The quantum Wasserstein distance of order 1

Giacomo De Palma

Milad Marvian

Dario Trevisan

Seth Lloyd

Motivations

- Hamming distance ubiquitous in classical probability, information theory, machine learning

- Yet no quantum version for qudits!!

- Bit flip small change wrt Hamming distance, but can generate orthogonal state

- Orthogonal states maximally far for any unitarily invariant distance

- Desired properties:
  - Recovery of Hamming distance for canonical basis states
  - One-qudit channels induce small changes
  - Global quantities (*e.g.*, entropy) continuous

The classical Wasserstein distance

- Probability distributions on metric space interpreted as distributions of unit amount of mass
- Moving unit mass from *x* to *y* has cost *d(x,y)*
- $W_1(p,q)$: minimum cost to transport *p* onto *q*
- Induced by a norm
- Countless applications in geometric analysis, probability, information theory, machine learning
- We consider bit strings (strings of symbols from finite alphabet) with Hamming distance

Quantum W_1 norm

- Neighboring states: coincide after discarding one qudit
- Require: neighboring states have distance at most one
- Minimum unit ball: convex hull of differences between neighboring states

$$\mathcal{B}_n = \left\{ \sum_{i=1}^{n} p_i \left(\rho^{(i)} - \sigma^{(i)} \right) : \mathrm{Tr}_i \rho^{(i)} = \mathrm{Tr}_i \sigma^{(i)} \right\}$$

- Maximum norm

$$\| \rho - \sigma \|_{W_1} = \min \left(t \geq 0 : \rho - \sigma \in t \, \mathcal{B}_n \right)$$

Properties

- Recovers classical W_1 distance for states diagonal in canonical basis, Hamming distance for canonical basis vectors

- Symmetries: local unitaries, qudit permutations

- Contractive wrt one-qudit quantum channels

- Additive wrt tensor product

$$\|\rho \otimes \rho' - \sigma \otimes \sigma'\|_{W_1} = \|\rho - \sigma\|_{W_1} + \|\rho' - \sigma'\|_{W_1}$$

- Relation with trace distance

$$\frac{1}{2}\|\rho - \sigma\|_1 \leq \|\rho - \sigma\|_{W_1} \leq \frac{n}{2}\|\rho - \sigma\|_1$$

- Local operations: if Φ acts on k qudits,

$$\|\Phi(\rho) - \rho\|_{W_1} \leq 2\,k$$

Contraction coefficient

- Contraction of trace distance

$$\eta(\Phi) = \max_{\rho \neq \sigma \in \mathcal{S}_n} \frac{\|\Phi(\rho) - \Phi(\sigma)\|_1}{\|\rho - \sigma\|_1}$$

- Trivial for $n \to \infty$ for tensor power channels

$$\Phi(\rho) \neq \Phi(\sigma) \implies \frac{1}{2} \lim_{n \to \infty} \left\|\Phi^{\otimes n}\left(\rho^{\otimes n}\right) - \Phi^{\otimes n}\left(\sigma^{\otimes n}\right)\right\|_1 = \lim_{n \to \infty} \eta\left(\Phi^{\otimes n}\right) = 1$$

- Contraction of quantum W_1 distance

$$\|\Phi\|_{W_1 \to W_1} = \max_{\rho \neq \sigma \in \mathcal{S}_n} \frac{\|\Phi(\rho) - \Phi(\sigma)\|_{W_1}}{\|\rho - \sigma\|_{W_1}}$$

Contraction coefficient

- At most twice the size of the largest light-cone of a qudit
- Tensor power channels: Φ one-qudit channel with fixed point ω, $\mathcal{E}$ replaces input with ω
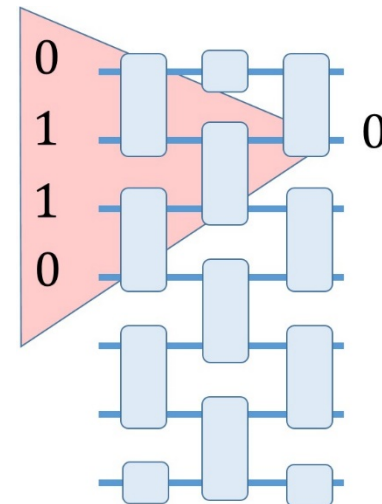
$$\left\|\Phi^{\otimes n}\right\|_{W_1 \to W_1} \leq \left\|\Phi - \mathcal{E}\right\|_\diamond$$

- Amplitude damping channel (decay probability $1\text{-}p$)

$$\left\|\Phi_p^{\otimes n}\right\|_{W_1 \to W_1} \leq 2\sqrt{\frac{p}{1-p}}$$

- Depolarizing channel (input replaced with probability $1\text{-}p$)

$$\left\|\mathcal{E}_p^{\otimes n}\right\|_{W_1 \to W_1} = p$$

Continuity of the von Neumann entropy

- Continuity bounds wrt trace distance / fidelity void for orthogonal states, but one-qudit channel can turn state into orthogonal state with entropy change at most 2 ln d

- Continuity bound wrt quantum W_1 distance

$$|S(\rho) - S(\sigma)| \leq g\left(\|\rho - \sigma\|_{W_1}\right) + \|\rho - \sigma\|_{W_1} \ln\left(d^2\, n\right)$$

$$g(t) = (t+1)\ln(t+1) - t\ln t \leq \ln(t+1) + 1$$

$$\|\rho - \sigma\|_{W_1} = o\left(\frac{n}{\ln n}\right) \implies |S(\rho) - S(\sigma)| = o(n)$$

- ln n not present for classical W_1 distance

The quantum Lipschitz constant

- Quantum Lipschitz constant

$$\|H\|_L = 2 \max_{i \in [n]} \min \left(\|H - H_{\bar{i}}\|_\infty : H_{\bar{i}} \text{ does not act on } i\text{-th qudit} \right)$$

- Recovers classical Lipschitz constant for operators diagonal in canonical basis
- Quantum W_1 distance as SDP

$$\|\rho - \sigma\|_{W_1} = \max_{\|H\|_L \leq 1} \mathrm{Tr}\left[H \left(\rho - \sigma \right) \right]$$

Relation with relative entropy

- Pinsker's inequality

$$\frac{1}{2}\,\|\rho - \sigma\|_1 \leq \sqrt{\frac{1}{2}\,S(\rho\|\sigma)}$$

- Improvement to quantum Marton's transportation inequality

$$\|\rho - \sigma_1 \otimes \ldots \otimes \sigma_n\|_{W_1} \leq \sqrt{\frac{n}{2}\,S(\rho\|\sigma_1 \otimes \ldots \otimes \sigma_n)}$$

Quantum Gaussian concentration inequality

- In high dimension, smooth functions are essentially constant

- Bound on partition function

$$\frac{1}{n} \ln \operatorname{Tr} \mathrm{e}^{tH} \leq \ln d + \frac{t^2}{8} \|H\|_L^2 \qquad \operatorname{Tr} H = 0$$

- Spectrum of H lies in interval with size $O\left(\sqrt{n}\, \|H\|_L\right)$

$$\dim\left(H \geq \left(\frac{\operatorname{Tr} H}{d^n} + \delta\sqrt{n}\, \|H\|_L\right)\mathbb{I}\right) \leq d^n\, \mathrm{e}^{-2\delta^2}$$

Perspectives

- Robustness of quantum machine learning
- Quantum Generative Adversarial Networks (arXiv:2101.03037)
- Quantum rate-distortion theory
- Mixing time of quantum Markov semigroups
- Shallow quantum circuits
- Quantum spin systems on infinite lattices

arXiv:2009.04469