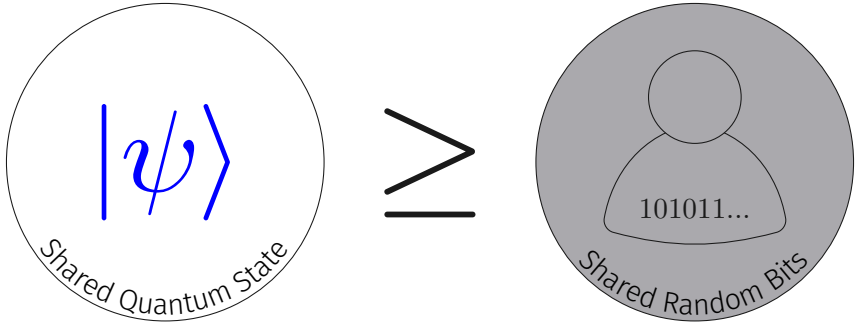
Tight Limits on Nonlocality from Nontrivial Communication Complexity

Noah Shutty, joint work with Mary Wootters and Patrick Hayden
Stanford University

Entanglement is strictly better than *shared random coins* for some tasks.



Such tasks include ***nonlocal games*** and ***distributed computation***.

CHSH[1]*, a Nonlocal Game:*

[1][CHSH69]

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Players Win if $xy = a + b \mod 2$

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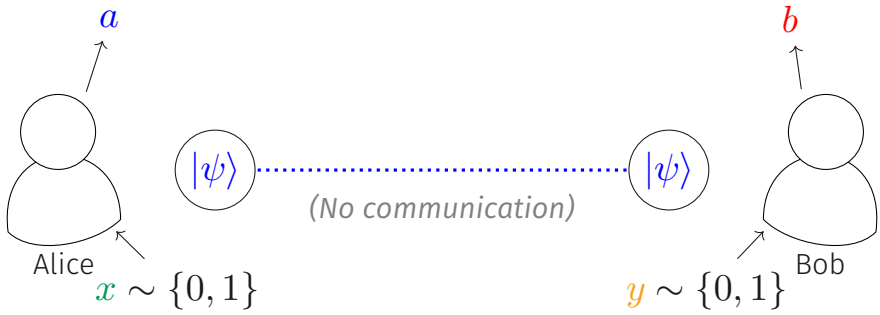
CHSH[2], *a Nonlocal Game:*



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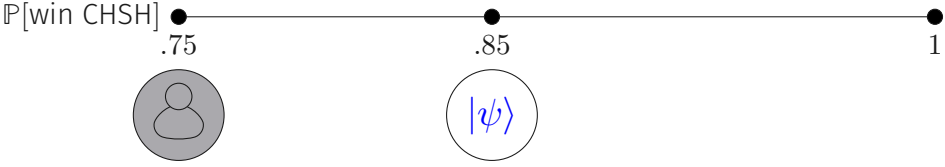
Entangled Players win $\approx$ 85.3% of games

[2][CHSH69]


ℙ[win CHSH]
.75
.85
1
$|\psi\rangle$

Distributed Computation:

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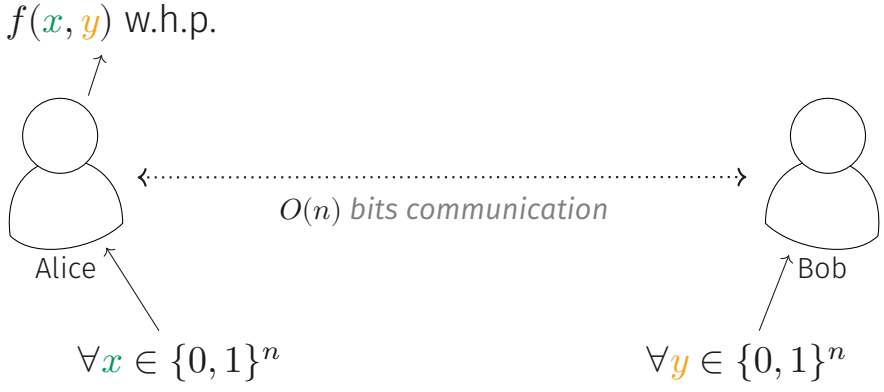
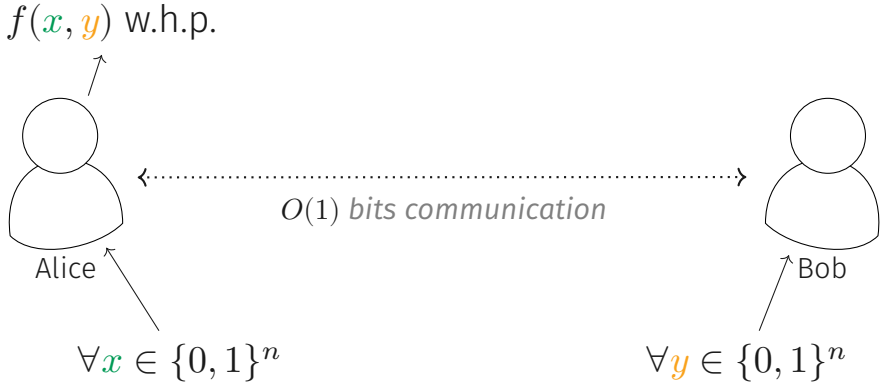
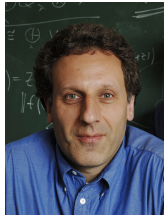


Trivial Communication Complexity:



What is the relationship between *nonlocal games* and *trivial communication complexity*?






Brassard Buhrman Linden Méthot Tapp Unger

Theorem [BBL+06] (informal)

Communication complexity is trivial if two players win more than $\approx 90.8\%$ of CHSH games.

