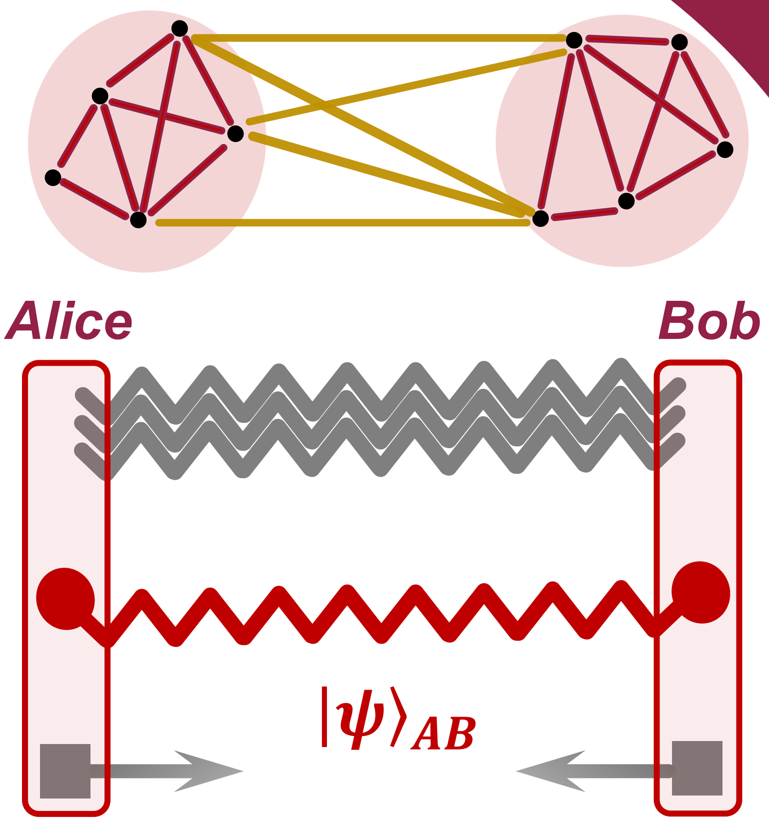
From Communication Complexity to an Entanglement Spread Area Law



Mehdi Soleimanifar (MIT)

(arxiv: 2004.15009)

Joint work with

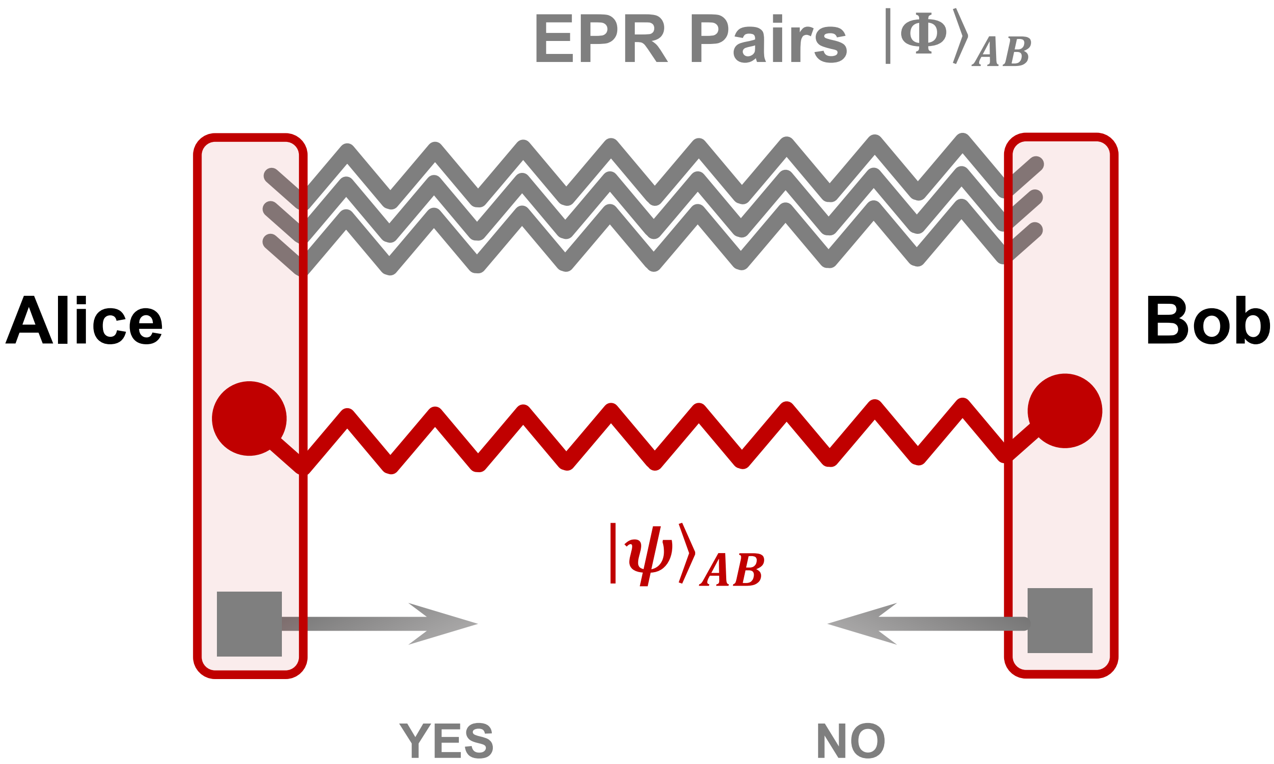
Anurag Anshu (UC Berkeley)

Aram Harrow (MIT)

