The semiring of dichotomies and asymptotic relative submajorization

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MOMENTUM OF INNOVATION

Dichotomies

- quantum system with Hilbert space $\mathcal{H}$
- pair of states (ρ, σ) on $\mathcal{H}$
- interpretations:
    1. "black-box", possible preparations
    2. ρ state, σ reference

Pair transformations

- Let $\rho_1, \sigma_1 \in \mathcal{S}(\mathcal{H})$ and T a channel from $\mathcal{H}$ to $\mathcal{K}$
- T transforms (ρ_1, σ_1) to $(T(\rho_1), T(\sigma_1)) = (\rho_2, \sigma_2)$
- T linear, convenient to allow trace $\neq 1$

$$
\begin{aligned}
T(\rho_1) = \rho_2 &\iff T(c\rho_1) = c\rho_2 \\
T(\sigma_1) = \sigma_2 &\iff T(c\sigma_1) = c\sigma_2
\end{aligned}
$$

Example: majorization

Let $p_1, p_2 \in \mathbb{R}^d_{\geq 0}$. p_1 **majorizes** p_2 if $\sum_{i=1}^{k} (p_1^{\downarrow})_i \geq \sum_{i=1}^{k} (p_2^{\downarrow})_i$ for $k = 1, 2, \ldots, d$, equality if $k = d$.

- $\rho_1 = \mathrm{diag}(p_{1,1}, \ldots, p_{1,d}) \in \mathbb{C}^{d \times d}$
- $\rho_2 = \mathrm{diag}(p_{2,1}, \ldots, p_{2,d}) \in \mathbb{C}^{d \times d}$
- $\sigma_1 = \sigma_2 = I \in \mathbb{C}^{d \times d}$ identity matrix

p_1 majorizes p_2 iff $T(\rho_1) = \rho_2$ and $T(I) = I$ for some channel T

Applications

- mixedness
- pure bipartite entanglement
- coherence

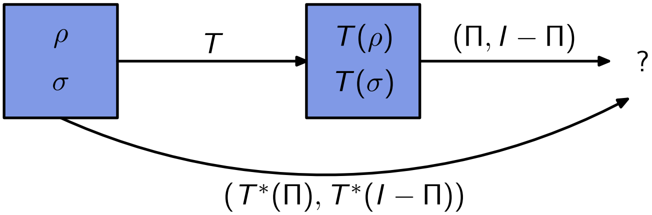
Example: Gibbs preserving maps

- thermal operations model interaction with heat bath at temperature β^{-1}
- T is Gibbs preserving if $T(2^{-\beta H}) = 2^{-\beta H}$
- $(\rho_1, 2^{-\beta H}) \to (\rho_2, 2^{-\beta H})$
- thermal operations are Gibbs preserving
- classically $\rho_1 \mapsto \rho_2$ possible with Gibbs preserving T iff possible with thermal operation

Example: hypothesis testing

Box transformations

- "black-box" either containing ρ or σ
- apply completely positive trace preserving map T to unknown state: new pair $(T(\rho), T(\sigma))$
- (ρ, σ) more distinguishable than $(T(\rho), T(\sigma))$



Relative submajorization

- allow unnormalized ρ_2, σ_2, traces represent probabilities
- T completely positive trace nonincreasing such that

$$T(\rho_1) \geq \rho_2$$
$$T(\sigma_1) \leq \sigma_2$$

relative submajorization [Ren16], notation: $(\rho_1, \sigma_1) \succcurlyeq (\rho_2, \sigma_2)$

- interpretation: probabilistic transformations

Example: hypothesis testing

- two-outcome measurement $(\Pi, I - \Pi)$, probabilities $\operatorname{Tr} \rho\Pi$, $\operatorname{Tr} \rho(I - \Pi)$
- redundant: probabilities sum to 1
- test: $T(\rho) = \operatorname{Tr}(\rho\Pi)$ for some $0 \leq \Pi \leq I$
- $(\rho, \sigma) \succcurlyeq (a, b)$ iff there is a test Π with

$$\alpha(\Pi) = 1 - \operatorname{Tr}(\rho\Pi) \leq 1 - a \qquad \text{type I error}$$
$$\beta(\Pi) = \operatorname{Tr}(\sigma\Pi) \leq b \qquad \text{type II error}$$

