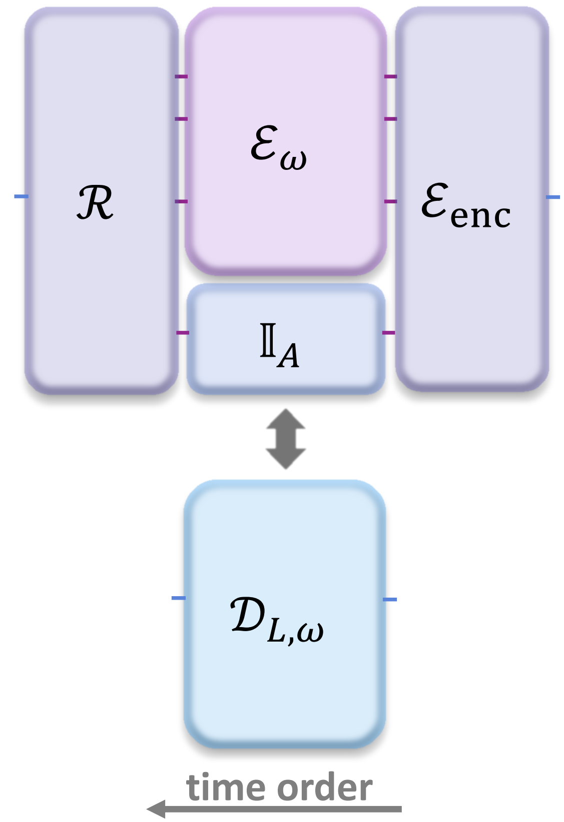
Two-dimensional code:

Encoding:

$|0_L\rangle = \sum_{ij}(C_0)_{ij}\,|i\rangle_S|j\rangle_{A_1}|0\rangle_{A_2},\quad |1_L\rangle = \sum_{ij}(C_1)_{ij}\,|i\rangle_S|j\rangle_{A_1}|1\rangle_{A_2}$

Decoding:

$\mathcal{R}(\cdot) = \sum_m R_m(\cdot)R_m^\dagger,\quad R_m = P_{|0\rangle_L}R_m P_{|0\rangle_{A_2}} + P_{|1\rangle_L}R_m P_{|1\rangle_{A_2}}$



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Attainability: the QEC protocol

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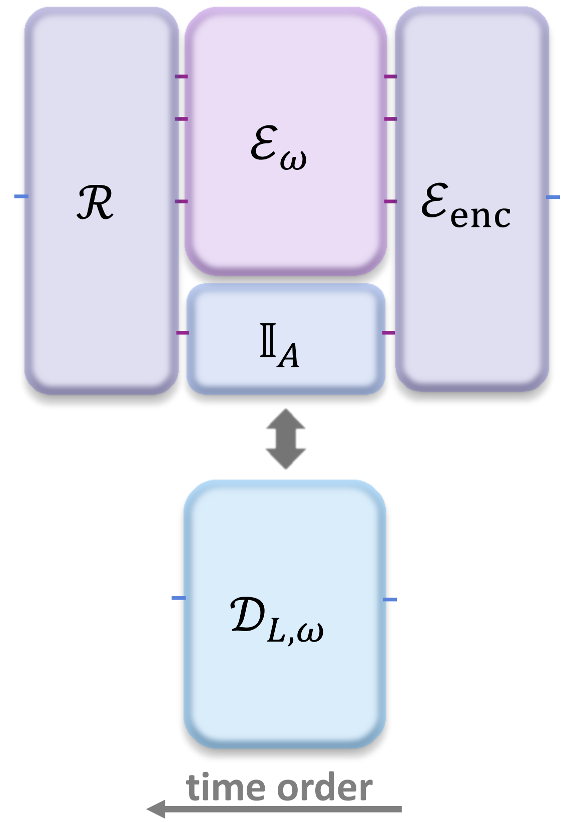
Encoding:

$$|0_L\rangle = \sum_{ij} (C_0)_{ij} \, |i\rangle_S |j\rangle_{A_1} |0\rangle_{A_2}, \quad |1_L\rangle = \sum_{ij} (C_1)_{ij} \, |i\rangle_S |j\rangle_{A_1} |1\rangle_{A_2}$$

Decoding:

$$\mathcal{R}(\cdot) = \sum_m R_m (\cdot) R_m^\dagger, \quad R_m = P_{|0\rangle_L} R_m P_{|0\rangle_{A_2}} + P_{|1\rangle_L} R_m P_{|1\rangle_{A_2}}$$

The optimization of $F_{\mathrm{HL,SQL}}(\mathcal{D}_{L,\omega})$ over C_0, C_1 and $\mathcal{R}$ gives Theorem 2 and 3.