Communication Complexity

Ground State Entanglement

Testing Bipartite States

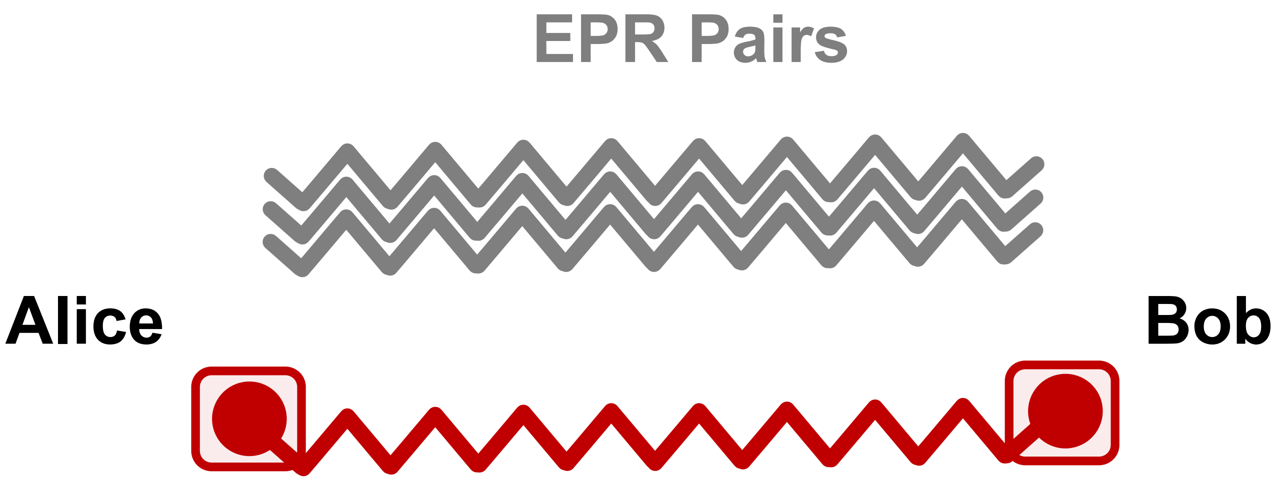


$$\{|\Omega\rangle\langle\Omega|_{AB}, I - |\Omega\rangle\langle\Omega|_{AB}\}$$

Two-Outcome Measurement

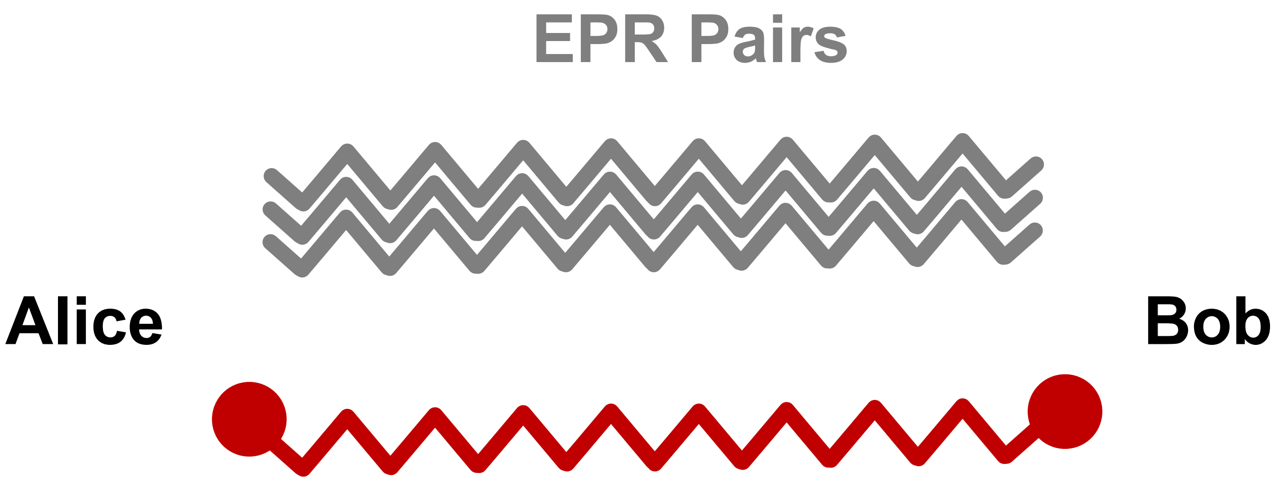
$C_\varepsilon(\Omega_{\mathrm{AB}}) =$ **Minimum # of exchanged qubits to perform ε approximation of** $\{|\Omega\rangle\langle\Omega|_{AB}, I - |\Omega\rangle\langle\Omega|_{AB}\}$

Testing Bipartite States



What property of $|\Omega\rangle_{AB}$ determines $C_\varepsilon(\Omega_{AB})$?

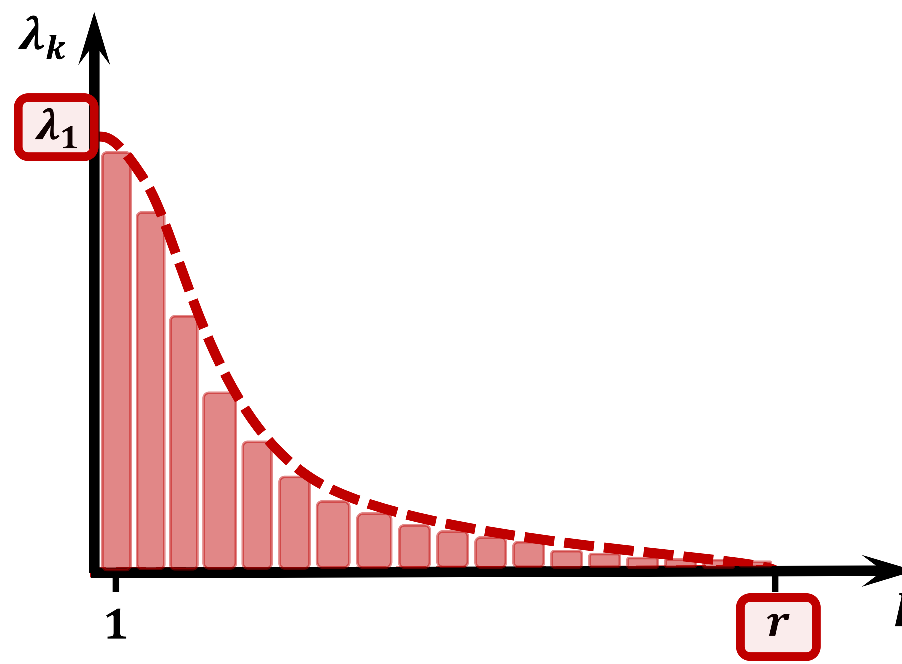
Testing Bipartite States



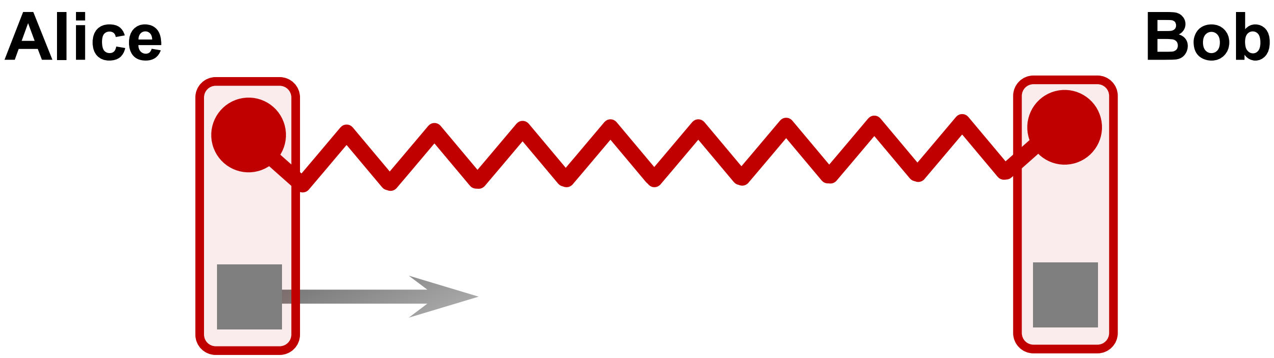
$$|\Omega\rangle_{AB} = \sum_{k=1}^{r} \sqrt{\lambda_k}|k\rangle_A|k\rangle_B$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$$
$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = 1$$

