Quantum Preparation Games

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joint work with Edgar A. Aguilar and Miguel Navascués

Presentation based on arXiv:2011.02216

Motivation: Certifying and Quantifying Quantum Resources

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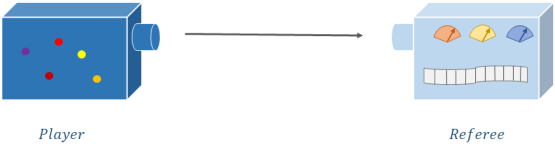
Quantum Preparation Games – General Setting



Basic setting: player prepares resources, referee scores player's resources after n rounds

Player's strategy $\mathcal{P}$

- Prepare quantum systems from $\mathcal{C}$

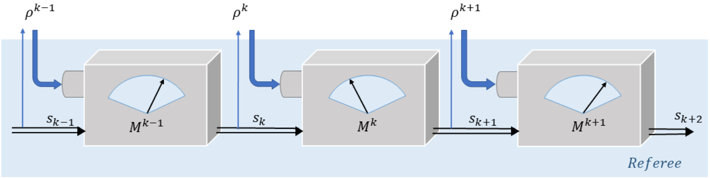
Protocols: Referee's strategies $(\mathcal{M}, \mathcal{S}, g)$

- Measuring devices $\mathcal{M}$
- Classical memory with states $\mathcal{S}$
- Scoring rule g

Expected score for a player with strategy $\mathcal{P}$

$$G(\mathcal{P}) = \sum_{s \in \mathcal{S}} p(s|\mathcal{P}) \langle g(s) \rangle .$$

Quantum Preparation Games – Referee



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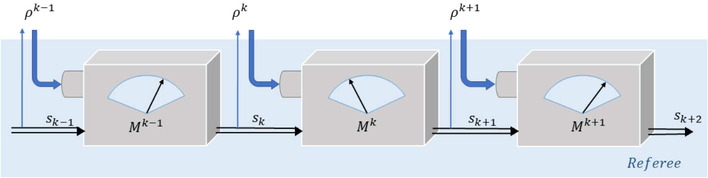


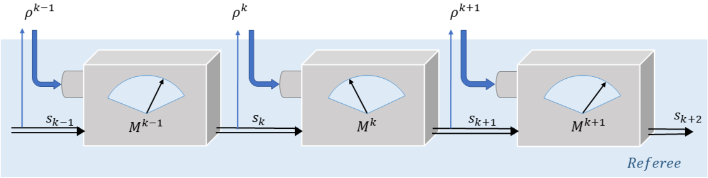
- Recursive computation (and optimisation) of the score of a player:

$$\mu_s^{(n)} = \max_{\rho \in \mathcal{C}} \sum_{s'} \mathrm{tr}(M_{s'|s}^{(n)} \rho) \langle g(s') \rangle,$$

$$\mu_s^{(k)} = \max_{\rho \in \mathcal{C}} \sum_{s'} \mathrm{tr}(M_{s'|s}^{(k)} \rho) \mu_{s'}^{(k+1)},$$

$$G_{\max} = \mu^{(1)}.$$

Quantum Preparation Games – Referee



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G_{\max} &= \mu^{(1)}.
\end{aligned}
$$

- Maxwell-demon games: finite number of measurements and full information about previous states of the experiment in memory $\mathcal{S}_k$.

Example: Entanglement Certification as a Preparation Game

Task: certify entanglement of a state ρ in n rounds; distinguish $\mathcal{E}_{\mathrm{ENT}} = \{\rho^{\otimes n}\}$ from $\mathcal{E}_{\mathrm{SEP}}$.

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- Binary final outcome $s \in \{\mathrm{ent}, \mathrm{sep}\}$ with $g(\mathrm{ent}) = 1$, $g(\mathrm{sep}) = 0$.
- Quality of the protocol given by the worst-case errors

$$e_I = \max_{\mathcal{P} \in \mathcal{E}_{\mathrm{SEP}}} p(\mathrm{ent}|\mathcal{P})$$

$$e_{II} = \max_{\mathcal{P} \in \mathcal{E}_{\mathrm{ENT}}} p(\mathrm{sep}|\mathcal{P})$$

Maximal expected score for separable strategy

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Maximal expected score for separable strategy

Goal: design optimal protocols $(\mathcal{M}, \mathcal{S}, g)$ for ρ taking restrictions on $\mathcal{M}$ of the referee into account.

Optimising Quantum Preparation Games

- Solving problems of the following type (1-round version)

$$\begin{aligned} \min_{(M_s)_s, e_{II}} \quad & e_{II} \\ \text{s.t.} \quad & 1 - \sum_s \mathrm{tr}(M_s \rho) \langle g(s) \rangle \le e_{II}, \\ & \sum_s \mathrm{tr}(M_s \sigma) \langle g(s) \rangle \le e_I \qquad \forall \sigma \in \mathrm{SEP}, \\ & (M_s)_s \in \mathcal{M}, \end{aligned}$$

$p(\mathrm{sep}|\rho)$

$p(\mathrm{ent}|\sigma)$

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& e_I \mathbb{I} - \sum_s M_s \langle g(s) \rangle \in \mathrm{SEP}^*, \\
& (M_s)_s \in \mathcal{M},
\end{aligned}
$$

- For SEP, the dual to the Doherty-Parillo-Spedalieri hierarchy approximates SEP^* (from the inside).