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Example 1: Pauli Z signal with bit-flip noise

$$\mathcal{E}_\omega(\rho) = \mathcal{N} \circ \mathcal{U}_\omega(\rho),$$

$$\mathcal{N}(\rho) = (1-p)\rho + pX\rho X, \quad U_\omega = e^{-\frac{i\omega Z}{2}},$$

$$H = Z \notin \mathcal{S} = \mathrm{span}\{I, X\}$$

Optimal code: $|0_L\rangle = |0\rangle_S \otimes |0\rangle_A$, $|1_L\rangle = |1\rangle_S \otimes |1\rangle_A$.

After error:

$(X \otimes I)|0_L\rangle = |1\rangle_S \otimes |0\rangle_A$, $(X \otimes I)|1_L\rangle = |0\rangle_S \otimes |1\rangle_A$.

Recovery:

Measure $S = Z \otimes Z$. If $S = -1$, flip the probe with X; otherwise do nothing.

Kessler *et al.* 2014, Arrad *et al.* 2014, Dur *et al.* 2014, Unden *et al.* 2016

Example 1: Pauli Z signal with bit-flip noise

$$\mathcal{E}_\omega(\rho) = \mathcal{N} \circ \mathcal{U}_\omega(\rho),$$

$$\mathcal{N}(\rho) = (1-p)\rho + pX\rho X, \quad U_\omega = e^{-\frac{i\omega Z}{2}},$$

$$H = Z \notin \mathcal{S} = \mathrm{span}\{I, X\}$$

The asymptotic QFI is:

$$F_{\mathrm{HL}}(\mathcal{E}_\omega) = 1.$$

(equal to the $F_{\mathrm{HL}}(\mathcal{U}_\omega)$)

The single-channel QFI is:

$$F(\mathcal{E}_\omega) = 1.$$

(equal to the $F(\mathcal{U}_\omega)$)

Kessler *et al.* 2014, Arrad *et al.* 2014, Dur *et al.* 2014, Unden *et al.* 2016

Example 2: Pauli Z signal with dephasing noise

$$\mathcal{D}_\omega(\rho) = \mathcal{N} \circ \mathcal{U}_\omega(\rho),$$

$$\mathcal{N}(\rho) = (1-p)\rho + pZ\rho Z, \quad U_\omega = e^{-\frac{i\omega Z}{2}},$$

$$H = Z \in \mathcal{S} = \mathrm{span}\{I, Z\}$$

The asymptotic QFI is:

$$F_{\mathrm{SQL}}(\mathcal{D}_\omega) = \frac{(1-2p)^2}{4p(1-p)} = O(1/p).$$

The single-channel QFI is:

$$F(\mathcal{D}_\omega) = (1-2p)^2 = O(1).$$

QEC is not necessary in this case.

Ulam-Orgikh & Kitagawa 2001, Demkowicz-Dobrzanski *et al.* 2012

Example 3: Pauli Z signal with depolarizing noise

$$\mathcal{E}_\omega(\rho) = \mathcal{N} \circ \mathcal{U}_\omega(\rho),$$

$$\mathcal{N}(\rho) = \left(1 - \tfrac{3}{4}p\right)\rho + \tfrac{p}{4}(X\rho X + Y\rho Y + Z\rho Z), \quad U_\omega = e^{-\frac{i\omega Z}{2}},$$

$$H = Z \in \mathcal{S} = \mathrm{span}\{I, X, Y, Z\}$$

The asymptotic QFI is:

$$F_{\mathrm{SQL}}(\mathcal{E}_\omega) = \frac{2(1-p)^2}{p(3-2p)} = O(1/p).$$

The single-channel QFI is:

$$F(\mathcal{E}_\omega) = \frac{2(1-p)^2}{2-p} = O(1).$$

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Conclusions and outlook

- The structure of the noise and the Hamiltonian determines the estimation precision limit in quantum metrology.
- Quantum error correction is powerful – recovering the HL, achieving the optimal SQL.
- Future directions:
    - parallel strategies vs sequential strategies when the HL is achievable
    - multi-parameter estimation for general quantum channels
    - fault-tolerant QEC for quantum metrology

Thank you!

Liang Jiang

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