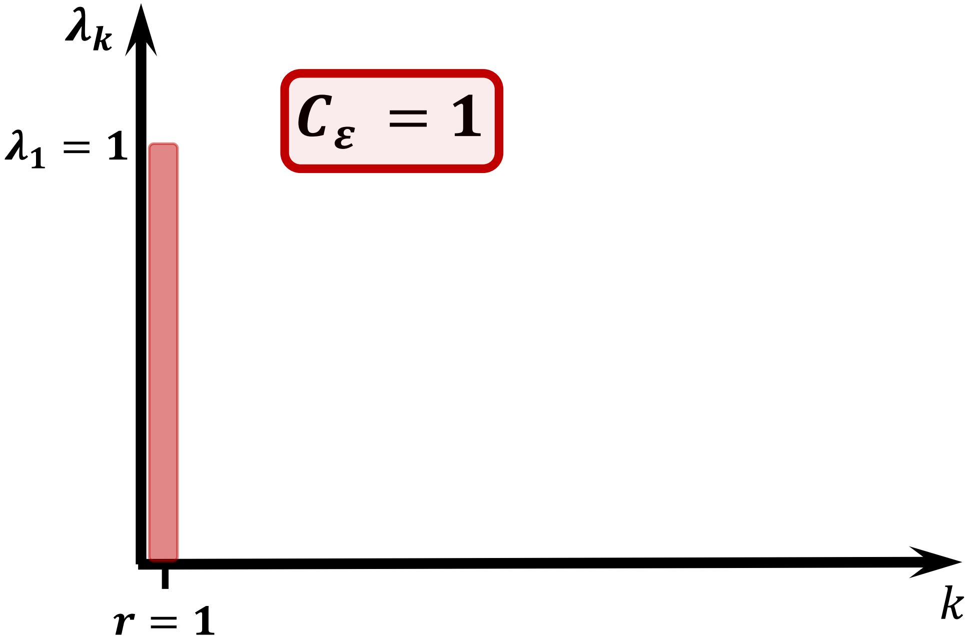
Schmidt Form



Testing Bipartite States



Testing $|0\rangle_A^{\otimes n}|0\rangle_B^{\otimes n}$

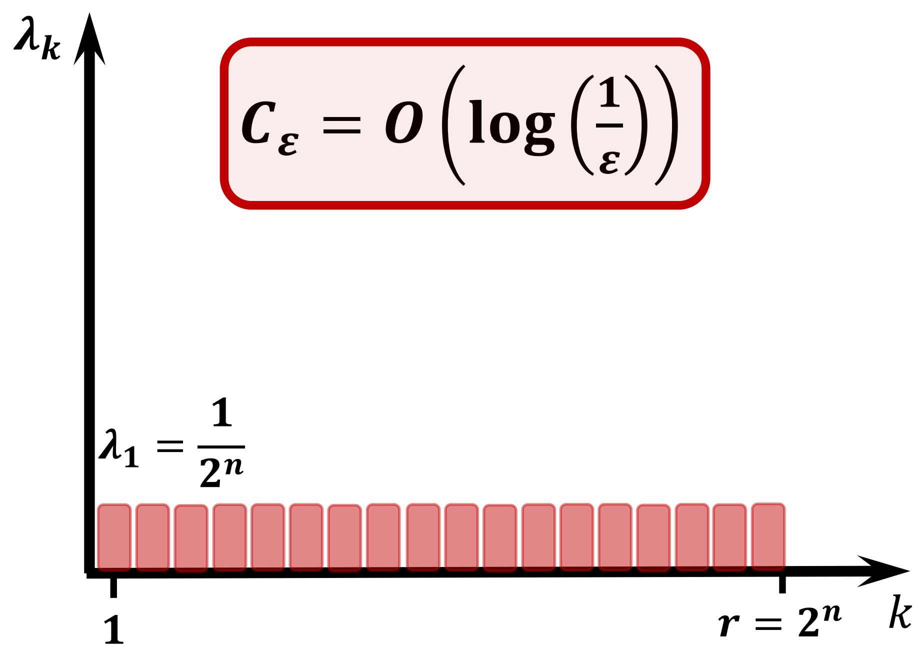


Testing Bipartite States

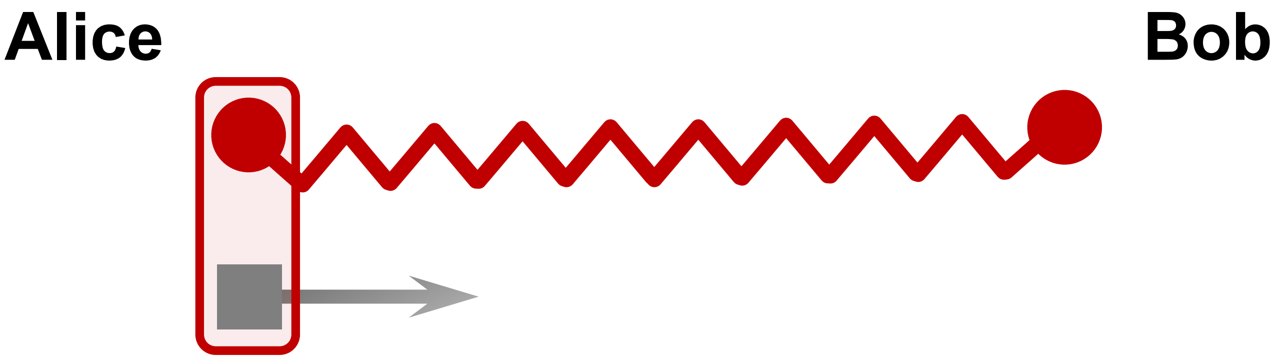
[AHL+14]