Doherty, Parillo, Spedalieri, PRL 88, 2002 and PRA 69, 2004.

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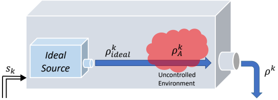
- For SEP, the dual to the Doherty-Parillo-Spedalieri hierarchy approximates SEP^* (from the inside).

- Applies to cases where

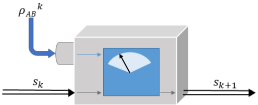
$$E = \{\rho' \mid \rho' \geq 0,\; \mathrm{tr}(\rho') = 1,\; \|\rho' - \rho\|_1 \leq \epsilon\}.$$

- Applies to finitely correlated strategies where correlations between rounds may build up in the uncontrolled environment.

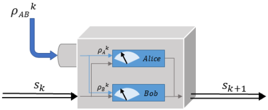


Optimising Protocols under Restrictions on Measurements

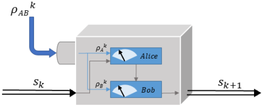
Global measurements



Non-adaptive measurements



Adaptive measurements



- Global measurements: any POVM $(M_s)_s$.
- Non-adaptive Pauli measurements: $M_s = \sum_{x,y} P(x,y,s|a,b)\sigma_{x,a} \otimes \sigma_{y,b}$ s.t.

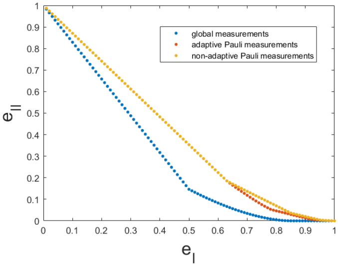
$$\sum_s P(x,y,s|a,b) = P(x,y).$$

- Adaptive Pauli measurements: $M_s = \sum_{x,y} P(x,y,s|a,b)\sigma_{x,a} \otimes \sigma_{y,b}$ s.t.

$$\sum_{y,s} P(x,y,s|a,b) = P(x) \quad \text{and} \quad \sum_s P(x,y,s|a,b) = P(x,y|a).$$

Entanglement Detection with Various Types of Measurements (few rounds)

- Single-shot entanglement certification of $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |1+\rangle)$



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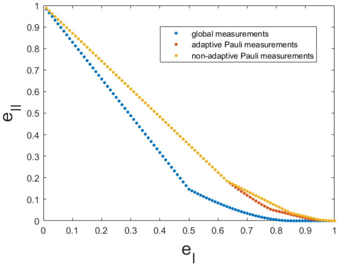
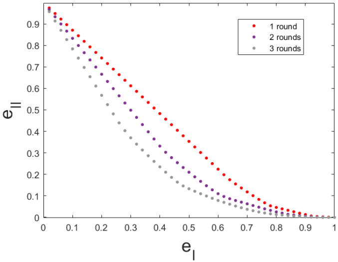
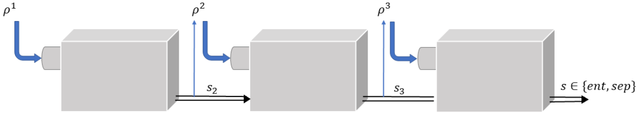


- Maxwell demon game for entanglement certification of $|\phi\rangle$ with adaptive Pauli measurements



Adaptiveness in Maxwell Demon Games



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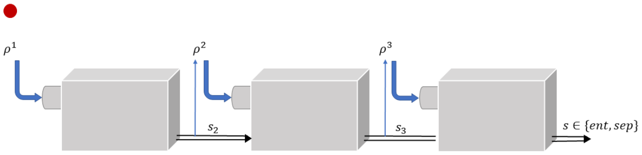

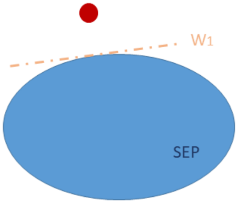

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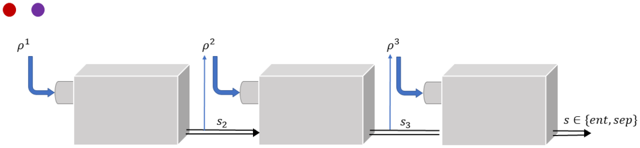

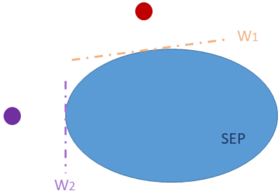

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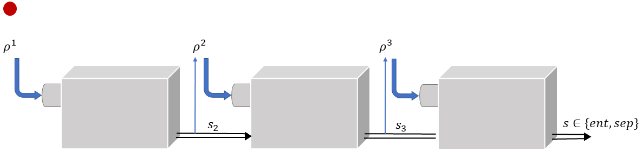

