Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs

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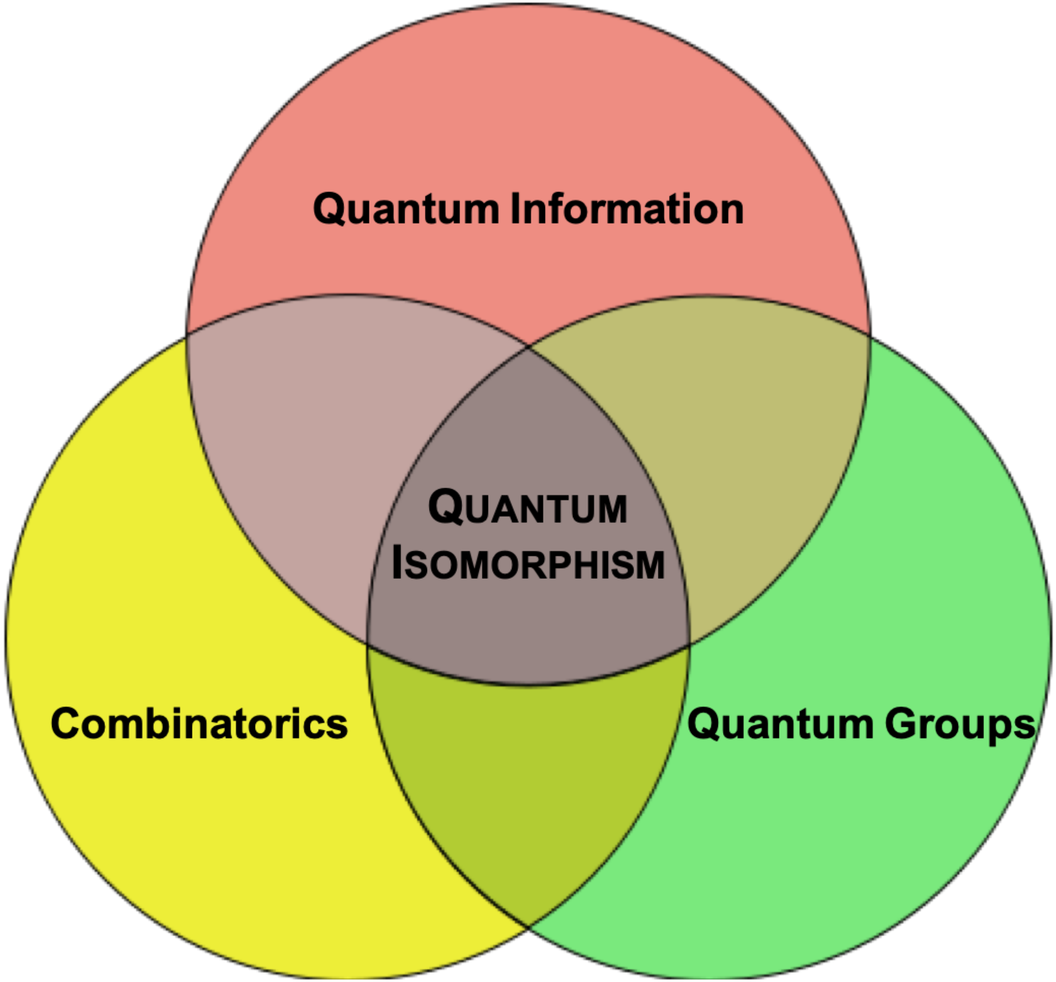
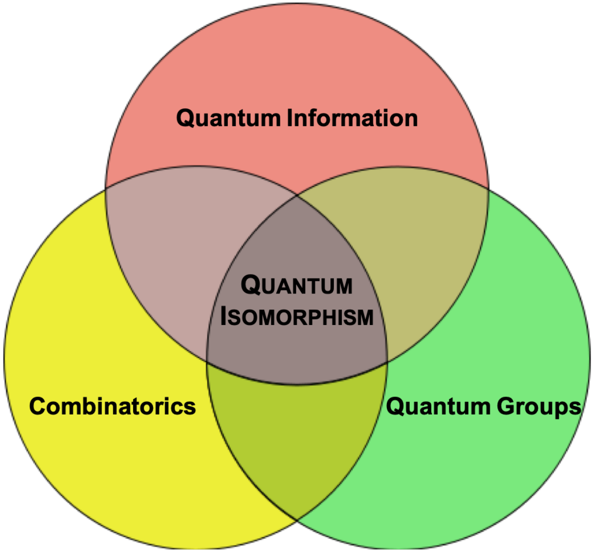
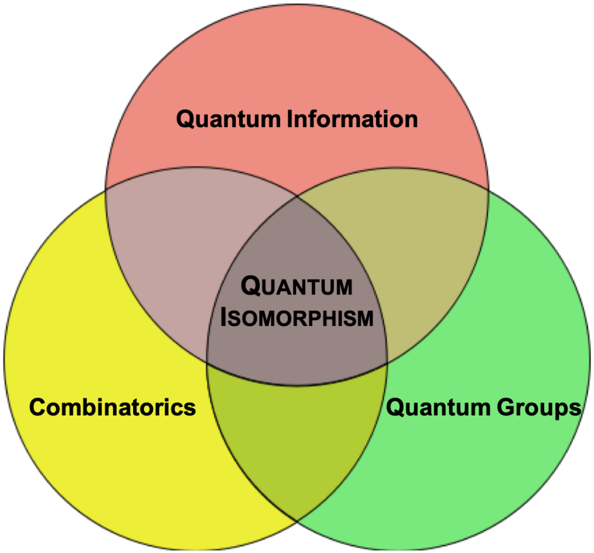
QIP 2021