Testing n EPR pairs $|\mathrm{EPR}\rangle_{AB}^{\otimes n}$

Using Quantum Expanders

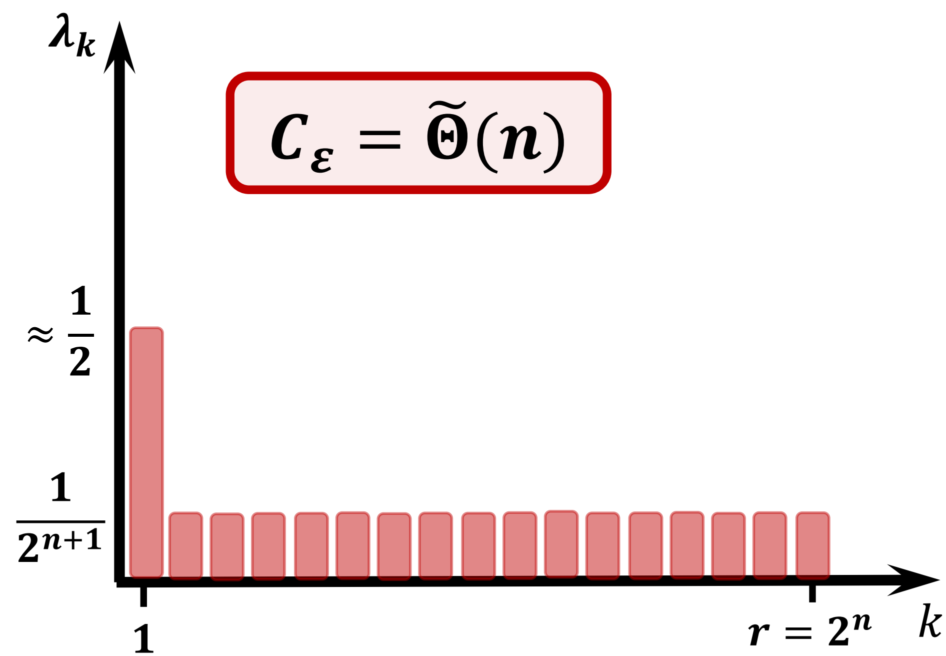


Testing Bipartite States

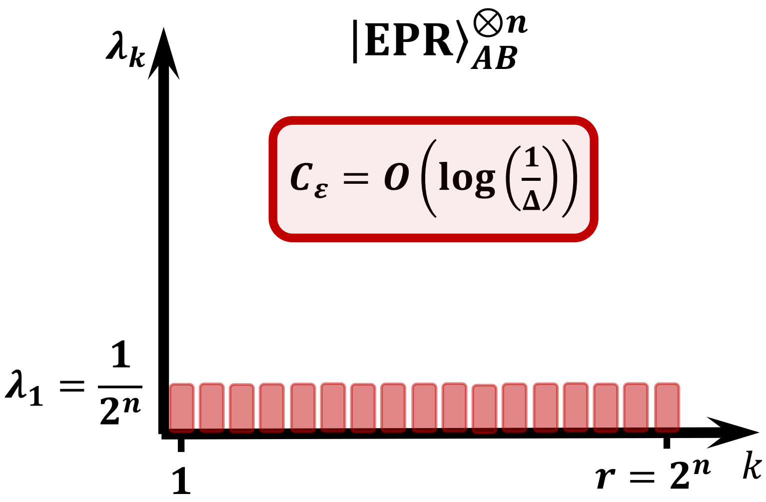


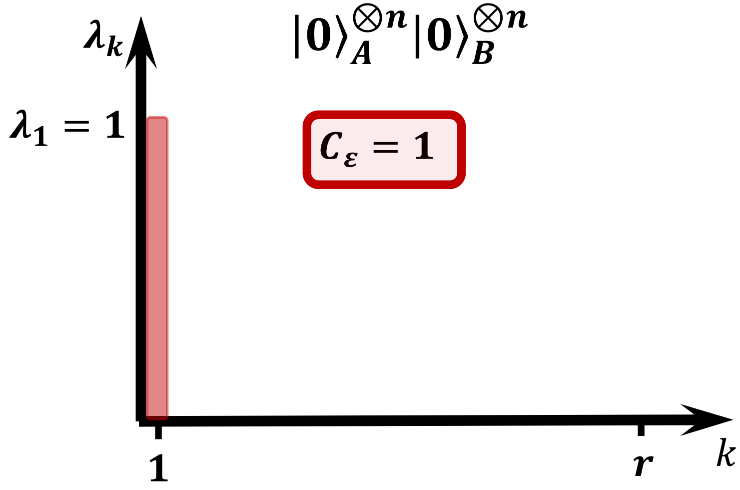
$$|\Omega\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle^{\otimes n} + |\mathrm{EPR}\rangle^{\otimes n}\right)$$

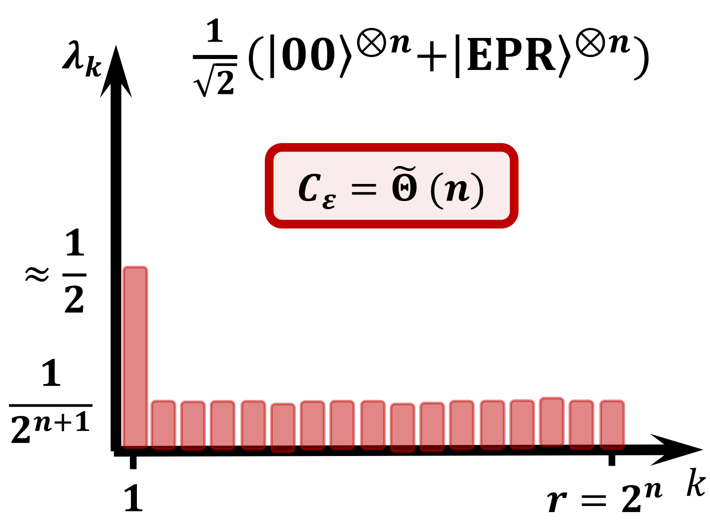
$$(I - 2|\Omega\rangle\langle\Omega|_{AB})|00\rangle\langle 00|^{\otimes n}$$
$$\approx |\mathrm{EPR}\rangle\langle \mathrm{EPR}|^{\otimes n}$$



Recap


$|\mathrm{EPR}\rangle_{AB}^{\otimes n}$
λ_k
$C_\varepsilon = O\left(\log\left(\frac{1}{\Delta}\right)\right)$
$\lambda_1 = \frac{1}{2^n}$
1
$r = 2^n$
k