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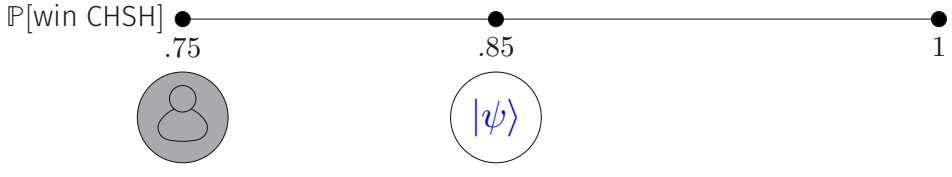

Theorem [Cleve], [van Dam], [Brassard, Buhrman, Linden, Méthot, Tapp, and Unger]

If it were possible to win CHSH with $\mathbb{P}[\text{win}] > .908$, communication complexity would become trivial.

$\frac{1}{2} + \frac{1}{\sqrt{6}} \approx 0.908$

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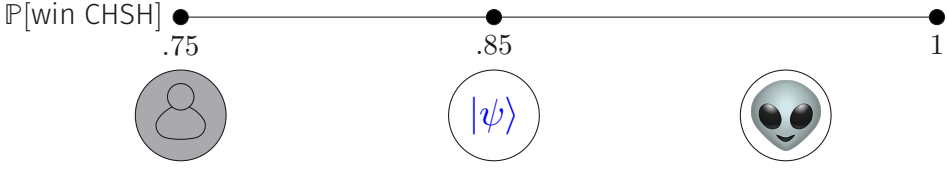

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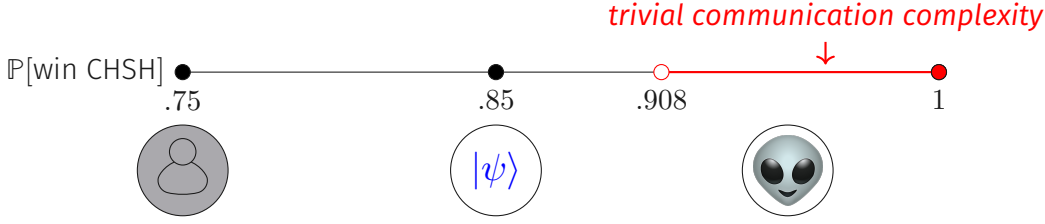

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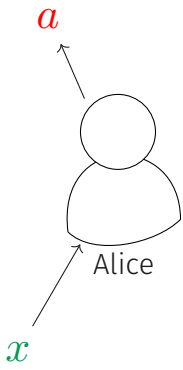
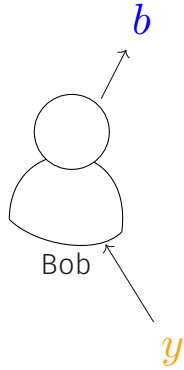


$$\mathbb{P}[a, b | x, y] \in \mathbb{R}^{16}$$



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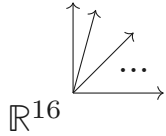

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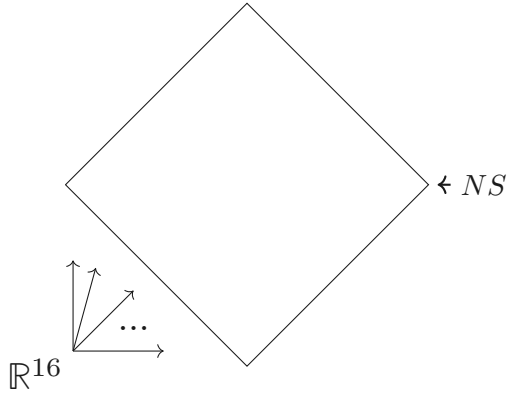

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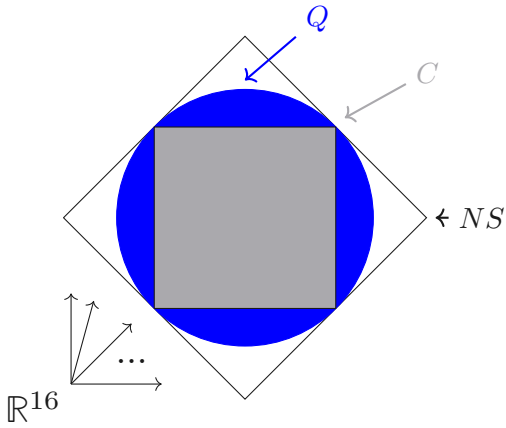

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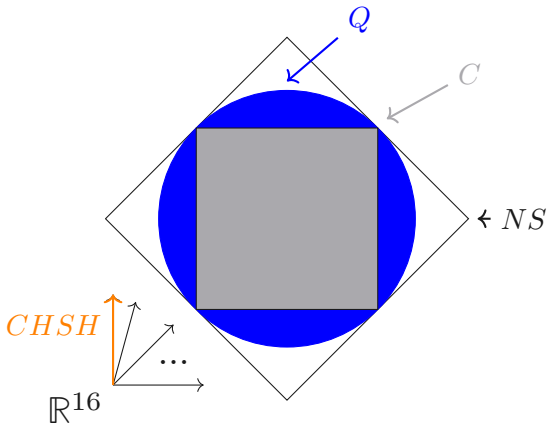

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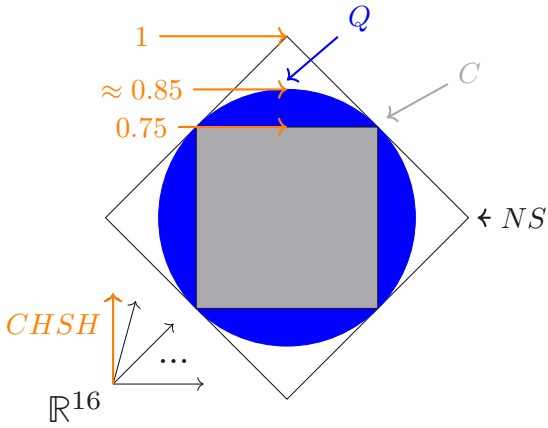