February 4, 2021


Quantum Information
Quantum Isomorphism
Combinatorics
Quantum Groups



- **Nonlocal games** provide a general framework for studying entanglement



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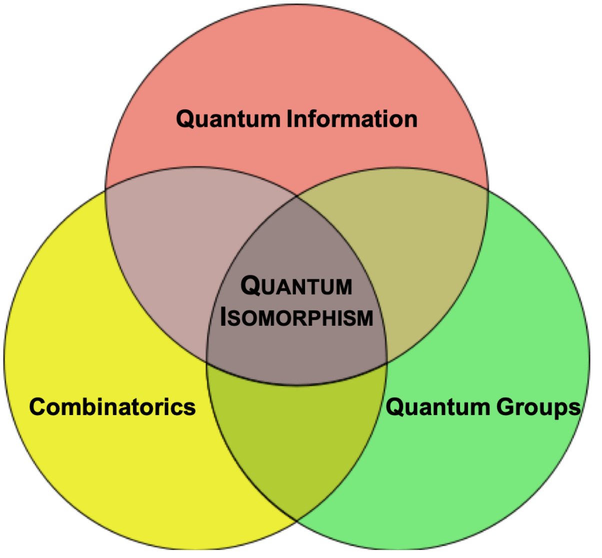

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- **Problem:** Entanglement-assisted strategies for arbitrary nonlocal games are **hard to analyze**
- **Line of attack:** Focus on a **well-behaved** class of games

This talk

PART I

Quantum isomorphism and different ways to think about it:

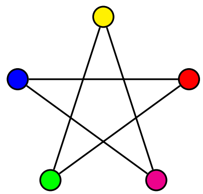
- Nonlocal games
- Matrix formulations
- Homomorphism counts

PART II

Elements of the proof:

- Intertwiners of quantum groups
- Bi-labeled graphs
- Homomorphism matrices

Graph isomorphism



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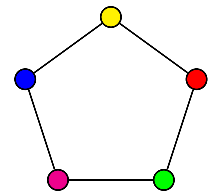

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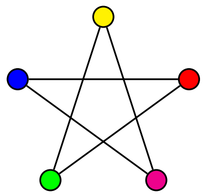

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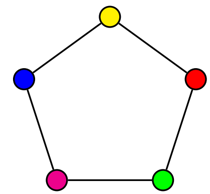

A map $f : V(G) \to V(H)$ is an **isomorphism** from G to H if

- f is a bijection and
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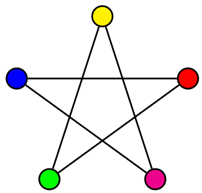

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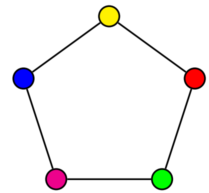

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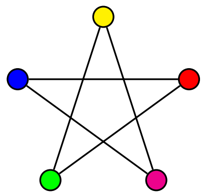

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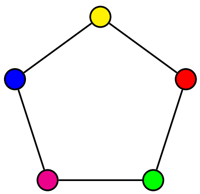

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Matrix formulation: $PA_G P^\dagger = A_H$ for some **permutation** matrix P

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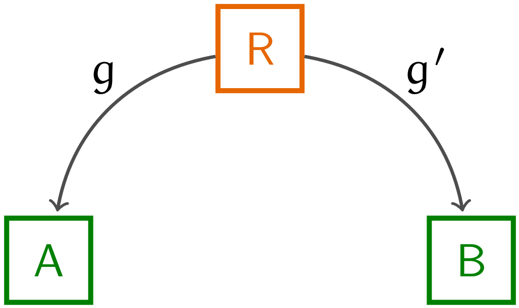

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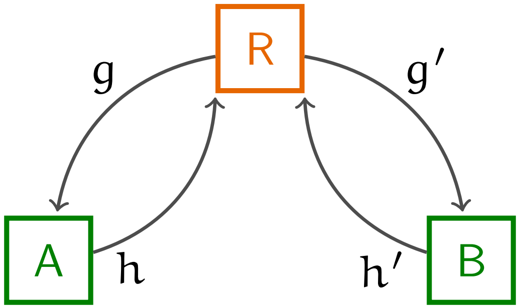

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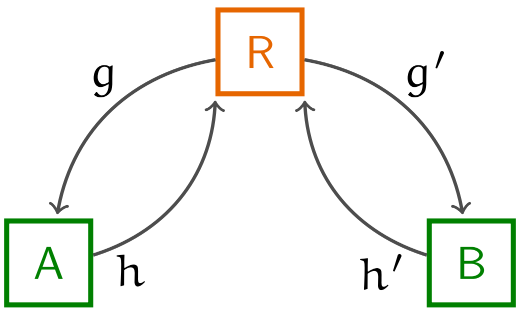

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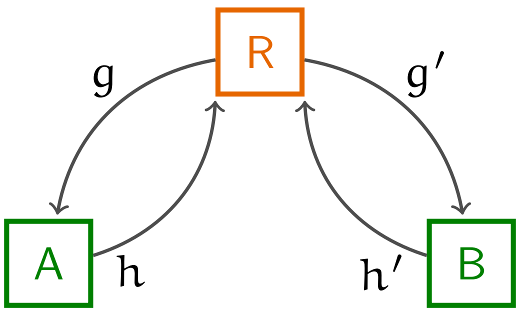

