Distinguishing unitaries with finite energy and a Gaussian Solovay-Kitaev theorem

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Quantum continuous variable (CV) systems

- **Quantum CV systems** are candidates for many protocols of **quantum computation** and **communication**

- CV systems can be hard to analyse (infinite dimensional objects, unboundedness, etc…)

- Important class of CV operations are **Gaussian states**, **channels** and **measurements**

- Advantage: characterized by **covariance matrices**, i.e. admit a **finite dimensional reduction**

How can one construct a theory of approximation of quantum Gaussian circuits?

Outline and motivation

- **From QIT/Resource theory:** Can one quantify the amount of entanglement/energy needed to distinguish quantum channels over arbitrary (separable) Hilbert spaces

  -> Restriction over input states required: **finite energy states**.

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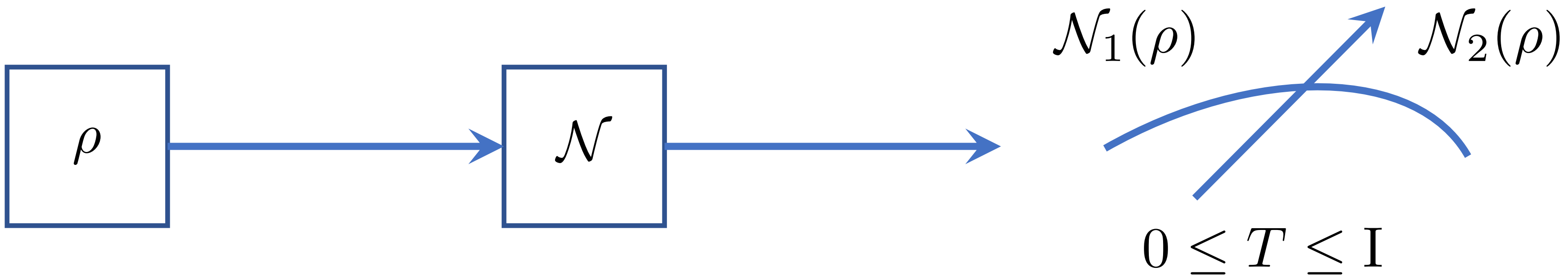
- **From quantum optics:** Given two Gaussian channels, how well can one distinguish them in terms of their defining properties?

  -> Restriction over input states required: **finite energy states**.

- **From CV quantum computing:** How can one efficiently approximate the action of a Gaussian circuit **on low energy states** with precision from a given set of primitive gates?

Discriminating quantum channels



$$\mathcal{N} = \begin{cases} \mathcal{N}_1 & \text{w.p. } \frac{1}{2} \\ \mathcal{N}_2 & \text{w.p. } \frac{1}{2} \end{cases}$$

$$\mathbb{P}^{(1)}_{\mathrm{succ}}(\mathcal{N}_1, \mathcal{N}_2) = \sup_{\rho} \mathbb{P}_{\mathrm{succ}}(\mathcal{N}_1(\rho), \mathcal{N}_2(\rho)) = \frac{1 + \frac{1}{2}\|\mathcal{N}_1 - \mathcal{N}_2\|_{1\to 1}}{2}$$

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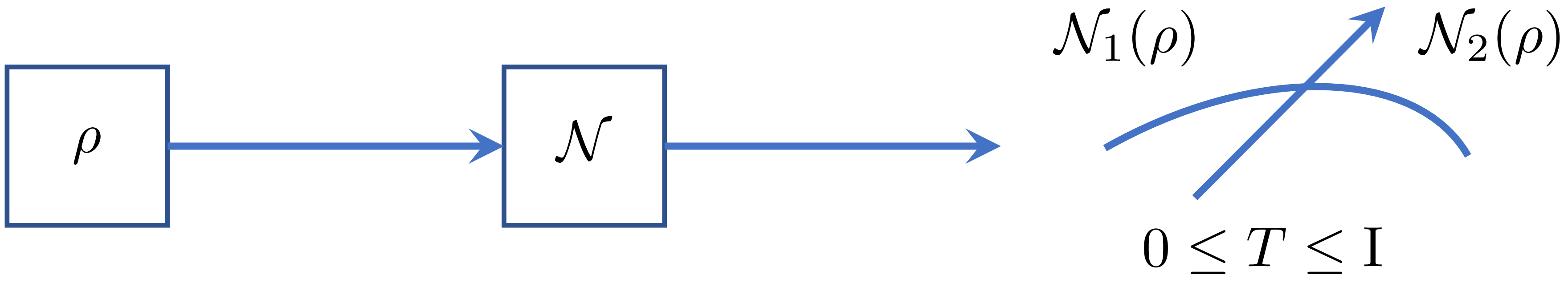