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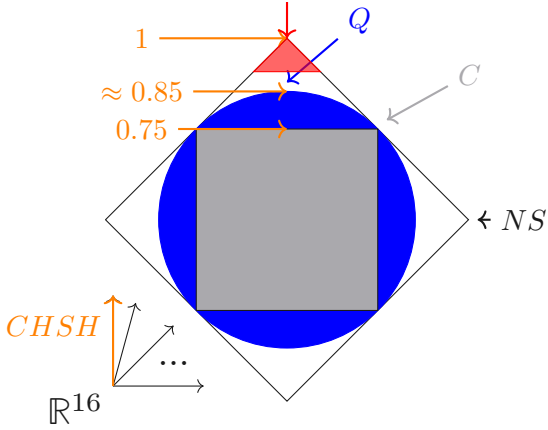
Alice + Bob's strategy as a conditional probability distribution:

trivial communication complexity [BBL+06, vD13, Cle]



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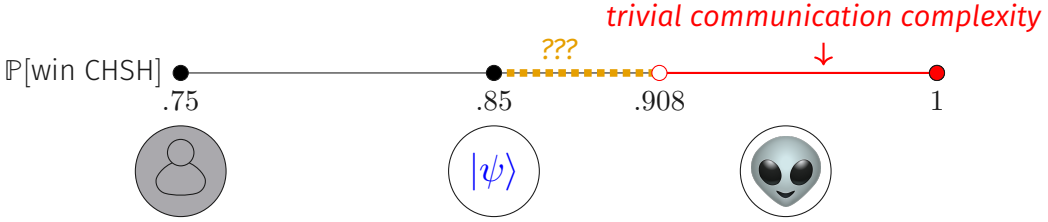
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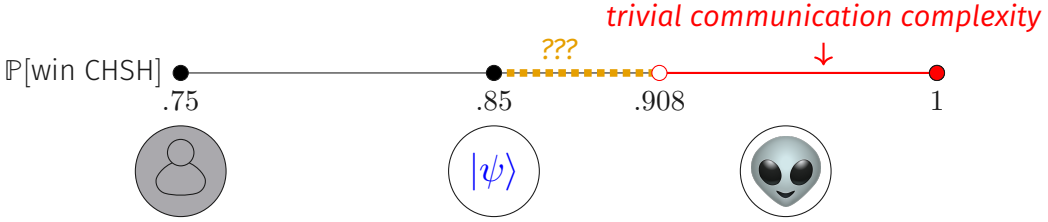
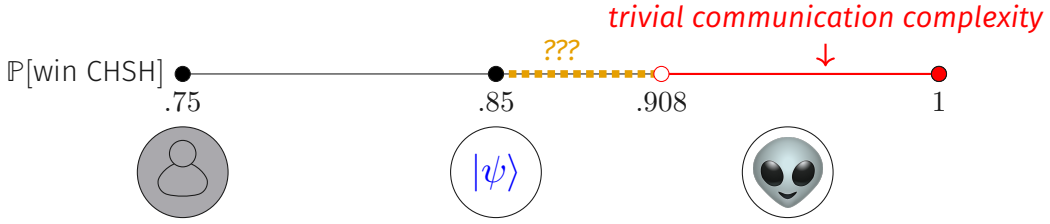


The [BBLMTU] proof used the fact that formulas of $\mathrm{AND}_\epsilon, \mathrm{XOR}_0$ gates for $\epsilon < 1/6$ support reliable computation.



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We give a tight upper bound that limits this approach:

Theorem 1

Reliable computation by formulas of ϵ-noisy AND gates and noise-free XOR gates is impossible for $\epsilon \geq 1/6$

Question

Is there a nonlocal game for which *any super-quantum success probability* causes communication complexity to become trivial?

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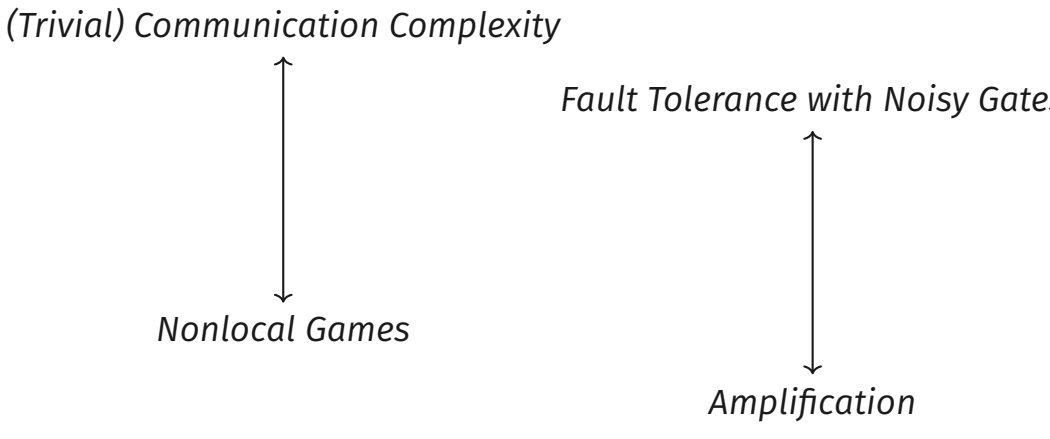
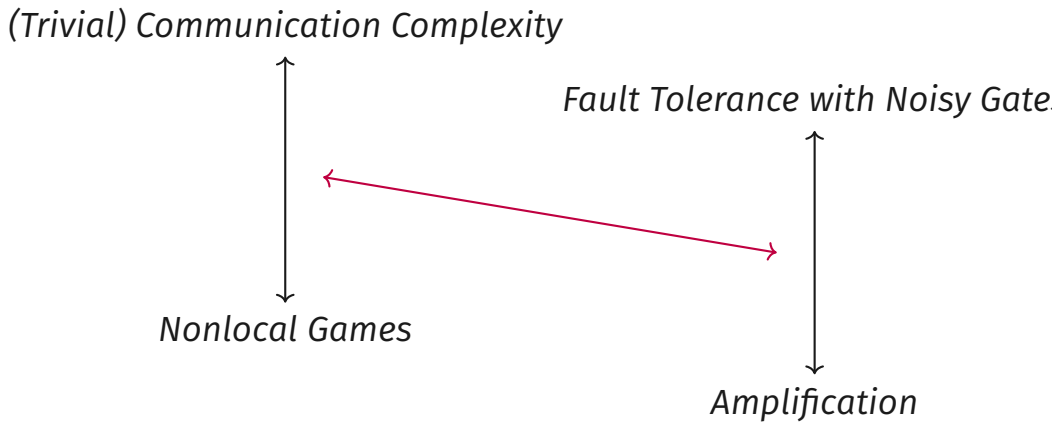
Theorem 3