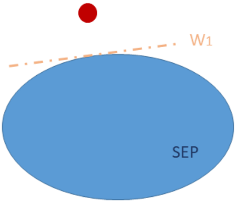

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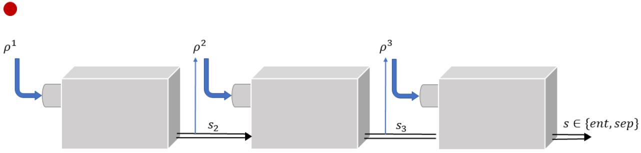

Surprise: Adaptiveness between rounds helps for entanglement detection of single states (e.g. for $|\phi\rangle$).

Beyond Maxwell Demon Games: Referees with Restricted Memory

- Memory state of restricted size and dependency, e.g. of fixed size and only dependent on last round.
- Independent repetition of (optimised) few-round Maxwell demon games.

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Example: Preparation game $(\mathcal{M}, \mathcal{S}, g)$ with memory requirements growing as $O(k^N)$ with round number k.

- Choose one out of N ± 1-outcome measurements $M_1(k), M_2(k), \ldots M_N(k)$.
- Counter of positive vs. negative outcomes for measurement separately $(s_1(k), \ldots, s_N(k)) \in \{-k, \ldots k\}^N$.
- Scoring rule in terms of final memory state $g(s_1(n), s_2(n), \ldots, s_N(n))$.

Example: Preparation Game for Quantifying 2-qubit Entanglement

Task: Find game to quantify the entanglement of $|\psi_\theta\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$ in n-rounds.

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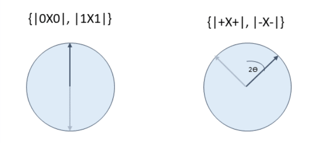
- Alice and Bob perform adaptive measurements ,

$$M_1(k) = \left\{ \frac{\mathbb{I} + W(\theta)}{2}, \frac{\mathbb{I} - W(\theta)}{2} \right\},$$



$$|\psi_\theta\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$$
$$= |+\rangle \left(\frac{\cos(\theta)|0\rangle + \sin(\theta)|1\rangle}{\sqrt{2}} \right) + |-\rangle \left(\frac{\cos(\theta)|0\rangle - \sin(\theta)|1\rangle}{\sqrt{2}} \right)$$

$$W(\theta) = \frac{1}{2}[\ |0\rangle\langle 0| \otimes Z + |1\rangle\langle 1| \otimes (-Z)$$
$$+ |+\rangle\langle +| \otimes (\sin(2\theta)X + \cos(2\theta)Z) + |-\rangle\langle -| \otimes (-\sin(2\theta)X + \cos(2\theta)Z)]$$

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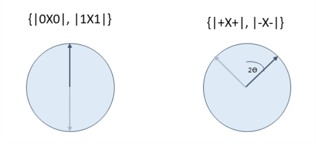
- Alice and Bob perform adaptive measurements for $\theta = \theta(s_2(k))$,

$$M_1(k) = \left\{ \frac{\mathbb{I} + W(\theta)}{2}, \frac{\mathbb{I} - W(\theta)}{2} \right\}, \quad M_2(k) = \left\{ \frac{\mathbb{I} + \frac{\partial}{\partial\theta} W(\theta)}{2}, \frac{\mathbb{I} - \frac{\partial}{\partial\theta} W(\theta)}{2} \right\}.$$



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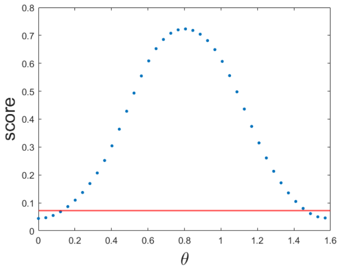
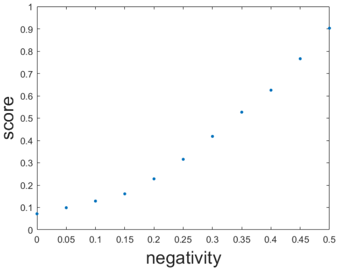
- Final score: $g(s_1(n), \theta = \theta(s_2(n))) = h\left(\cos^2(\theta)\right) \Theta\left(s_1(n) - \delta(\theta)\right)$

Quantifying two-qubit entanglement in 40 rounds

Player preparing states $|\psi_\theta\rangle\langle\psi_\theta|^{\otimes n}$



Player preparing states $\sigma \in \mathcal{E}_\mathcal{N}$



Negativity $\mathcal{N}(\sigma) = \frac{\|\sigma^{T_B}\|_1 - 1}{2}$.

Further research and open questions

- Implications for other resources ? (High-dimensional entanglement, multi-party entanglement, non-locality, magic states)

- Allow referee to conduct resourceful operations ?

- Application in NISQ devices (allow referee with quantum memory of fixed dimension) ?

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Thank you!

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