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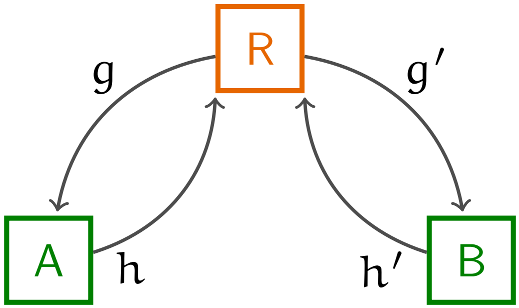

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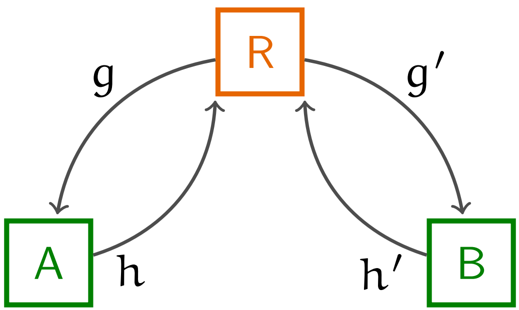

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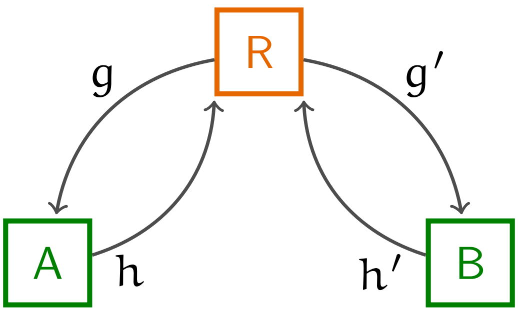

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[1] We work in the **commuting** rather than the tensor-product model.

Quantum commuting strategies

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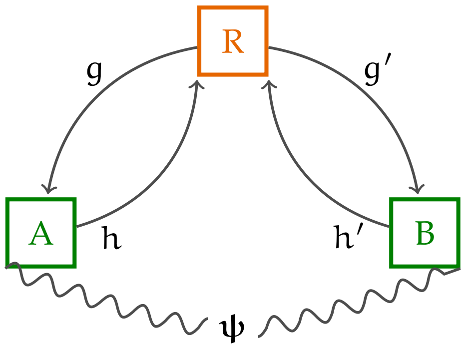

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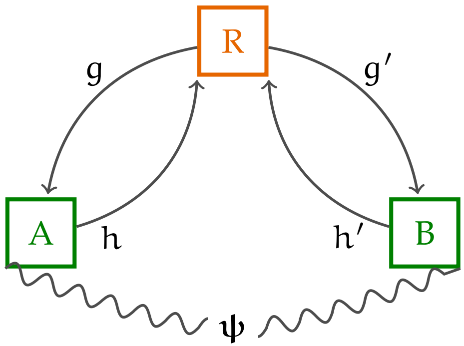

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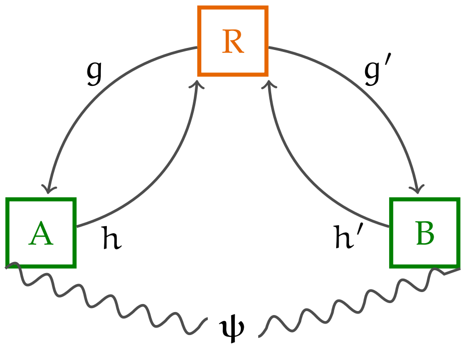

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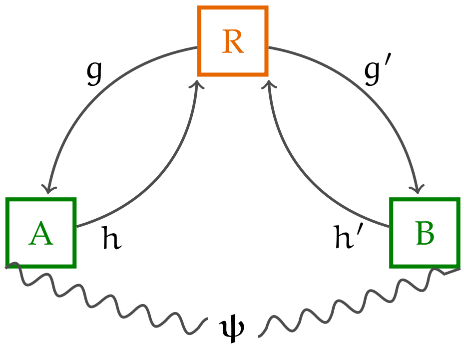

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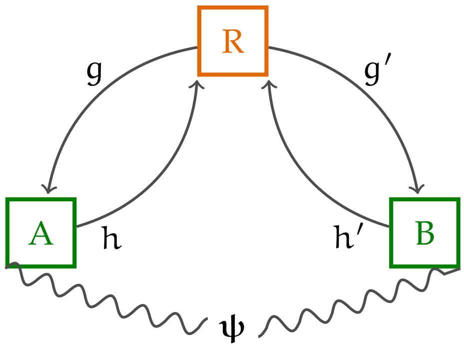

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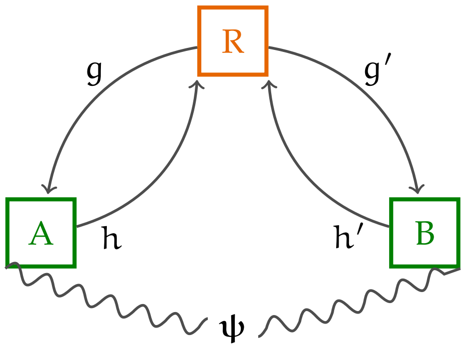