Yes.

(Trivial) Communication Complexity

Nonlocal Games


(Trivial) Communication Complexity
Nonlocal Games
Fault Tolerance with Noisy Gates
Amplification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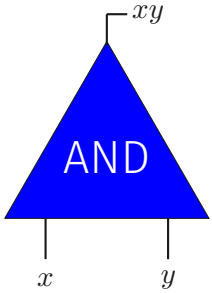
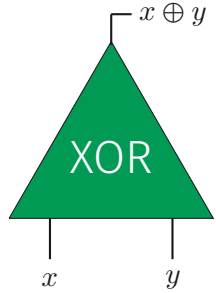
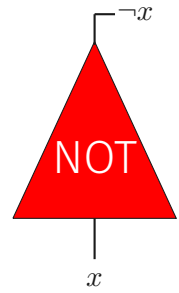
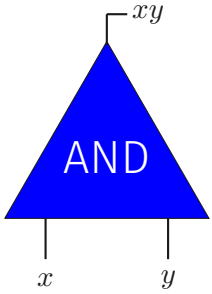
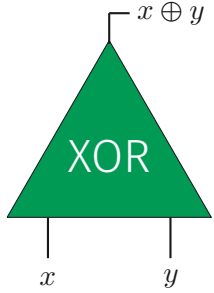
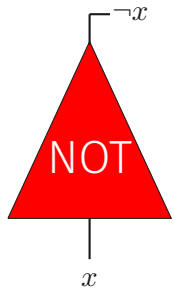

xy
AND
x
y


$x \oplus y$
XOR
x
y


$\neg x$
NOT
x







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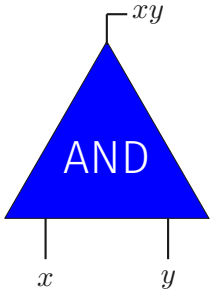
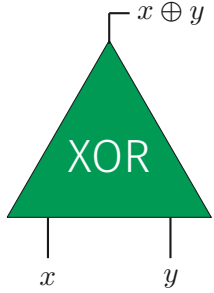
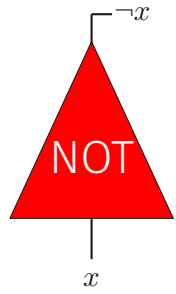
Can you compute an arbitrary function using a formula of AND, XOR, and NOT gates?







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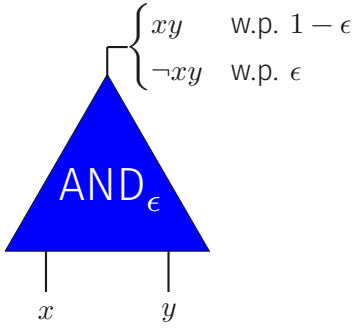
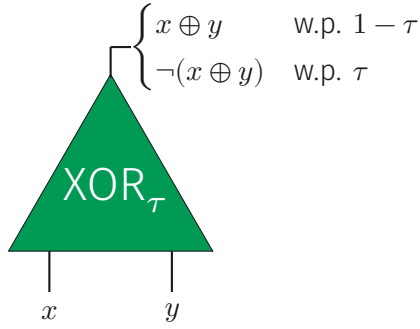
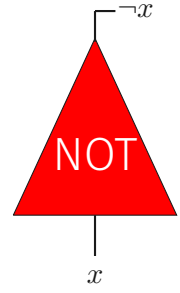
Answer

Yes! This set of gates is functionally universal.


xy w.p. $1-\epsilon$
$\neg xy$ w.p. ϵ
AND$_\epsilon$
x
y


$x \oplus y$ w.p. $1-\tau$
$\neg(x \oplus y)$ w.p. τ
XOR$_\tau$
x
y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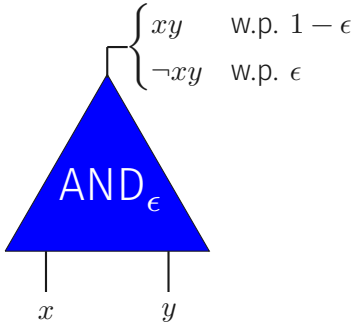