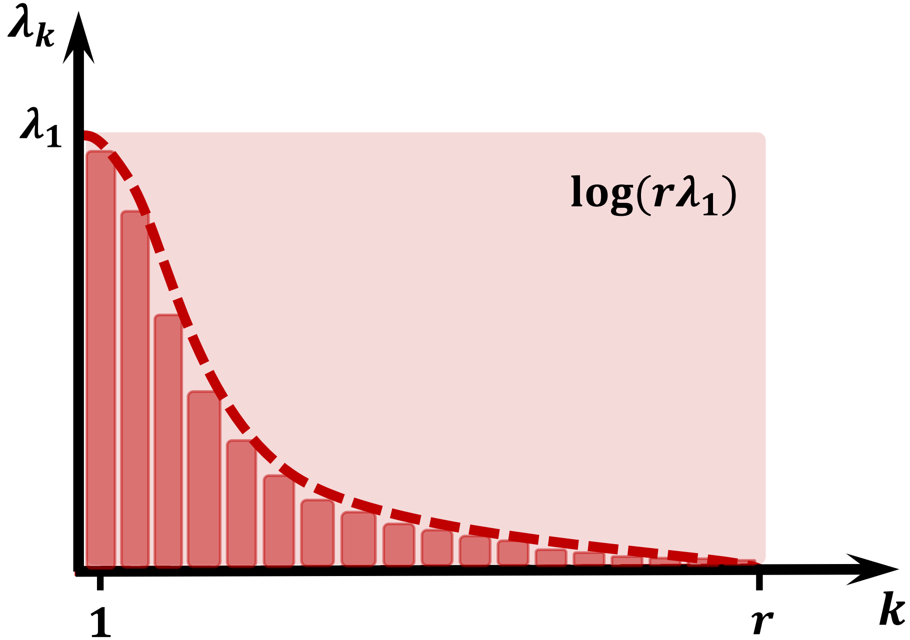

$|0\rangle_A^{\otimes n}|0\rangle_B^{\otimes n}$
λ_k
$\lambda_1 = 1$
$C_\varepsilon = 1$
1
r
k


$\frac{1}{\sqrt{2}}(|00\rangle^{\otimes n} + |\mathrm{EPR}\rangle^{\otimes n})$
λ_k
$C_\varepsilon = \widetilde{\Theta}(n)$
$\approx \frac{1}{2}$
$\frac{1}{2^{n+1}}$
1
$r = 2^n$
k

Entanglement Spread

[HW03] $$\mathrm{ES}(\Omega_A) = \log(r\lambda_1) \approx \log\left(\frac{\lambda_1}{\lambda_r}\right)$$

$$\Omega_A = \mathrm{Tr}_\mathrm{B}|\Omega\rangle\langle\Omega|_{AB}$$



Entanglement Spread

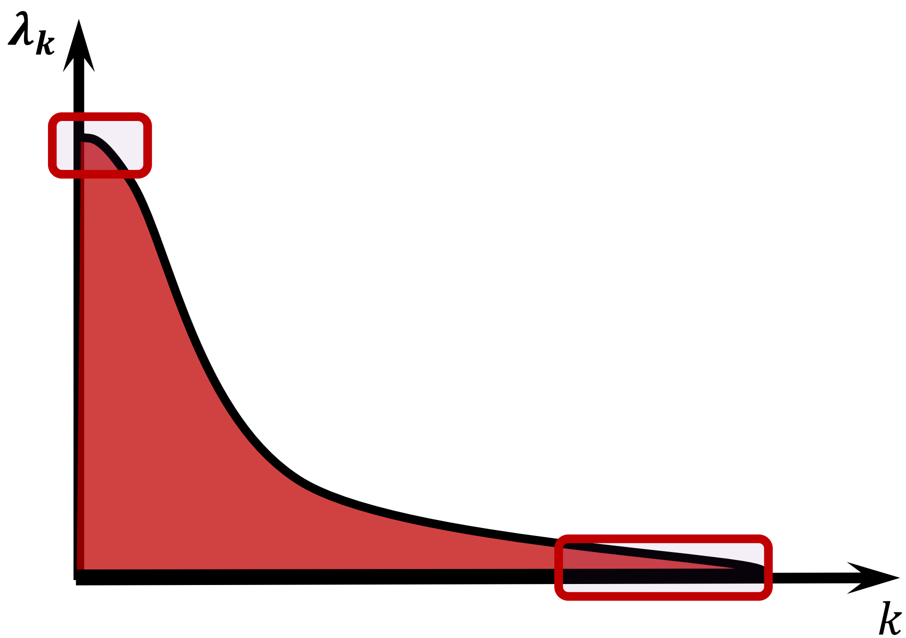
[HW03] $$\mathrm{ES}(\Omega_A) = \log(r\lambda_1) = \log(r) - \log(1/\lambda_1)$$

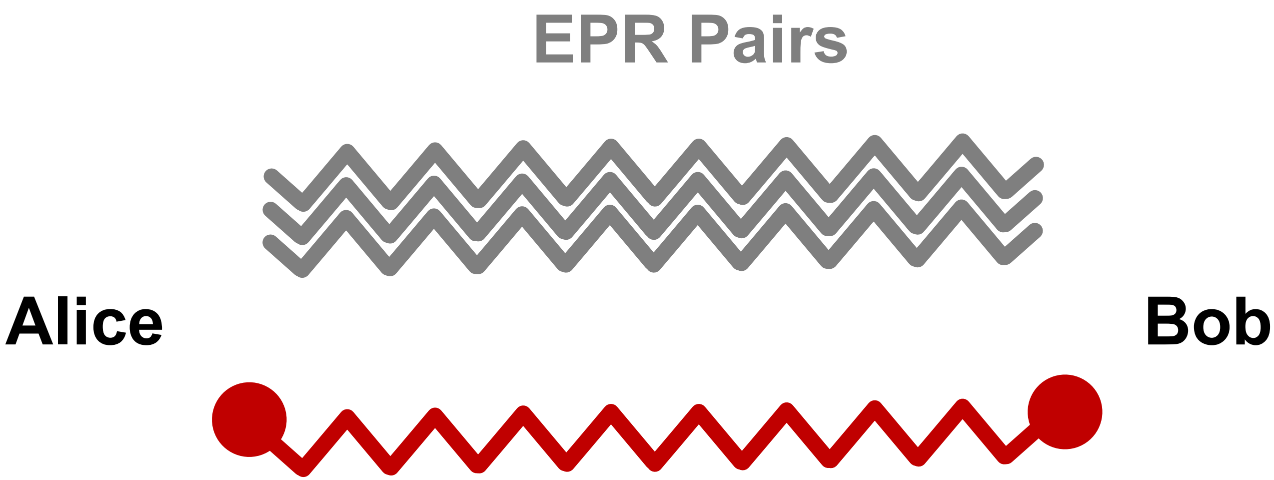
$$\Omega_A = \mathrm{Tr}_{\mathrm{B}} |\Omega\rangle\langle\Omega|_{AB} \qquad = \mathrm{S}_{\max}(\Omega_A) - \mathrm{S}_{\min}(\Omega_A)$$

ε −Smooth Entanglement Spread

$$\mathrm{ES}_\varepsilon(\Omega_A) = S^\varepsilon_{\max}(\Omega_A) - S^\varepsilon_{\min}(\Omega_A)$$