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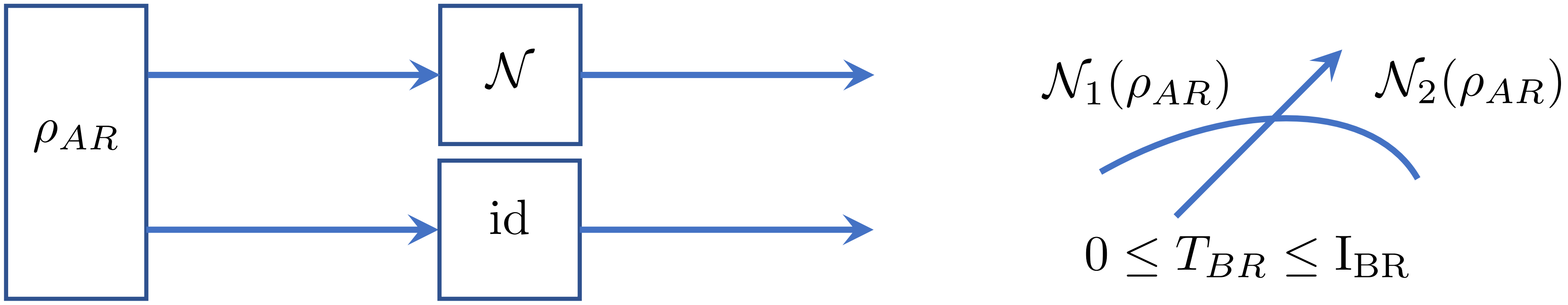

$$\mathbb{P}^{(\mathrm{e.a.})}_{\mathrm{succ}}(\mathcal{N}_1, \mathcal{N}_2) = \sup_{R} \sup_{\rho_{AR}} \mathbb{P}_{\mathrm{succ}}(\mathcal{N}_1(\rho_{AR}), \mathcal{N}_2(\rho_{AR})) = \frac{1 + \frac{1}{2}\|\mathcal{N}_1 - \mathcal{N}_2\|_{\diamond}}{2}$$

The diamond norm and its flaws

$\Lambda : \mathcal{T}_1(\mathcal{H}) \to \mathcal{T}_1(\mathcal{H})$ Hermitian preserving[1] $\|\Lambda\|_\diamond := \sup_R \sup_{|\psi\rangle \in \mathcal{H}_{AR}} \|(\Lambda \otimes \mathrm{id})(|\psi\rangle\langle\psi|)\|_1$

(≤ 2 for channel differences)

Problem: The diamond norm is trivial in simple infinite dimensional situations

Example: Bosonic quantum limited attenuator: $\forall |\alpha\rangle \in L^2(\mathbb{R})$ coherent state, $\mathcal{A}_\eta(|\alpha\rangle\langle\alpha|) := |\eta\alpha\rangle\langle\eta\alpha|$

$\eta \leq 1$

Lemma [Winter 2017] $\|\mathcal{A}_\eta - \mathcal{A}_{\eta'}\|_\diamond = 2\,, \forall \eta \neq \eta'$ $\Rightarrow \mathbb{P}^{(\text{e.a.})}_{\text{succ}}(\mathcal{A}_\eta, \mathcal{A}_{\eta'}) = 1$