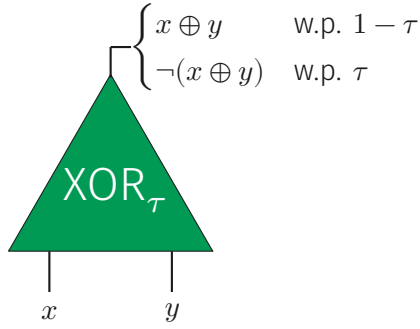

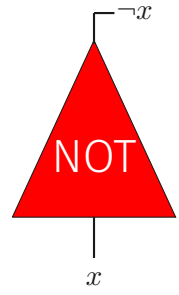

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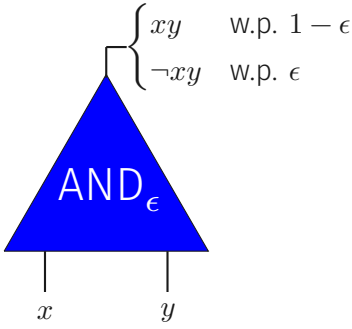
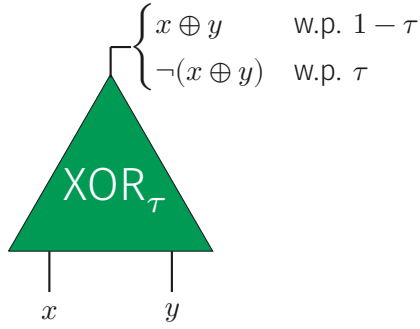
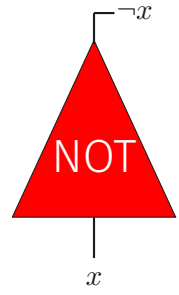
Can you compute an arbitrary function with *bounded probability of error* using a formula of AND_ϵ, XOR_τ, and NOT gates?







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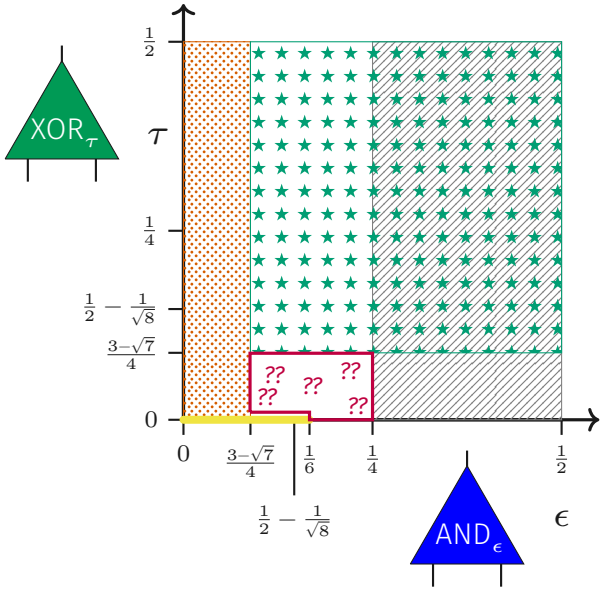
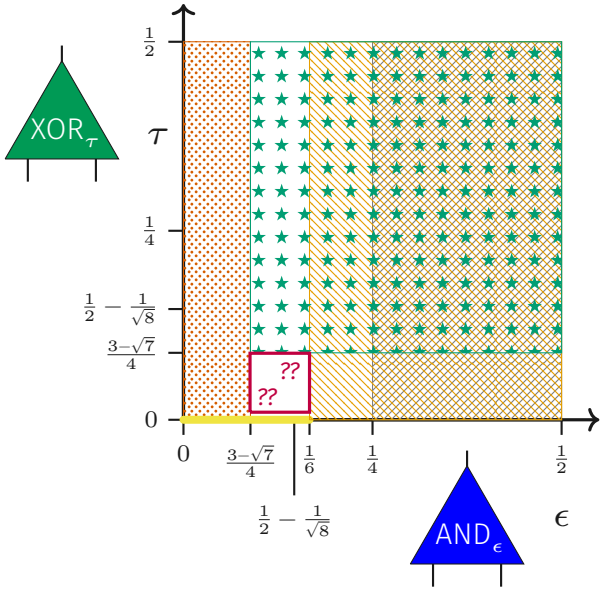
It depends on ϵ and τ!



Yes
- [EP98]
- [BBL$^+$06]

No
- folklore
- [ES99, EP98, Ung07]



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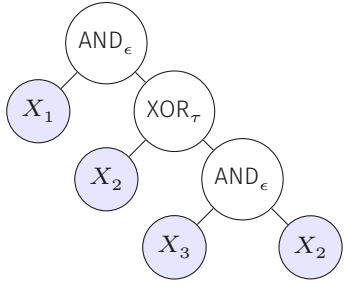

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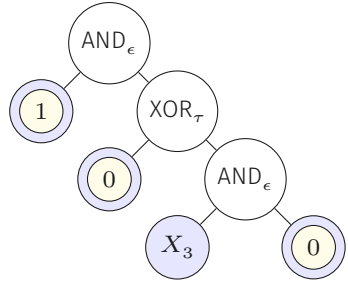

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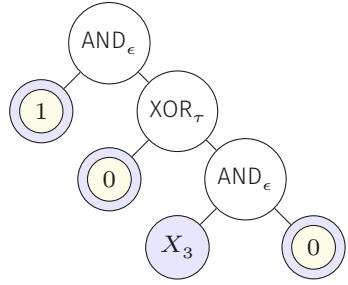

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Answer: Yes, there is a simple characterization.