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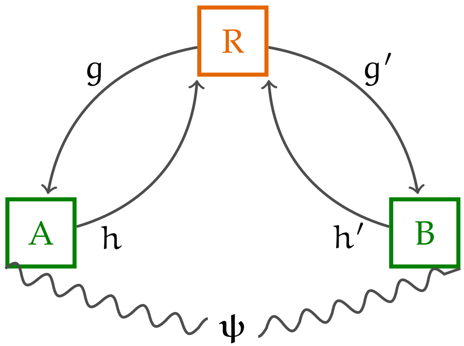

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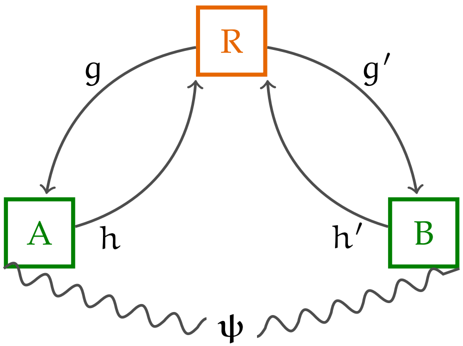

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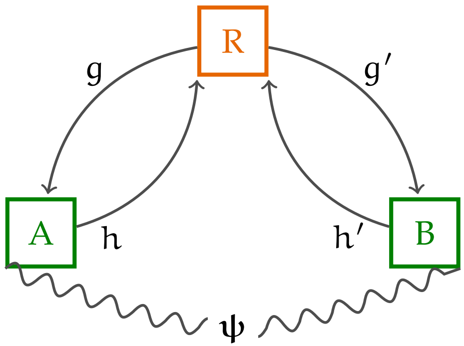

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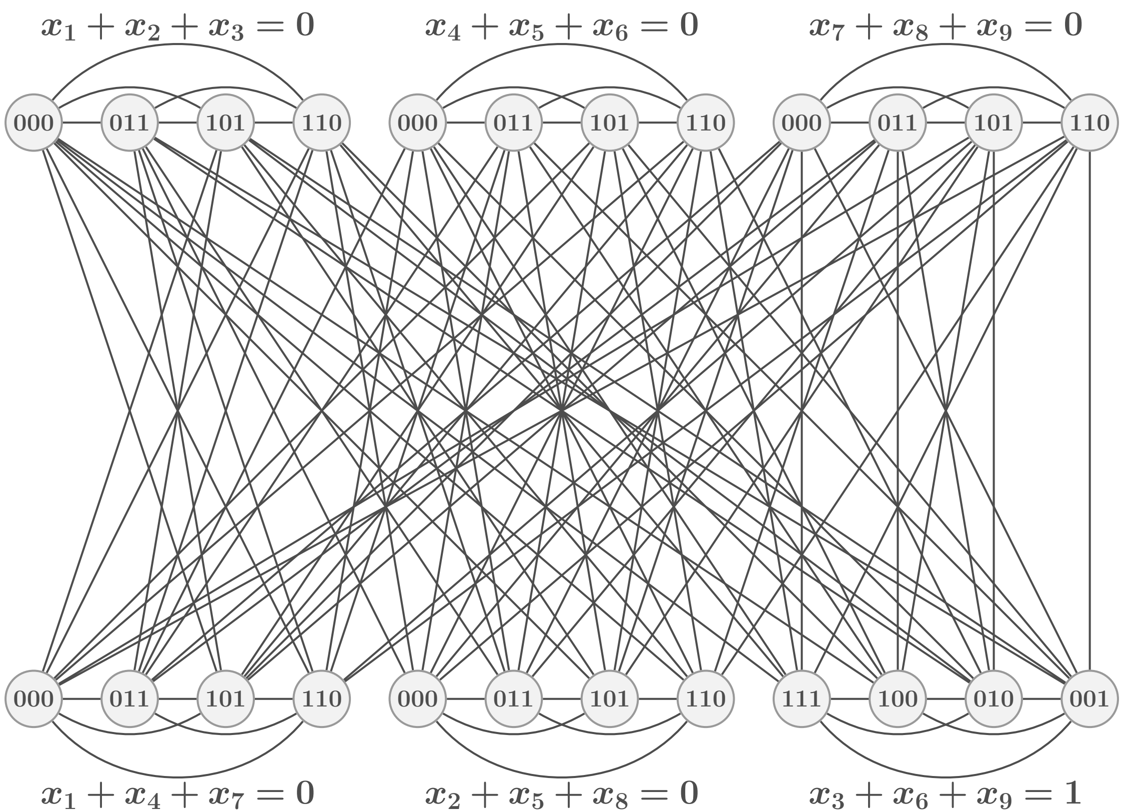
Example: G $\not\cong$ H but G $\cong_{qc}$ H



Construction based on reduction from linear system games.