Proof: $\| |\eta\alpha\rangle\langle\eta\alpha| - |\eta'\alpha\rangle\langle\eta'\alpha| \|_1 = 2\sqrt{1 - e^{-|(\eta-\eta')\alpha|^2}} \to 2\,, |\alpha| \to \infty$

Physical interpretation: $N := a^\dagger a$ $\quad \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$

$$\mathbb{P}^{(\text{e.a.})}_{\text{succ}}(\mathcal{N}_1, \mathcal{N}_2) = \frac{1 + \frac{1}{2}\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond}{2}$$

$\Rightarrow$ Perfect discrimination requires **infinite average energy!**

1 Aharonov et al 1998

Energy constrained diamond (ECD) norm

Fix a Hamiltonian H_A which we assume grounded ($\min \mathrm{Sp}(H_A) = 0$) and average energy $E \geq 0$

Definition (ECD norm) [Shirokov 2017, Winter 2017, Pirandola et. Al. 2017]

$$\|\Lambda\|_{\diamond}^{H,E} := \sup_{R} \sup_{\substack{\rho_{AR} \\ \mathrm{Tr}[\rho_A H_A] \leq E}} \|(\Lambda \otimes \mathrm{id})(\rho_{AR})\|_1$$

$$\leq \sup_{R} \sup_{|\psi\rangle \in \mathcal{H}_{AR}} \|(\Lambda \otimes \mathrm{id})(|\psi\rangle\langle\psi|)\|_1 \equiv \|\Lambda\|_{\diamond}$$

(i) Energy constrained discrimination of unitaries

The ECD distance of unitaries is achieved on product states

[Aharonov, Kitaev, Nisan 1998] No entanglement needed to achieve EC distance between unitaries:

$$2\sqrt{1-\inf_{\rho}|\operatorname{Tr}[\rho U^\dagger V]|^2} = \|\mathcal{U}-\mathcal{V}\|_\diamond = \|\mathcal{U}-\mathcal{V}\|_{1\to 1} = 2\sqrt{1-\inf_{|\psi\rangle}|\langle\psi|U^\dagger V|\psi\rangle|^2}$$

$$\mathcal{U} := U(.)U^\dagger$$
$$\mathcal{V} := V(.)V^\dagger$$

The result extends to the ECD norm:

Theorem: Let U, V be two unitaries, $E \geq 0$, $H \geq 0$

$$\Rightarrow \quad \|\mathcal{U}-\mathcal{V}\|_\diamond^{H,E} = 2\sqrt{1-\inf_{\langle\psi|H|\psi\rangle\leq E}|\langle\psi|U^\dagger V|\psi\rangle|^2}$$

$$\forall E \geq 0,\ \mathbb{P}_{\mathrm{succ}}^{(1),E}(\mathcal{U},\mathcal{V}) = \mathbb{P}_{\mathrm{succ}}^{(\mathrm{e.a.}),E}(\mathcal{U},\mathcal{V})$$

Perfect discrimination with EC and multiple queries

$$2\sqrt{1-\inf_{\rho}|\operatorname{Tr}[\rho U^\dagger V]|^2} = \|\mathcal{U}-\mathcal{V}\|_\diamond = \|\mathcal{U}-\mathcal{V}\|_{1\to 1} = 2\sqrt{1-\inf_{|\psi\rangle}|\langle\psi|U^\dagger V|\psi\rangle|^2}$$

[Acín 2001] Given two unitaries $U \neq V$, there exists $n \in \mathbb{N}$ such that $\|\mathcal{U}^{\otimes n}-\mathcal{V}^{\otimes n}\|_\diamond = 2$

Again, the result extends to the ECD norm: denoting $H^{(n)} := \sum_{j=1}^{n} H_j$ on $\mathcal{H}^{\otimes n}$,