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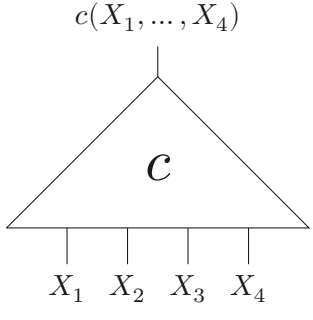

Amplifier away from $1/2$

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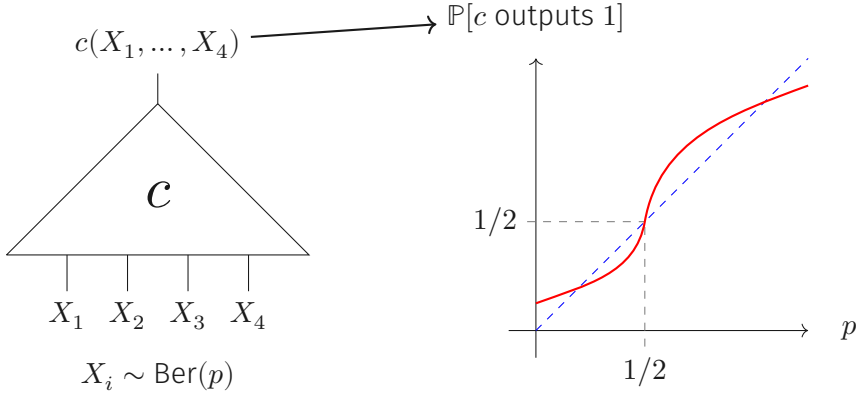
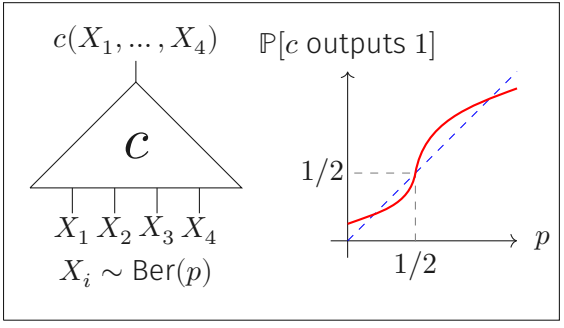


Amplifier away from $1/2$

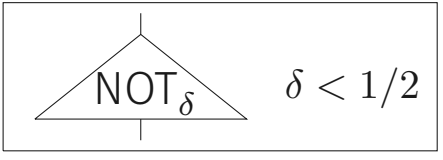
Theorem 2

A set of noisy circuits closed under convex combinations allows fault-tolerant computation if and only if it contains an amplifier away from $1/2$ and a (noisy) NOT gate.

$\Leftrightarrow$



&



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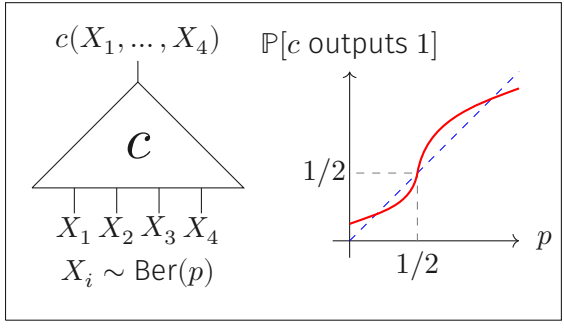

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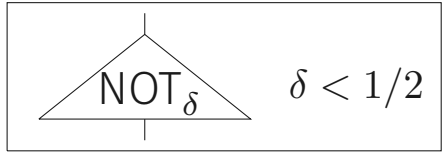
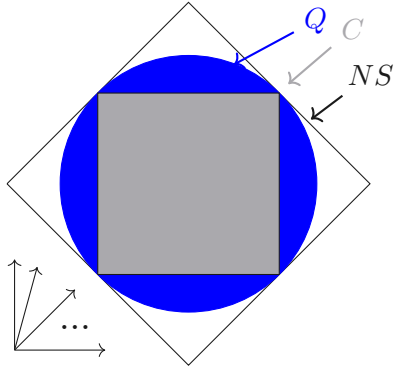


$$\mathbb{P}[a, b|x, y] \in \mathbb{R}^N$$



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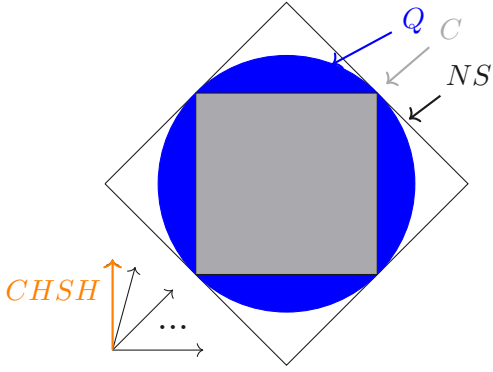

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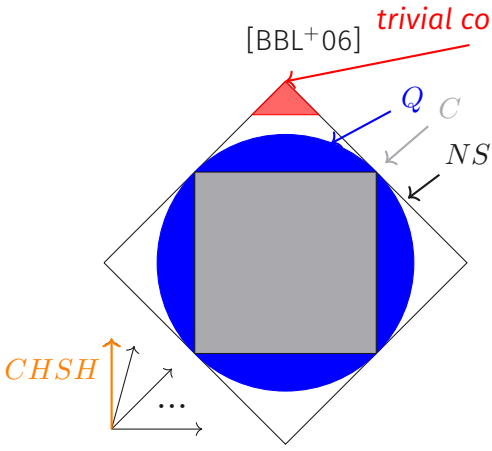

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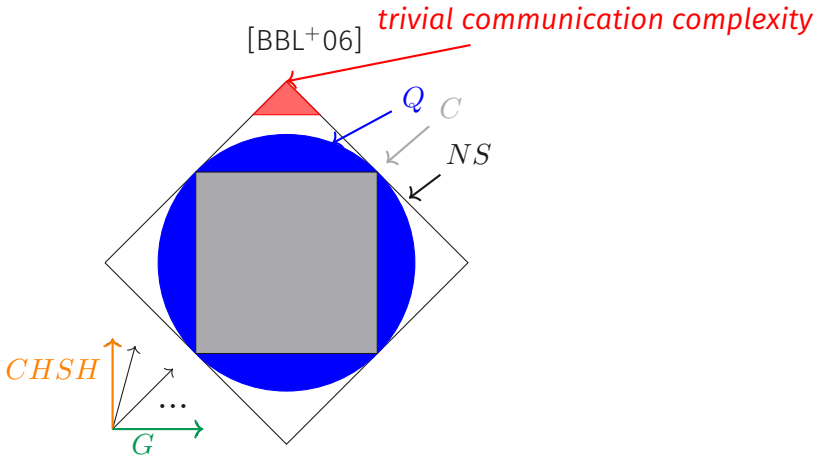

