Degree vs. Approximate Degree and Quantum Implications of Huang's Sensitivity Theorem

Scott Aaronson[1] Shalev Ben-David[2] Robin Kothari[3]
Shravas Rao[4] Avishay Tal[5]

[1]The University of Texas at Austin [2]University of Waterloo
[3]Microsoft Quantum [4]Northwestern University
[5]University of California, Berkeley

February 3, 2021

Summary of Results

Definition (Boolean Functions)

$f : \{0,1\}^n \to \{0,1\}$ (total).

Our Results

- $\deg(f) = O(\mathrm{Q}(f)^2)$
  - $\mathrm{D}(f) = O(\mathrm{Q}(f)^4)$
  - The quantum query complexity of any non-trival monotone graph property on n vertices is $\Omega(n)$.

Summary of Results

Definition (Boolean Functions)

$f : \{0,1\}^n \to \{0,1\}$ (total).

Our Results

- $\deg(f) = O(\mathrm{Q}(f)^2)$
  - $\mathrm{D}(f) = O(\mathrm{Q}(f)^4)$
  - The quantum query complexity of any non-trival monotone graph property on n vertices is $\Omega(n)$.
- $\deg(f) = O(\widetilde{\deg}(f)^2)$
  - The approximate degree of any read-once formula on n variables is $\Omega(\sqrt{n})$.

Boolean Functions

Definition (Boolean Functions)

$f : \{0,1\}^n \to \{0,1\}$ (total).

Examples

- $\text{DICTATOR}(x_1, \ldots, x_n) = x_1$
- $\text{OR}(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } x_1 = x_2 = \cdots = x_n = 0 \\ 1 & \text{otherwise} \end{cases}$
- $\text{XOR}(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } x_1 + x_2 + \cdots + x_n \text{ is even} \\ 1 & \text{otherwise} \end{cases}$

Deterministic Query Complexity

Definition (Deterministic Query Complexity, $\mathrm{D}(f)$)

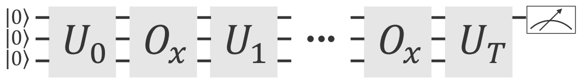
The deterministic query complexity of a Boolean function f, is the number of deterministic queries required to compute f on any input. (Compute $f(x)$ by reading as few bits as possible.)

	DICTATOR	OR	XOR
$\mathrm{D}(f)$	1	n	n

Quantum Query Complexity

Definition (Quantum Query Complexity, Q(f))

The quantum query complexity of a Boolean function f, is the number of quantum queries required to compute f on any input with error probability at most $1/3$.



	DICTATOR	OR	XOR
D(f)	1	n	n
Q(f)	1	$\Theta(\sqrt{n})$ (Grover, BBBV)	$n/2$

Determinstic vs Quantum Query Complexity

Theorem (Nisan 1991, Nisan Szegedy 1994, Beals Buhrman Cleve Mosca de Wolf 2001)

For all total Boolean functions f, $\mathrm{D}(f) = O(\mathrm{Q}(f)^6)$

Theorem (This work)

For all total Boolean functions f, $\mathrm{D}(f) = O(\mathrm{Q}(f)^4)$.

Remark

This relationship is tight, due to (Ambainis Balodis Belovs Lee Santha Smotrovs 2017).

Degree

Theorem

Every Boolean function f can be represented exactly by a polynomial, that is,

$$f(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} a_S \prod_{i \in S} x_i$$

Definition (Degree, $\deg(f)$)

The degree of a Boolean function f, is the degree of its polynomial representation.

Theorem

$\deg(f) \leq \mathrm{D}(f)$

Degree of OR

$$\mathrm{OR}(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } x_1 = x_2 = \cdots = x_n = 0 \\ 1 & \text{otherwise} \end{cases}$$
$$= 1 - \prod_{i=1}^{n}(1 - x_i).$$

- $\deg(\mathrm{OR}) = n$

Spectral Sensitivity

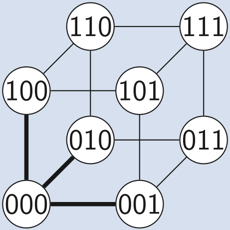
Definition (Spectral Sensitivity, $\lambda(f)$)

The spectral sensitivity of a Boolean function f is the largest eigenvalue of the matrix $A_f \in \mathbb{R}^{\{0,1\}^n \times \{0,1\}^n}$ defined by

$$A_f(x,y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ differ in 1 coordinate and } f(x) \neq f(y) \\ 0 & \text{otherwise.} \end{cases}$$

Example (Spectral Sensitivity of OR)

$$A_{\text{OR}} = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$



$\lambda(\text{OR}) = \sqrt{n}$

Proof Overview

Theorem

For all total Boolean functions f, $\deg(f) = O(\mathrm{Q}(f)^2)$.