Theorem: Let $U \neq V$ be two unitaries on $H \geq 0$,

$$\exists E < \infty,\ n \in \mathbb{N} \text{ s.t. } \|\mathcal{U}^{\otimes n}-\mathcal{V}^{\otimes n}\|_\diamond^{H^{(n)},E} = 2$$

(ii) Discriminating Gaussian unitaries

Phase space formalism, quick recap

$\mathcal{H} = L^2(\mathbb{R}^m),\ m \in \mathbb{N}$: number of modes

$[a_j, a_k^\dagger] = \delta_{jk}\,\mathrm{I}, [a_j, a_k] = 0$: creation and annihilation operators,

$a_j = \dfrac{x_j + ip_j}{\sqrt{2}}$, x_j, p_j position/momentum operators, $\mathbf{a} = (a_1, ..., a_m)$

$\mathbf{R} = (x_1, p_1, ..., x_m, p_m)$, $[R_j, R_k] = i\,\Omega_{jk}$, $\Omega := \bigoplus_{s=1}^{m} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: symplectic form

$N := \sum_{j=1}^{m} a_j^\dagger a_j$: number operator

Gaussian unitary channels

Phase space representation: $\mathbb{R}^{2m} \ni \mathbf{z} \mapsto \mathcal{D}(\mathbf{z}) \equiv e^{i\mathbf{z}^T.\Omega\mathbf{R}}$ **displacement operators**

Gaussian unitary channel: characterized by their induced action on the phase space:

$$\mathrm{Sp}_{2m}(\mathbb{R}) := \{S \in \mathbb{M}_{2m}(\mathbb{R}) | \ \ S\,\Omega\,S^T = \Omega\}$$

$$\forall S \in \mathrm{Sp}_{2m}(\mathbb{R}),\ \mathbf{z} \mapsto S\mathbf{z} \quad \Rightarrow \quad U_S^\dagger \mathcal{D}(\mathbf{z}) U_S = \mathcal{D}(S\mathbf{z})$$

symplectic unitaries

Problem: How well can we distinguish S from S' ?

Toolbox: Quantum speed limits

Quantum speed limits: How long does it take for $|\psi_t\rangle = e^{itH}|\psi_0\rangle$ to satisfy $\langle\psi_t|\psi_0\rangle = 0$?

$$\begin{cases} U_t := e^{-itH} \\ V_t := e^{-itH'} \end{cases}$$

Conditions (relative boundedness):

$$\langle\psi||H||\psi\rangle \le \alpha\,\langle\psi|H_0|\psi\rangle + \beta\||\psi\rangle\|^2\,, \qquad \forall|\psi\rangle \in \mathrm{dom}(\sqrt{H_0})$$
$$\|(H - H')|\psi\rangle\| \le \gamma\,\|H|\psi\rangle\| + \delta\||\psi\rangle\|\,, \qquad \forall|\psi\rangle \in \mathrm{dom}(H_0)$$

Theorem: $$\|\mathcal{U}_t - \mathcal{V}_t\|_\diamond^{H_0,E} \le 2\sqrt{2}\,\sqrt{\alpha E + \beta}\,\sqrt{\gamma\, t} + \delta\, t$$

Recall:

$$\|\mathcal{U} - \mathcal{V}\|_\diamond^{H,E} = 2\sqrt{1 - \inf_{\langle\psi|H|\psi\rangle \le E} |\langle\psi|U^\dagger V|\psi\rangle|^2}$$

Back to symplectic unitaries

Quantum speed limits can certainly be used to distinguish symplectic unitaries $U_S, U_{S'}$ for $S \sim S'$

$$S \sim I \quad \Rightarrow U_S = \mathrm{e}^{iP_2(\mathbf{R})}$$

Problem: $\mathrm{Sp}_{2m}(\mathbb{R})$ non-compact $\Rightarrow$ $\exp : \mathrm{sp}_{2m}(\mathbb{R}) \to \mathrm{Sp}_{2m}(\mathbb{R})$ non-surjective