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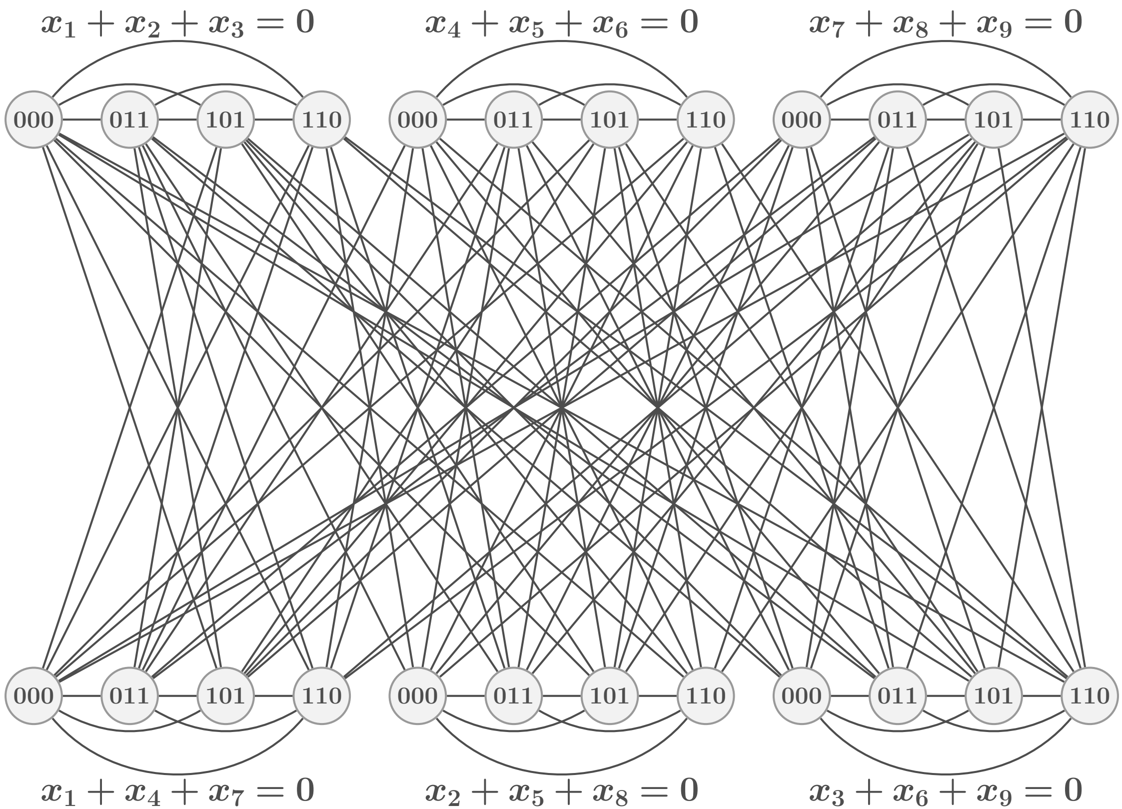

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Thm. (Lupini, M., Roberson)

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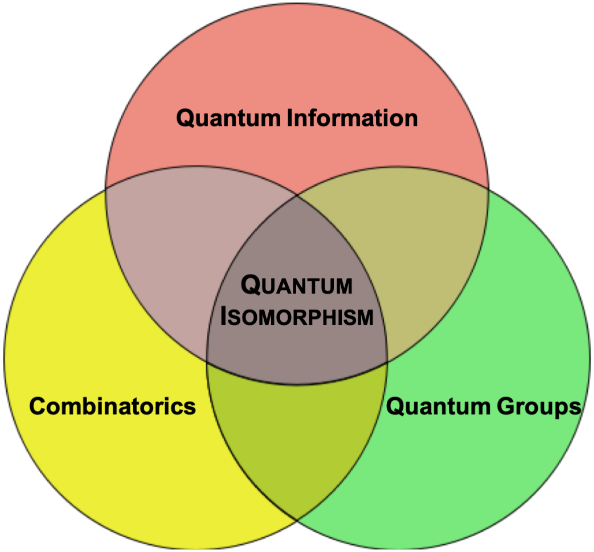
Can we describe quantum isomorphism in combinatorial terms?



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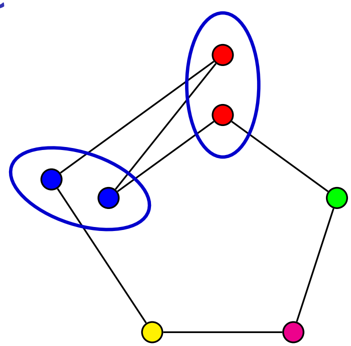

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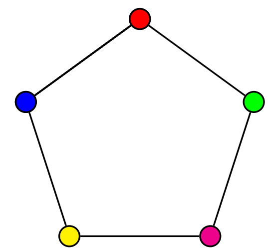

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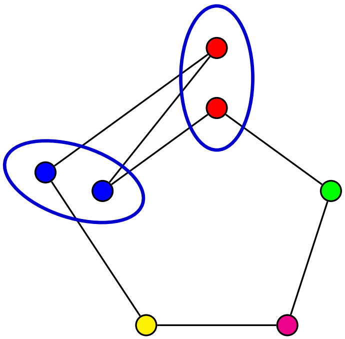

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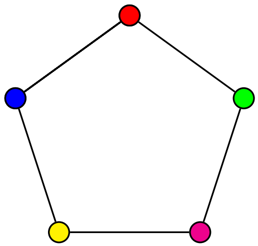

hom(F, G) := # of homomorphisms from F to G.

Counting homomorphisms

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Homomorphism counts determine a graph up to isomorphism, i.e.