Proof.

1. $\deg(f) \leq \lambda(f)^2$ (Huang 2019)
2. $\lambda(f) \leq \mathrm{SA}(f)$ (This work)
3. $\mathrm{SA}(f) = O(Q(f))$ (Barnum Saks Szegedy 2003)

$\square$

Deterministic vs Quantum Query Complexity

Corollary

$\mathrm{D}(f) = O(\mathrm{Q}(f)^4)$

Proof.

1. $\mathrm{D}(f) \leq \mathrm{bs}(f)\deg(f)$ (Midrijanis 2004)
2. $\mathrm{bs}(f) = O(\mathrm{Q}(f)^2)$ (Beals Buhrman Cleve Mosca de Wolf 2001)
3. $\deg(f) = O(\mathrm{Q}(f)^2)$ (This work)

$\square$

Aanderaa–Karp–Rosenberg Conjecture

Corollary

The quantum query complexity of any non-trival monotone graph property (e.g. Connectivity, k-Clique) on n vertices is $\Omega(n)$, which is tight.

Proof.

1. The degree of any non-trival monotone graph property is $\Omega(n^2)$ (Dodis Khanna 1999)
2. $\deg(f) = O(\mathrm{Q}(f)^2)$ (This work)

□

Remark

The best lower bound for the randomized query complexity of any non-trival monotone graph property on n vertices is $\Omega(n^{4/3})$ (conjectured $\Omega(n^2)$).

Approximate Degree

Definition

A polynomial p ϵ-approximates a Boolean function f if $|f(x) - p(x)| \leq \epsilon$ and $p(x) \in [0, 1]$ for all $x \in \{0, 1\}^n$.

Definition (Approximate Degree, $\widetilde{\deg}(f)$)

The approximate degree of a Boolean function f, is the smallest degree of a polynomial that 1/3-approximates f.

Theorem (Beals Buhrman Cleve Mosca de Wolf 2001)

$\widetilde{\deg}(f) = O(Q(f))$

Degree and Approximate Degree of OR

$$\mathrm{OR}(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } x_1 = x_2 = \cdots = x_n = 0 \\ 1 & \text{otherwise} \end{cases}$$
$$= 1 - \prod_{i=1}^{n} (1 - x_i).$$

$$\mathrm{OR}(x_1, x_2) \approx \frac{1}{3} + \frac{1}{3} x_1 + \frac{1}{3} x_2$$

- $\deg(\mathrm{OR}) = n$
- $\widetilde{\deg}(\mathrm{OR}) = \Theta(\sqrt{n})$ (Chebyshev polynomials)

Degree vs Approximate Degree

Theorem (Nisan Szegedy 1994, Beals Buhrman Cleve Mosca de Wolf 2001)

For all total Boolean functions f, $\deg(f) = O(\widetilde{\deg}(f)^6)$

Theorem (This work)

For all total Boolean functions f, $\deg(f) = O(\widetilde{\deg}(f)^2)$.

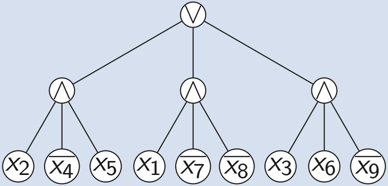
Remark

This relationship is tight, as $\deg(\mathrm{OR}) = n$ and $\widetilde{\deg}(\mathrm{OR}) = \Theta(\sqrt{n})$.

Read-once Formulas

Definition

A read-once formula is a formula of AND, OR, and NOT gates in which each variable appears exactly once.



Corollary

The approximate degree of any read-once formula on n variables is $\Omega(\sqrt{n})$, which is tight.

Proof.

The degree of any read-once formula on n variables is n. □

Proof Overview

Theorem

For all total Boolean functions f, $\deg(f) = O(\widetilde{\deg}(f)^2)$.

Proof.