Tensor product

- (ρ_1, σ_1) and (ρ_2, σ_2) dichotomies on $\mathcal{H}_1$ and $\mathcal{H}_2$

- product
$$(\rho_1, \sigma_1) \cdot (\rho_2, \sigma_2) = (\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2)$$
dichotomy on $\mathcal{H}_1 \otimes \mathcal{H}_2$

Example: independent subsystems

- H_1, H_2 Hamiltonians and ρ_1, ρ_2 states
- $(\rho_1, 2^{-\beta H_1})$ and $(\rho_2, 2^{-\beta H_2})$
- total Hamiltonian (no interaction): $H = H_1 \otimes I + I \otimes H_2$
- Gibbs state factorizes
$$2^{-\beta H} = 2^{-\beta H_1} \otimes 2^{-\beta H_2}$$
- independent preparation, no interaction: $(\rho_1 \otimes \rho_2, 2^{-\beta H_1} \otimes 2^{-\beta H_2})$

Example: asymptotic hypothesis testing

- discriminate $\rho^{\otimes n}$ and $\sigma^{\otimes n}$, test Π_n
- errors $\alpha_n = 1 - \mathrm{Tr}(\rho^{\otimes n}\Pi_n)$ and $\beta_n = \mathrm{Tr}(\sigma^{\otimes n}\Pi_n)$
- quantum Stein's lemma [HP91]: $\alpha_n \to 0$ possible if $\beta_n \geq 2^{-nD(\rho\|\sigma)+o(n)}$

Strong converse

- if $\beta_n \to 0$ faster, then $\alpha_n \to 1$
- exponential convergence, $\beta_n = 2^{-rn+o(n)}$ and $\alpha_n = 1 - 2^{-Rn+o(n)}$

$$R^*(r) = \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left[r - \widetilde{D}_\alpha(\rho\|\sigma)\right]$$

strong converse exponent [MO15]

Asymptotic relative submajorization

- compare many copies $(\rho_1^{\otimes n}, \sigma_1^{\otimes n})$ and $(2^{-Rn+o(n)}\rho_2^{\otimes n}, 2^{-rn}\sigma_2^{\otimes n})$
- ordered by relative submajorization
- can be made exact in second component if $\operatorname{Tr}\sigma_1 \geq 2^{-r}\operatorname{Tr}\sigma_2$

Goal

Characterize trade-off between R and r.

Result: characterization of asymptotic transformations

Theorem

Let (ρ_1, σ_1), (ρ_2, σ_2) *be pairs of states such that* $\operatorname{supp}\rho_i \subseteq \operatorname{supp}\sigma_i$. *Given* R, r, $(\rho_1^{\otimes n}, \sigma_1^{\otimes n})$ *relative submajorizes* $(2^{-Rn+o(n)}\rho_2^{\otimes n}, 2^{-rn}\sigma_2^{\otimes n})$ *for every* n *iff*

$$R \geq R^*(r) = \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left[r - \widetilde{D}_\alpha(\rho_1\|\sigma_1) + \widetilde{D}_\alpha(\rho_2\|\sigma_2)\right]$$

$$\widetilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\operatorname{Tr}(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha$$

Overview of proof

Dual approaches:

- ~~constructions: channels T such that $T(\rho_1^{\otimes n}) \geq \rho_2^{\otimes n}$ and $T(\sigma_1^{\otimes n}) \leq \sigma_2^{\otimes n}$~~
- obstructions: monotone quantities that disprove $(\rho_1, \sigma_1) \gtrsim (\rho_2, \sigma_2)$