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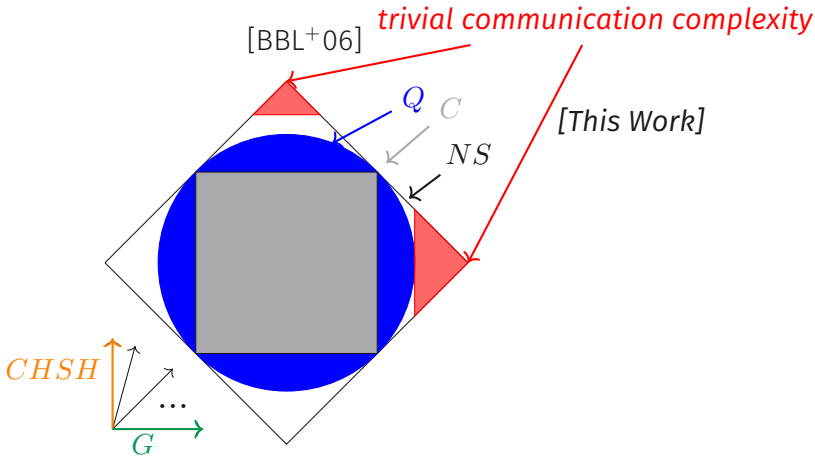

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Question

Is there a nonlocal game for which *any super-quantum success probability* causes communication complexity to become trivial?

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Theorem 3

$\forall k \geq 1$, there is a nonlocal game G_k, such that a win probability above the quantum value causes communication complexity to become trivial.

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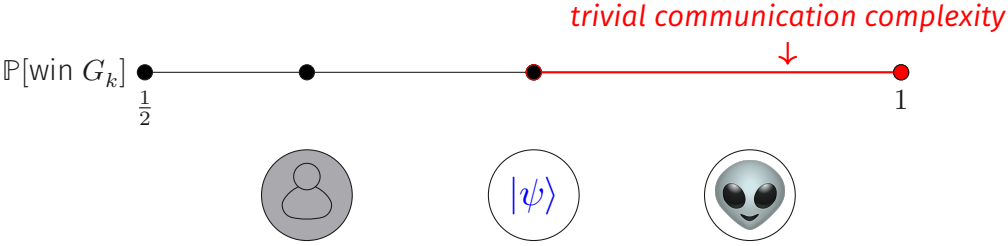
Is there a nonlocal game for which *any super-quantum success probability* causes communication complexity to become trivial?

Theorem 3

$\forall k \geq 1$, there is a nonlocal game G_k, such that a win probability above the quantum value causes communication complexity to become trivial. Conversely, if communication complexity is trivial, then $\exists k$ such that the win probability for G_k is above the quantum value.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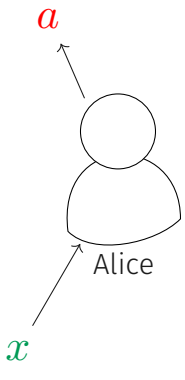
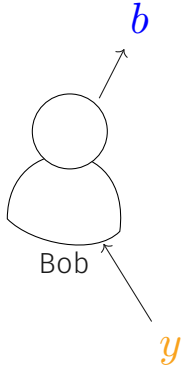
$\mathbb{P}[\text{win } G_k]$
$\frac{1}{2}$
1
$|\psi\rangle$

Alice + Bob as a conditional probability distribution:





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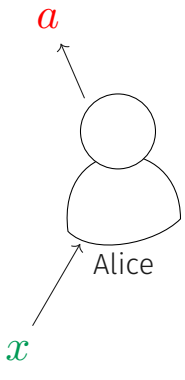

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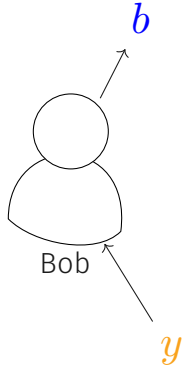

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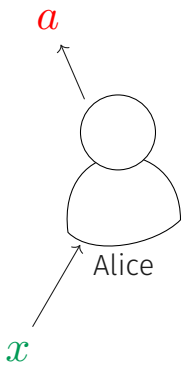

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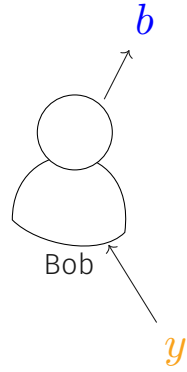
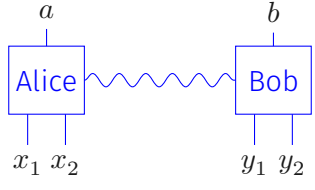


$$\mathbb{P}[\text{win game}] = \sum_{x,y,a,b} \mathbb{P}[a, b|x, y]\mathbb{P}[x, y]V(a, b; x, y)$$

Turning Strategies (Conditional Probability Distributions) Into Gates:



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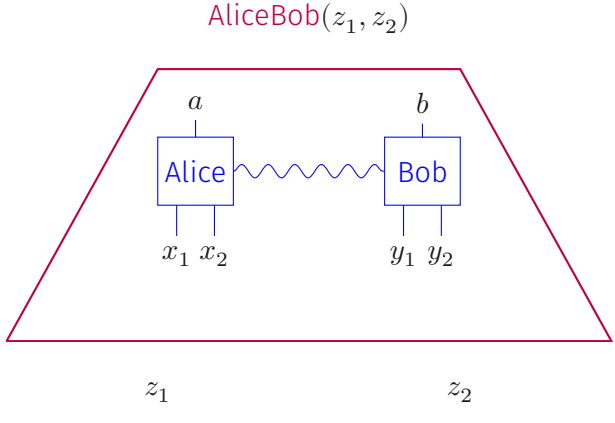

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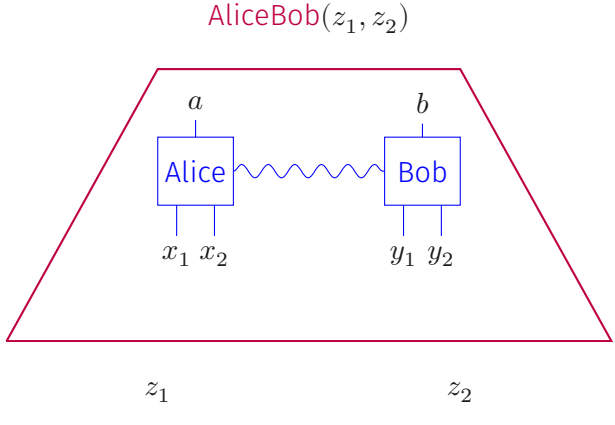

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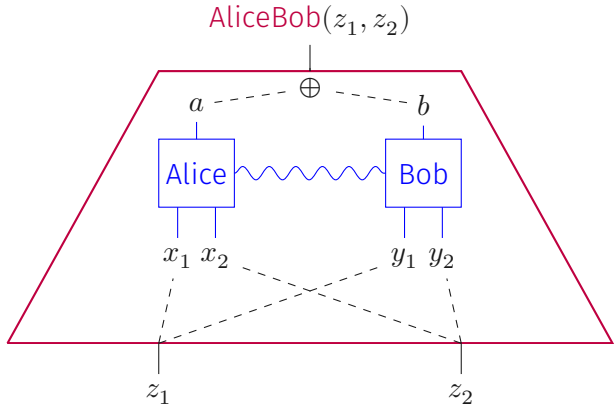

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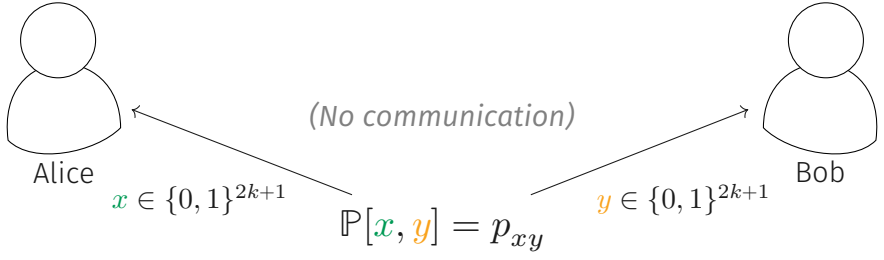

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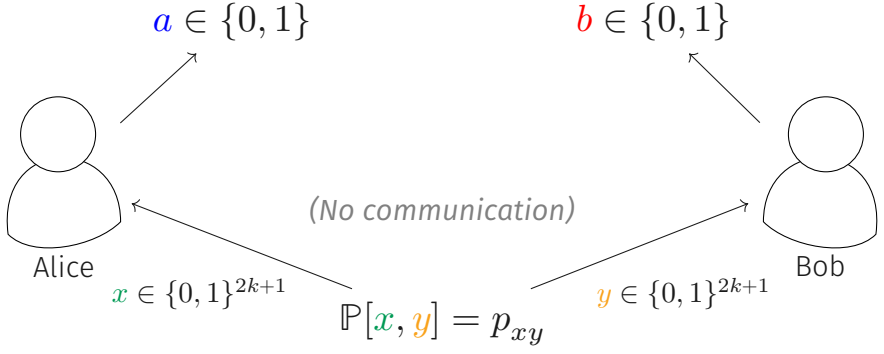

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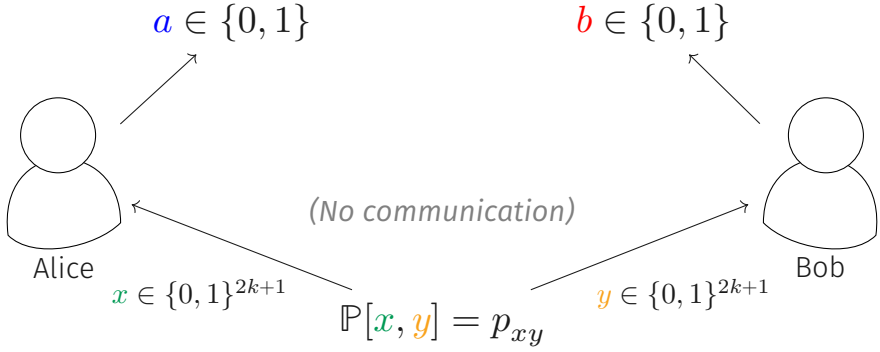

Players Win if $\mathrm{Maj}_{2k+1}(x+y) = a+b$

For some choice of p_{xy}, any $\mathbb{P}[\text{win } G_k]$ above quantum implies that AliceBob is an amplifier away from $1/2$.

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This implies that communication complexity is trivial for such Alice and Bob.

How to modify nonlocal game to have quantum advantage:

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$$G \longrightarrow qG + (1-q)\text{Magic}$$


Gilles Brassard, Harry Buhrman, Noah Linden, André Allan Méthot, Alain Tapp, and Falk Unger.
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Thank you for listening.

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