Polar decomposition:

$$S = PO\,, P \in \mathrm{Pos}_{2m}(\mathbb{R}) \cap \exp(\mathrm{sp}_{2m}(\mathbb{R})),\; O \in O_{2m}(\mathbb{R}) \cap \exp(\mathrm{sp}_{2m}(\mathbb{R}))$$

Theorem: For any two symplectic unitaries $U_S, U_{S'}$, $\|S\|_\infty, \|S'\|_\infty \leq r$

$$\|\mathcal{U}_S - \mathcal{U}_{S'}\|_\diamond^{N,E} \leq c\sqrt{m\,E}\,r\,\sqrt{\|(S')^{-1}S - I\|_2}$$

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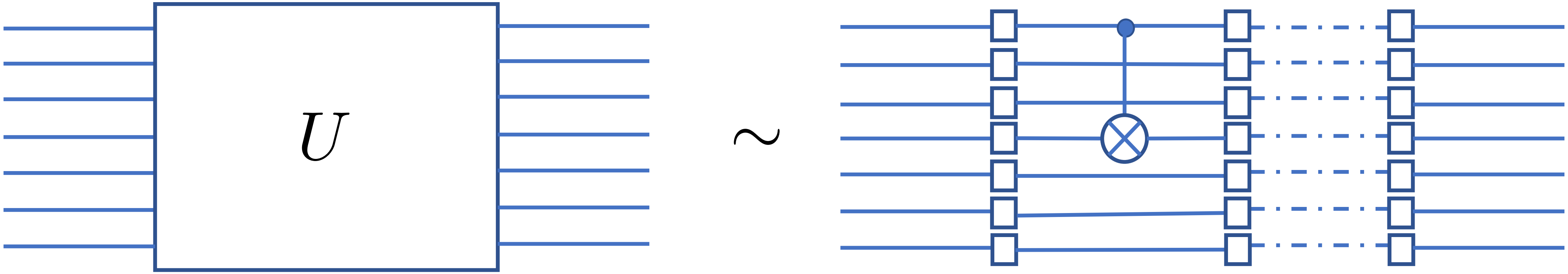
Maximum squeezing

(iii) A Gaussian Solovay-Kitaev theorem

Quantum computing with qubits

Universal gate set: ex: $\mathcal{G}$ = {H, π/8, CNOT} [Boykin et. Al 1999]



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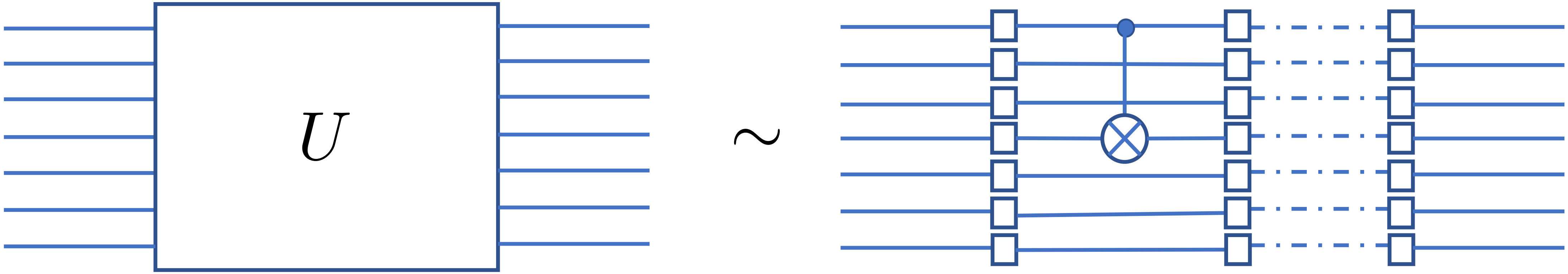


Efficiency: Assume $\mathcal{G}$ is closed under taking inverses.