Strategy

1. show that special obstructions suffice
2. classify these obstructions

Preordered semirings

- preordered semiring: $(S, +, \cdot, 0, 1, \preccurlyeq)$, operations and preorder behave like in $\mathbb{N}$ in detail:
  - $+, \cdot$ associative, commutative, distributive
  - 0 and 1 neutral elements for $+$ and $\cdot$
  - $\preccurlyeq$ reflexive, transitive
  - $x \preccurlyeq y$ implies $x + z \preccurlyeq y + z$ and $x \cdot z \preccurlyeq y \cdot z$
- **asymptotic preorder** $\gtrsim$: for suitable $u \in S$, define $x \gtrsim y$ as $u^{o(n)} \cdot x^n \succcurlyeq y^n$
- if $f : S \to \mathbb{R}_{\geq 0}$ multiplicative, $\succcurlyeq$-monotone, then also $\gtrsim$-monotone:

$$u^{o(n)} \cdot x^n \succcurlyeq y^n \implies f(u^{o(n)} \cdot x^n) \geq f(y) \implies \underbrace{f(u)^{o(1)}}_{\to 1} f(x) \geq f(y)$$

Duality

- **spectrum**: $\Delta(S) = \{f : S \to \mathbb{R}_{\geq 0} | f \text{ monotone homomorphism}\}$
  in detail:
  - $f(1) = 1$
  - $f(x + y) = f(x) + f(y)$
  - $f(x \cdot y) = f(x)f(y)$
  - $x \succcurlyeq y \implies f(x) \geq f(y)$
- $x \gtrsim y \implies \forall f \in \Delta(S) : f(x) \geq f(y)$

Theorem (informal, see precise forms in [Str88] and [Vra20b])

Let S be preordered semiring satisfying additional conditions. Then

$$x \gtrsim y \iff \forall f \in \Delta(S) : f(x) \geq f(y)$$

Examples of preordered semirings

semiring	dual char	spectrum
2-tensors	yes	trivial
k-tensors [Str88]	yes	hard (but see recent progress [CVZ18])
graphs [Zui19]	yes	hard (but see recent progress [Vra19])
graphs, q/ea [LZ20]	yes	hard
nc graphs [LZ20]	?	hard
nc graphs, ea [LZ20]	yes	hard
bipartite LOCC [JV19]	yes	nontrivial but solved
k-partite LOCC [JV19]	yes	hard (but see recent progress [Vra20a])
dichotomies [PVW20]	yes [Vra20b]	nontrivial but solved (this talk)
multiple states [BV20]	yes [Vra20b]	nontrivial but solved (this talk)

Semiring of dichotomies

- elements: (equivalence classes of) pairs (ρ, σ) where $\rho, \sigma \geq 0$, $\operatorname{supp} \rho \subseteq \operatorname{supp} \sigma$

- operations: direct sum, tensor product

$$(\rho_1, \sigma_1) + (\rho_2, \sigma_2) = \left(\begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}, \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \right)$$
$$(\rho_1, \sigma_1) \cdot (\rho_2, \sigma_2) = (\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2)$$

- preorder: relative submajorization

Characterization of sandwiched Rényi divergences

need to find spectrum, i.e. functions $f(\rho,\sigma)$ that satisfy

(i) $f(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) = f(\rho_1,\sigma_1) f(\rho_2,\sigma_2)$ (multiplicativity)

(ii) $f(\rho_1 \oplus \rho_2, \sigma_1 \oplus \sigma_2) = f(\rho_1,\sigma_1) + f(\rho_2,\sigma_2)$ (additivity)

(iii) $f(I_n, I_n) = n$ (normalization)

(iv) $f(T(\rho), T(\sigma)) \leq f(\rho,\sigma)$ when T is a completely positive trace-nonincreasing map (data processing inequality)

(v) f is increasing in the first and decreasing in the second argument (with respect to the semidefinite partial order). (monotonicity)

Theorem

These are

$$f_\alpha(\rho,\sigma) = \widetilde{Q}_\alpha(\rho,\sigma) = \operatorname{Tr}(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha$$

with $\alpha \geq 1$.

Summary

- characterization of asymptotic relative submajorization
- extends operational interpretation of sandwiched Rényi divergences
- axiomatic characterization of sandwiched Rényi divergences
- new technique for studying resource theories *directly in the asymptotic regime*
- robust tools, may be suitable for studying similar problems, e.g. channel discrimination, restricted transformations
- would be interesting to extend method to different asymptotic regimes (approximate, vanishing error, direct region, etc.)

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