ε −Smooth Min/Max Entropies





Communication Complexity ≥ Entanglement Spread

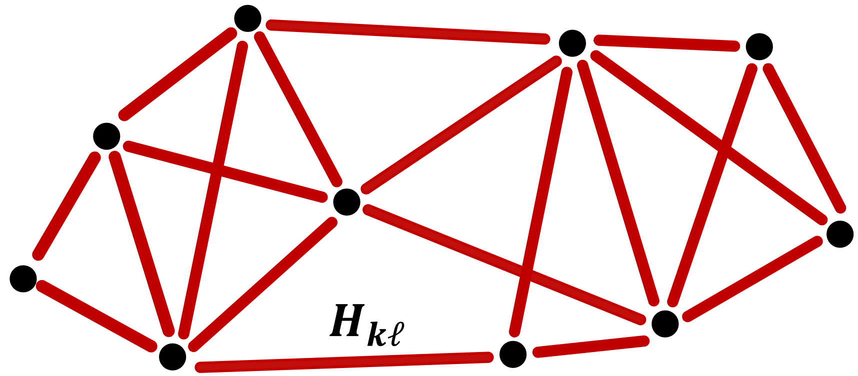
$$C_\varepsilon(\Omega_{AB}) \geq \mathbf{ES}_\varepsilon(\Omega_A) = S^\varepsilon_{\max}(\Omega_A) - S^\varepsilon_{\min}(\Omega_A)$$

[HW03, CH19, HL11]

Holds even with EPR-assistance

Communication Complexity

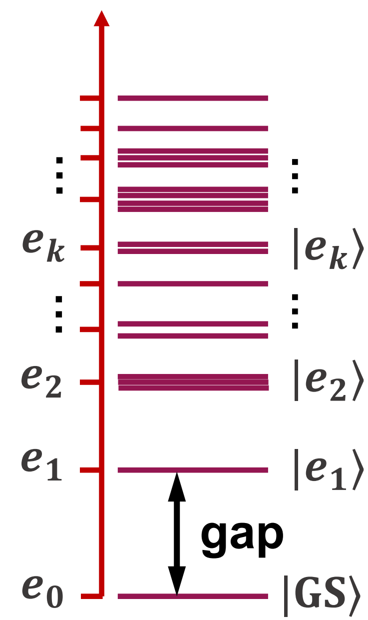
Ground State Entanglement



Local Hamiltonians $H = \sum_{k\sim\ell} H_{k\ell}$

(Hamiltonian need not be 2-local)

This Talk: Gapped Ground States



Ground State $|GS\rangle$

$e_0 = 0$

Gapped Ground States

- Connected to central problems in physics (e.g. low T properties and novel phases of matter)

- Inherit locality of Hamiltonians

  Low-degree $\mathbf{poly(H)} \approx |\mathbf{GS}\rangle\langle\mathbf{GS}|$ [AKLV13]

- Exhibit exponential decay of correlations

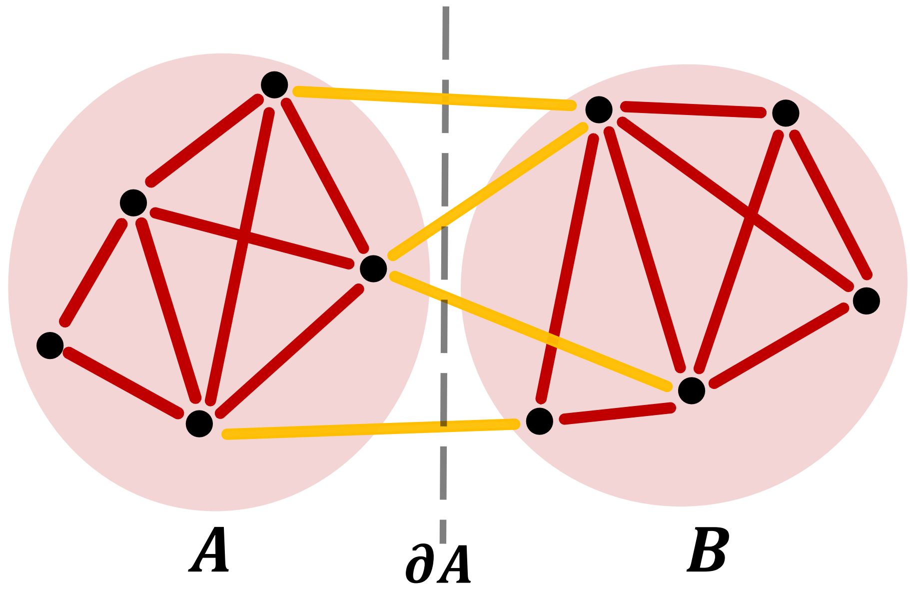
  [Hastings04, HK05]

$$\langle A\rangle = \mathrm{Tr}[A \cdot \mathrm{GS}]$$

$$|\langle A \otimes B\rangle - \langle A\rangle\langle B\rangle| \leq \|A\| \cdot \|B\| \cdot e^{-\mathrm{dist}(A,B)/\xi}$$

- *Short-range entanglement*

Ground State Entanglement



$$|\mathrm{GS}\rangle_{AB} = \sum_k \sqrt{\lambda_k}|k\rangle_A|k\rangle_B$$

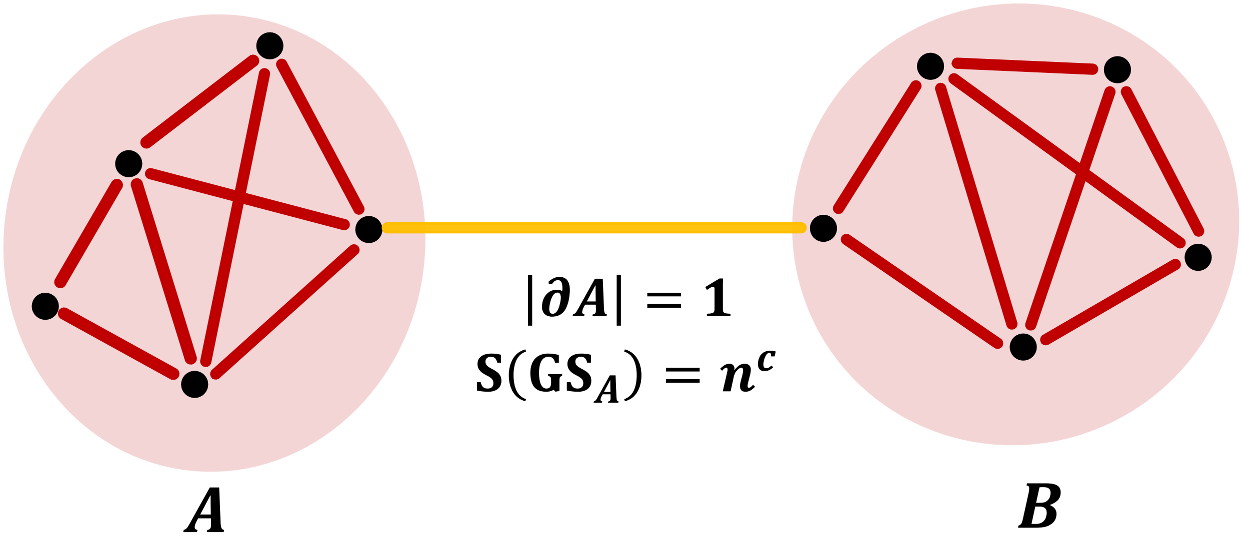
Entanglement Entropy $S(\mathrm{GS}_A) = -\sum_k \lambda_k \log(\lambda_k)$