Theorem [Solovay 1995, Kitaev 1997]

For any $U \in \mathcal{SU}(2)$, and all $\delta > 0$, there exists a finite concatenation U' of polylog(δ^{-1}) elements from a universal gate set $\mathcal{G}$, which can be found in time polylog(δ^{-1}) and such that

$$\|U - U'\|_\infty < \delta$$

Quantum computing with continuous variables

Universality:

[Lloyd & Braunstein 1999] : Universality in the sense of approximations of unitaries $U = e^{i\,\mathrm{Poly}(\mathbf{R})}$ of finite degree d.

$\mathcal{G}_2$ = {translations, phase shifts, squeezers} generates all quadratic Hamiltonians
+ { Q^3 }: any higher degree achievable.

[Barlett, Sanders, Braunstein & Nemoto 2002] d=2? “Gaussian circuits are classically simulatable”

Efficiency?

The Gaussian Solovay-Kitaev theorem

Here we are interested in efficiently approximating symplectic unitaries U_S where

$$S \in \mathrm{Sp}_{2m}^r := \{S \in \mathrm{Sp}_{2m}(\mathbb{R}),\ \|S\|_\infty \leq r\}$$

Physical interpretation: $\|S\|_\infty$ measures the amount of squeezing induced by S

Fix $\mathcal{G}$ to be any finite generating subset of Sp_{2m}^r containing inverses of its elements.

In practice, any gate set generating the group of passive Gaussian unitary and an arbitrary non-passive Gaussian unitary will do.

Theorem [Becker, Datta, Lami, CR 2020]

For any $S \in \mathrm{Sp}_{2m}^r$ and all $\delta > 0$, there exists a finite concatenation S' of polylog(δ^{-1}) elements from $\mathcal{G}$, which can be found in time polylog(δ^{-1}) and such that

$$\|\mathcal{U}_S - \mathcal{U}_{S'}\|_\diamond^{N,E} \leq c\sqrt{m}\, r^{\frac{3}{2}} \sqrt{E}\sqrt{\delta}$$

Gaussian SV theorem: proof idea

Theorem [Becker, Datta, Lami, CR 2020]

For any $S \in \mathrm{Sp}_{2m}^{r}$ and all $\delta > 0$, there exists a finite concatenation S' of $\mathrm{polylog}(\delta^{-1})$ elements from $\mathcal{G}$, which can be found in time $\mathrm{polylog}(\delta^{-1})$ and such that

$$\|\mathcal{U}_S - \mathcal{U}_{S'}\|_{\diamond}^{N,E} \leq c\sqrt{m}\, r^{\frac{3}{2}} \sqrt{E}\sqrt{\delta}$$

Proof: 2 ingredients:

1) Extension of the original SV theorem to[1] $\{X \in \mathrm{Sp}_{2m}(\mathbb{R}),\ \|X\|_\infty \leq r\}$

2) Use of our bounds on energy constrained diamond norm in order to reduce to 1)

$$\|\mathcal{U}_S - \mathcal{U}_{S'}\|_{\diamond}^{N,E} \leq c\sqrt{m\,E}\, r\,\sqrt{\|(S')^{-1}S - I\|_2}$$

1. [Aharonov, Arad, Eban, Landau 2008] Extension of SV to $\{X \in \mathrm{SL}_d(\mathbb{R}),\ \|X - \mathrm{I}\|_\infty \leq r\}$

Conclusions

Summary of results:

(i) **Entanglement is not necessary** when EC discriminating between unitaries

(ii) **Perfect discrimination** between unitaries possible at finite energy and with finite number of copies

(iii) **New quantum speed limits** measuring the drift between two unitaries of unbounded Hamiltonians

(iv) **A Solovay-Kitaev theorem for symplectic unitaries**

Future direction:

Necessary and sufficient conditions for perfect EC discrimination of non-unitary channels?
Bounds for Gaussian non-unitary (noisy) channels?
Etc…

Thank you for your attention!

$$\mathcal{H},\ \dim(\mathcal{H}) \geq 3 \quad \Rightarrow \quad \|\mathcal{U}-\mathcal{V}\|_\diamond^{H,E} = 2\sqrt{1 - \inf_{\langle\psi|H|\psi\rangle\leq E} |\langle\psi|U^\dagger V|\psi\rangle|^2}$$

Proof: By Schmidt decomposition, $\|\mathcal{U}-\mathcal{V}\|_\diamond^{H,E} = 2\sqrt{1 - \inf_{\mathrm{Tr}[\rho H]\leq E} |\,\mathrm{Tr}[\rho U^\dagger V]|^2}$ $(*)$

Remains to prove that infimum is achieved on pure states.