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$G \cong_{k} H \;\Leftrightarrow\; \hom(F,G) = \hom(F,H)$ for all F of **treewidth** $\leqslant k$

Complexity

Context: Homomorphism counting

Thm. (Lovász, 1967)
$G \cong H \Leftrightarrow \hom(F, G) = \hom(F, H)$ for **all graphs** F

Thm. (M., Roberson, 2019)
$G \cong_{qc} H \Leftrightarrow \hom(F, G) = \hom(F, H)$ for all **planar** graphs F

Folklore.
G and H cospectral $\Leftrightarrow \hom(F, G) = \hom(F, H)$ for all **cycles** F

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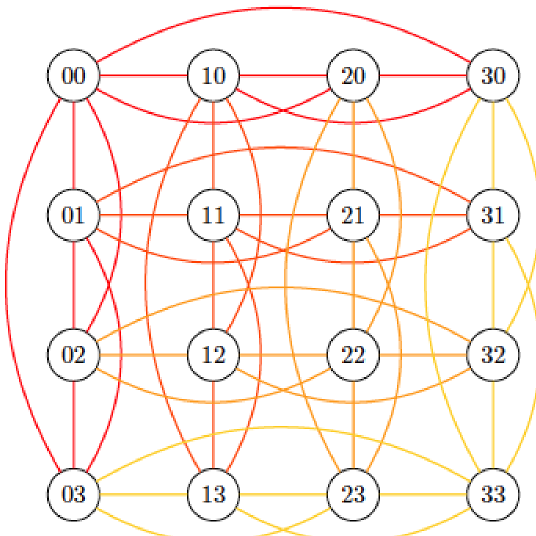
Complexity: Except for the class of planar graphs, equality of homomorphism counts from all of the above graph classes can be tested in at worst quasi-polynomial time.

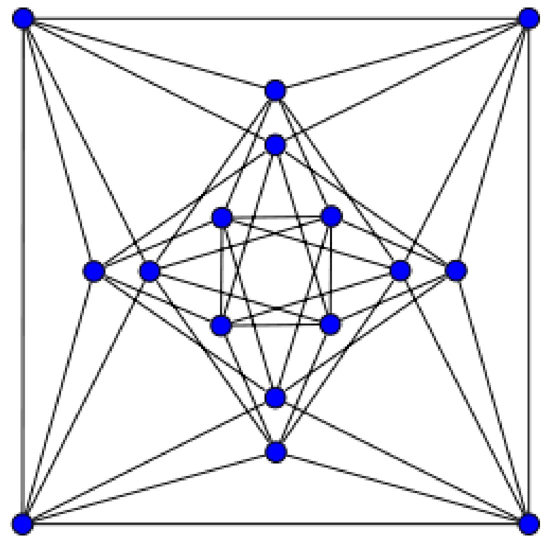
Application: Certificate for $G \not\cong_{qc} H$

Are these two graphs quantum isomorphic?

Rook graph



Shrikhande graph



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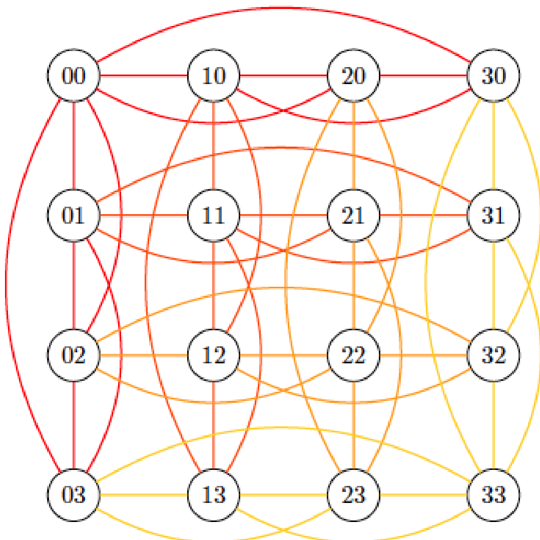

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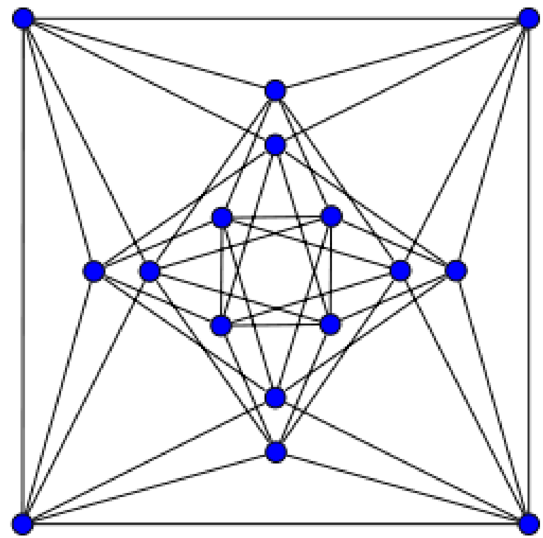

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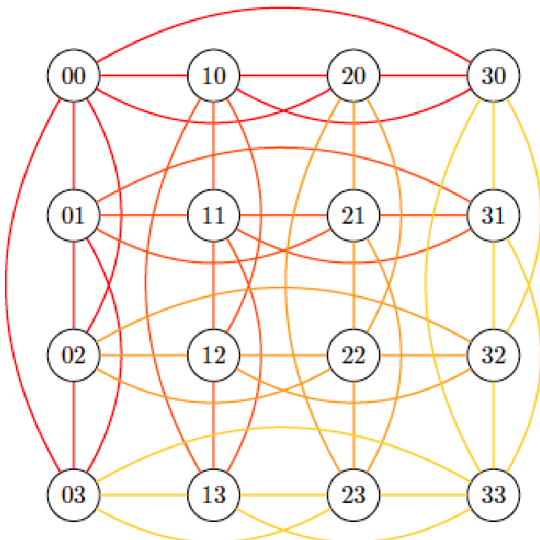