[Hast07, ALV12, AKLV13] - **Area Law in 1D** $S(\mathrm{GS}_A) \leq \widetilde{O}\left(\frac{|\partial A|}{\text{gap}}\right)$

Used to find efficient MPS approximation

[AAG20, Abr19,…] - **Progress in 2D and Trees**

Ground State Entanglement



$$|GS\rangle_{AB} = \sum_k \sqrt{\lambda_k}|k\rangle_A|k\rangle_B$$

Entanglement Entropy $S(GS_A) = -\sum_k \lambda_k \log(\lambda_k)$

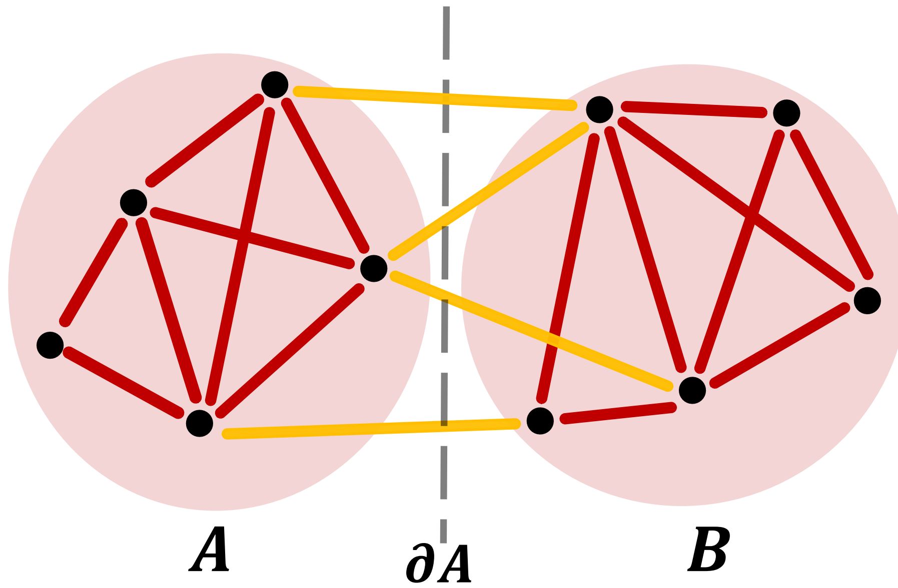
[Hast07, ALV12, AKLV13] - **Area Law in 1D** $\quad S(GS_A) \leq \widetilde{O}\left(\frac{|\partial A|}{\text{gap}}\right)$

Used to find efficient MPS approximation

[AAG20, Abr19,…] - **Progress in 2D and Trees**

[AHL+14] - **Counter Example on General Graphs**

Ground State Entanglement



$$|\mathrm{GS}\rangle_{AB} = \sum_k \sqrt{\lambda_k}|k\rangle_A|k\rangle_B$$

Other structural properties for ground state entanglement?

$$\mathrm{ES}_\varepsilon(\mathrm{GS}_A) = \mathrm{S}^{\varepsilon}_{\max}(\mathrm{GS}_A) - \mathrm{S}^{\varepsilon}_{\min}(\mathrm{GS}_A)$$

Our Result: Area law for Entanglement Spread on *any* Graph

$$\mathrm{ES}_\varepsilon(\mathrm{GS}_A) \le \widetilde{O}\left(\frac{|\partial A|}{\mathrm{gap}} \cdot \log\frac{1}{\varepsilon}\right)$$

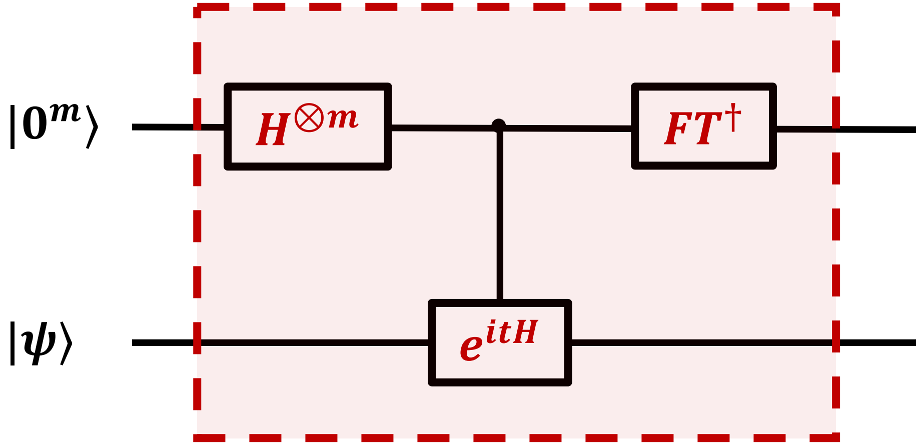
← By designing a testing protocol

Testing Gapped Ground States

Measure energy $\langle\psi|H|\psi\rangle$
- **Yes**: $\langle\psi|H|\psi\rangle \leq \mathbf{gap}/2$
- **No**: $\langle\psi|H|\psi\rangle > \mathbf{gap}/2$