1. $\deg(f) \leq \lambda(f)^2$ (Huang 2019)
2. $\lambda(f) = O(\widetilde{\deg}(f))$ (This work)

$\square$

Proof Overview

Theorem

$\lambda(f) = O(\widetilde{\deg}(f))$

Proof Idea

1. $2A_f(x, y) = 1 - (2f(x) - 1)(2f(y) - 1)$ when x and y differ in 1 coordinate.
2. $2A_f = A_H - \operatorname{diag}(2f - 1)A_H \operatorname{diag}(2f - 1)$ where A_H is the adjacency matrix of the hypercube.
3. If f is a parity function on d inputs, A_H and $\operatorname{diag}(2f - 1)A_H \operatorname{diag}(2f - 1)$ have the same eigenvectors with eigenvalues that differ by at most $2d$.
4. Generalize to all polynomials, and approximations of polynomials.

Take-home and Open Problems

Take-home

- $\mathrm{D}(f) = O(\mathrm{Q}(f)^4)$ for total functions f.
- $\deg(f) = O(\widetilde{\deg}(f)^2) = O(\mathrm{Q}(f)^2)$.
- Spectral sensitivity is a useful complexity measure.

Open Problems

- What is the relationship between $\mathrm{R}(f)$, randomized query complexity, and $\mathrm{Q}(f)$?
  (There exist f such that $\mathrm{R}(f) = \Omega(\mathrm{Q}(f)^3)$ due to Bansal Sinha 2021, Sherstov Storozhenko Wu 2021)
- What is the relationship between $\mathrm{bs}(f)$, block sensitivty, and $\lambda(f)$?
  (i.e., can $\mathrm{bs}(f) = O(\lambda(f)^4)$ due to Huang be improved?)

Open Problems

Table 1: Best known separations between complexity measures

	D	R_0	R	C	RC	bs	s	λ	Q_E	deg	Q	$\widetilde{\deg}$
D		2, 2 [ABB+17]	2, 3 [ABB+17]	2, 2 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	3, 6 [BHT17]	4, 6 [ABB+17]	2, 3 [ABB+17]	2, 3 [GPW18]	4, 4 [ABB+17]	4, 4 [ABB+17]
R_0	1, 1 $\oplus$		2, 2 [ABB+17]	2, 2 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	3, 6 [BHT17]	4, 6 [ABB+17]	2, 3 [ABB+17]	2, 3 [GJPW18]	3, 4 [ABB+17]	4, 4 [ABB+17]
R	1, 1 $\oplus$	1, 1 $\oplus$		2, 2 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	3, 6 [BHT17]	4, 6 [ABB+17]	$\frac{3}{2}$, 3 [ABB+17]	2, 3 [GJPW18]	3, 4 [BS20] [SSW20]	4, 4 [ABB+17]
C	1, 1 $\oplus$	1, 1 $\oplus$	1, 2 $\oplus$		2, 2 [GSS13]	2, 2 [GSS13]	2.22, 5 [BHT17]	2.44, 6 [BHT17]	1.15, 3 [Amb13]	1.63, 3 [NW95]	2, 4 $\wedge$	2, 4 $\wedge$
RC	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		$\frac{3}{2}$, 2 [GSS13]	2, 4 [Rub95]	2, 4 $\wedge$	1.15, 2 [Amb13]	1.63, 2 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
bs	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		2, 4 [Rub95]	2, 4 $\wedge$	1.15, 2 [Amb13]	1.63, 2 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
s	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		2, 2 $\wedge$	1.15, 2 [Amb13]	1.63, 2 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
λ	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$
Q_E	1, 1 $\oplus$	1.33, 2 $\bar{\wedge}$-tree	1.33, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	3, 6 [BHT17]	4, 6 [ABK16]		2, 3 [ABK16]	2, 4 $\wedge$	4, 4 [ABK16]
deg	1, 1 $\oplus$	1.33, 2 $\bar{\wedge}$-tree	1.33, 2 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2, 2 $\wedge \circ \vee$	2, 2 $\wedge \circ \vee$	2, 2 $\wedge \circ \vee$	2, 2 $\wedge$	1, 1 $\oplus$		2, 2 $\wedge$	2, 2 $\wedge$
Q	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	2, 2 [ABK16]	2, 3 [ABK16]	2, 3 [ABK16]	3, 6 [BHT17]	4, 6 [ABK16]	1, 1 $\oplus$	2, 3 [ABK16]		4, 4 [ABK16]
$\widetilde{\deg}$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	2, 2 [BT17]	2, 2 [BT17]	2, 2 [BT17]	2, 2 [BT17]	2, 2 [BT17]	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	