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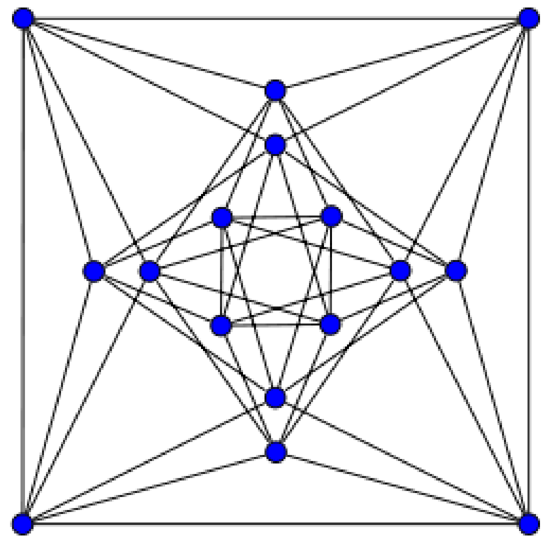

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Now: Only one (the Rook graph) contains K_4.

Part II
Elements of the proof

Thm. $G \cong_{qc} H \;\Leftrightarrow\; \hom(F, G) = \hom(F, H)$ for all **planar** graphs F

The starting point

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Main component of our proof: Provide a *combinatorial description* of the **intertwiners** of $\mathrm{Qut}(G)$.

Intertwiners of $\mathrm{Qut}(G) = \langle U, M, A_G \rangle_{\circ,\otimes,*,\mathrm{lin}}$, where

$$U = \sum_{i \in V(G)} e_i, \quad M(e_i \otimes e_j) = \delta_{ij} e_i \ \forall i, j \in V(G).$$

Bi-labeled graphs

Def. (Lovász, Large Networks and Graph Limits)
An (ℓ, k)-**bi-labeled graph** is a triple $\vec{F} = (F, \vec{a}, \vec{b})$ where

- F is a graph
- $\vec{a} = (a_1, \ldots, a_\ell)$ and $\vec{b} = (b_1, \ldots, b_k)$ are tuples of vertices of F.

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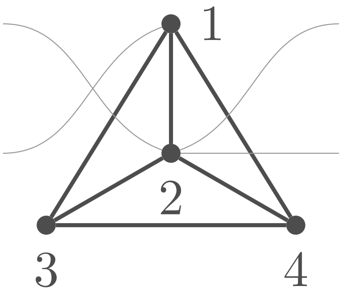

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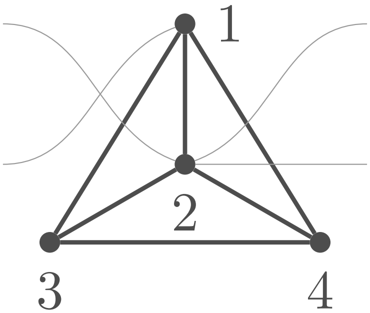

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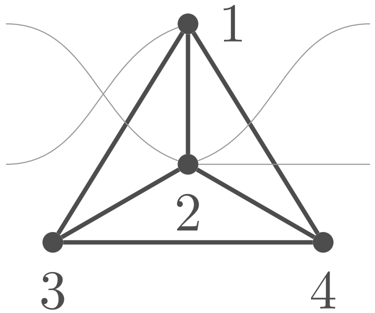

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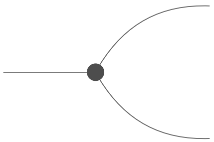

$\vec{U} = \big(K_1, (1), \varnothing\big)$ $\vec{M} = \big(K_1, (1), (1,1)\big)$ $\vec{A} = \big(K_2, (1), (2)\big)$

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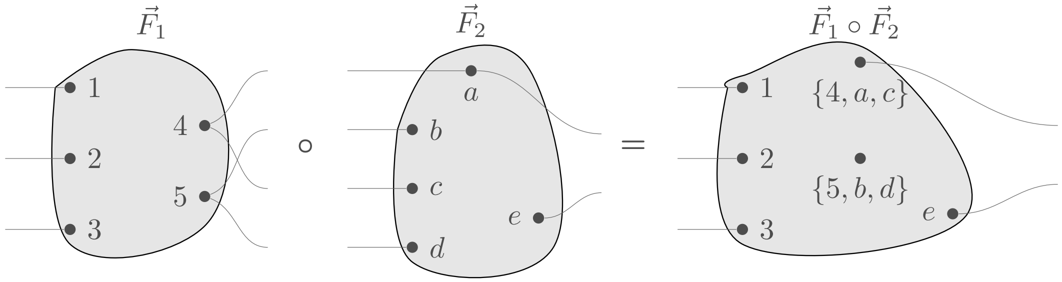
Thm. For a graph G and bi-labeled graphs $\vec{F}_1, \vec{F}_2$,