Theorem (Au-Yeung, Poon 1979) For $d \geq 3$ and Z_1, Z_2, Z_3 s.a.
$F_3 := \{(\langle\psi|Z_1|\psi\rangle,\ \langle\psi|Z_2|\psi\rangle, \langle\psi|Z_3|\psi\rangle);\ |\psi\rangle \in \mathbb{C}^d\}$ is convex.

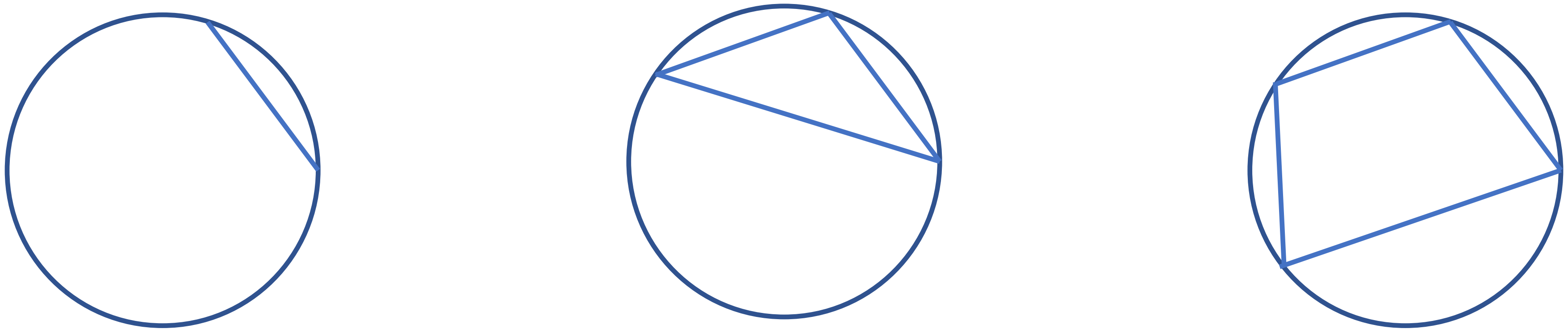
Use it for $(Z_1, Z_2, Z_3) = (\mathrm{Re}(U^\dagger V), \mathrm{Im}(U^\dagger V), H)$, and conclude by $(*)$ and

$$\inf_{\substack{(x,y,z)\in\mathrm{Conv}(R_3)\\ z\leq E}} x^2+y^2 = \inf_{\substack{(x,y,z)\in(R_3)\\ z\leq E}} x^2+y^2 \ \text{(Au-Yeung, Poon)}$$

$$\exists E < \infty,\, n \in \mathbb{N} \text{ s.t. } \|\mathcal{U}^{\otimes n} - \mathcal{V}^{\otimes n}\|_{\diamond}^{H^{(n)},E} = 2$$

Proof: By the previous theorem, we need to prove that there exists an energy such that

$$0 \in \{\langle\psi|(U^\dagger V)^{\otimes n}|\psi\rangle,\ \langle\psi|H^{(n)}|\psi\rangle \leq E\}$$



$U^\dagger V$ $(U^\dagger V)^{\otimes 2}$ $(U^\dagger V)^{\otimes 3}$

Take tensor products of three eigenvectors of $U^\dagger V$ whose eigenvalues triangularize 0, approximate them by vectors in the domain of H. Conclude by Toeplitz-Hausdorff theorem.