$$T^{\vec{F}_1} T^{\vec{F}_2} = T^{\vec{F}_1 \circ \vec{F}_2},$$

where $\vec{F}_1 \circ \vec{F}_2$ is defined as



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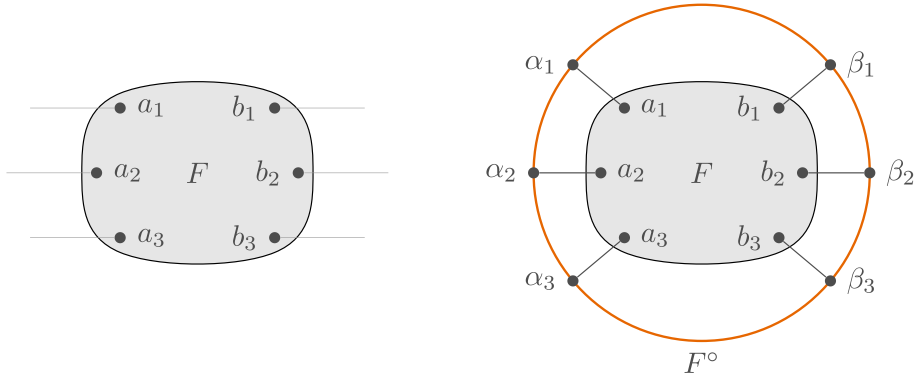

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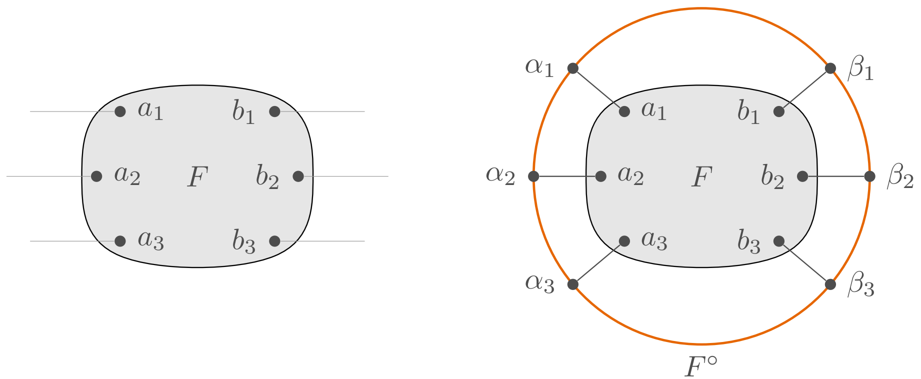

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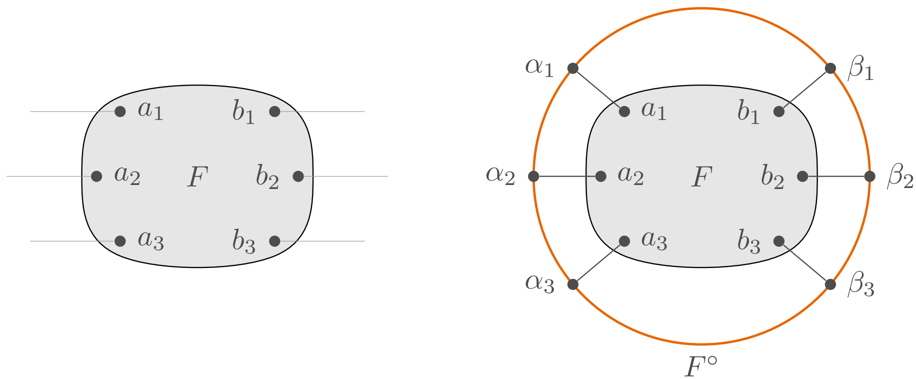

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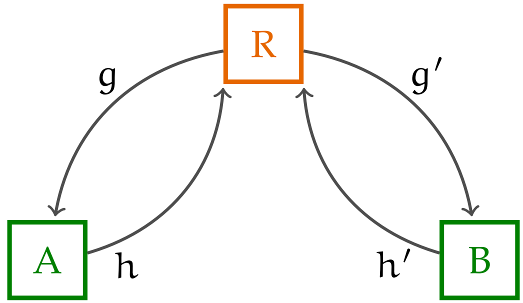
Thm. (informal) Intertwiners of $\mathrm{Qut}(G)$ are given by the span of homomorphism matrices of planar bi-labeled graphs.

Summary

Graph isomorphism can be formulated in terms of a **nonlocal game**.



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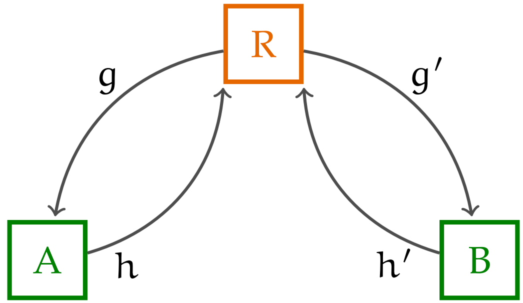

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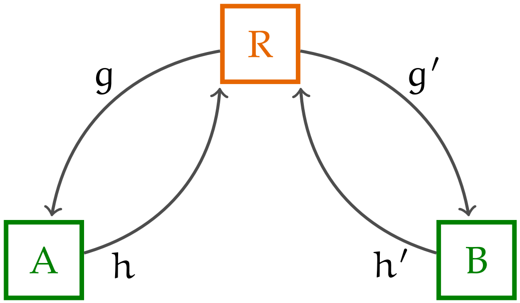

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Thank you!