Quantum Phase Estimation

$t = O\left(\frac{1}{\mathbf{gap}}\right)$

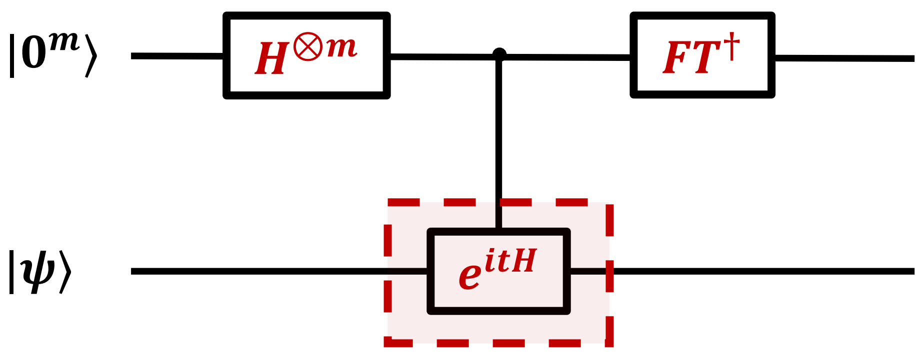


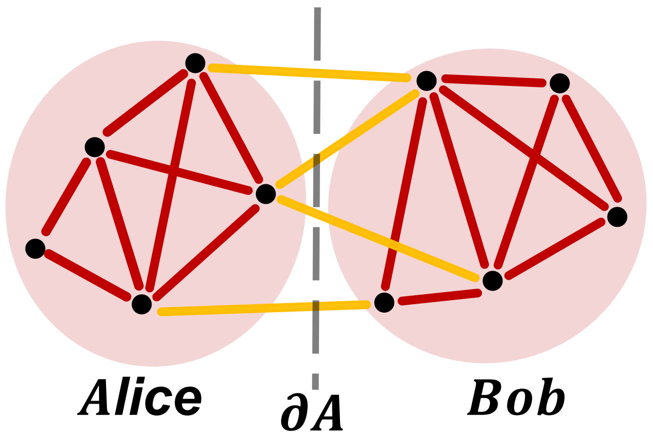
$\approx_\varepsilon \; |\mathrm{GS}\rangle\langle\mathrm{GS}|$

Repeat for $O\left(\log\frac{1}{\varepsilon}\right)$ to get ε approximation

Testing Gapped Ground States

Communication Protocol



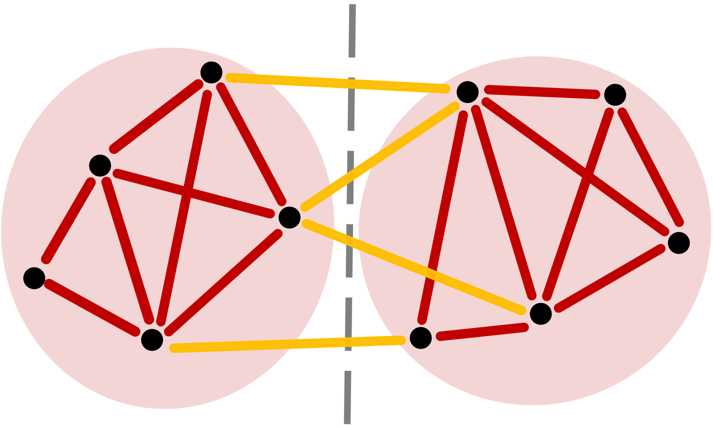


Alice and Bob jointly apply $e^{itH_{AB}}$

$O\left(\frac{|\partial A|}{\text{gap}}\right)$ communications for $t = O(1/\text{gap})$

Overall Communication Cost: $\widetilde{O}(|\partial A|/\text{gap} \cdot \log 1/\varepsilon)$

Hamiltonian Simulation (Performing $e^{itH_{AB}}$)



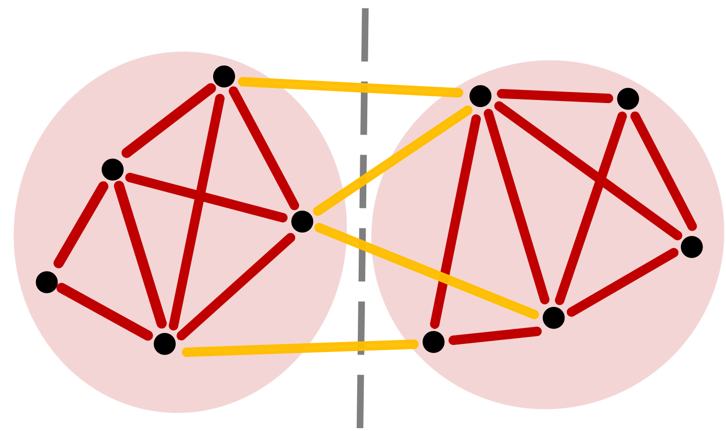
$$H_{AB} = H_A + H_{\partial A} + H_B$$

Depth of Hamiltonian simulation algorithms is $O(t\|H_{AB}\|)$

***Communication cost* of $e^{itH_{AB}}$ is $O(t\|H_{AB}\|)$**

How to improve this to $O(t\|H_{\partial A}\|)$?

Hamiltonian Simulation (Performing $e^{itH_{AB}}$)



$$H_{AB} = H_A + H_{\partial A} + H_B$$

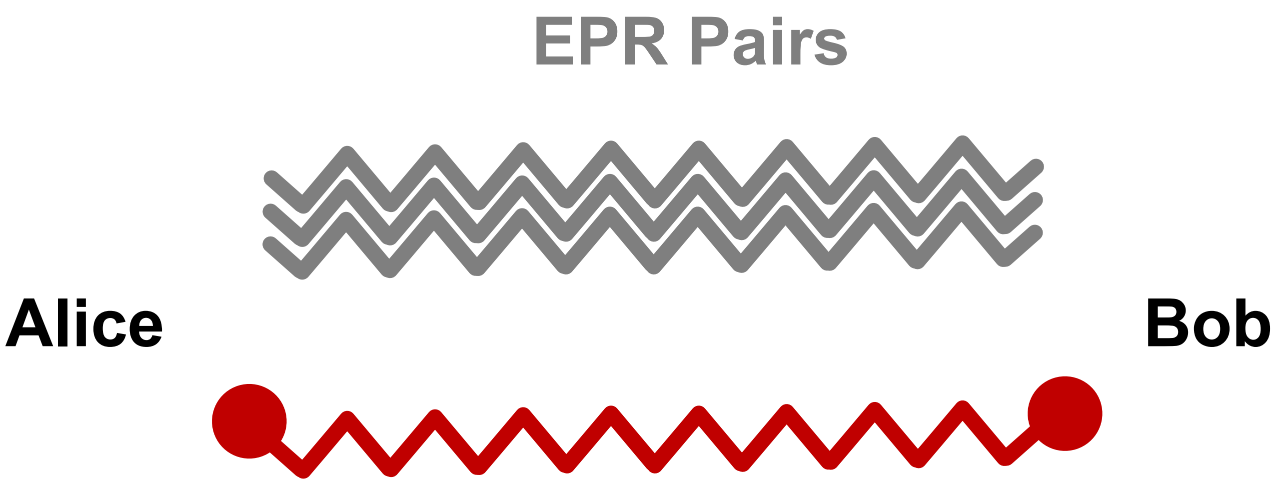
$$e^{itH_{AB}} = e^{itH_A} \cdot e^{itH_B} \cdot e^{itH_{\partial A}}$$ *when $H_A, H_B, H_{\partial A}$ Commute*

Interaction Picture: Time-dependent Hamiltonian [LW18]

$$H_I(t) = e^{-it(H_A+H_B)} \cdot H_{\partial A} \cdot e^{it(H_A+H_B)}$$

$$e^{itH_{AB}} = e^{itH_A} \cdot e^{itH_B} \cdot e^{\int_{\tau=0}^{t} iH_I(\tau)\, \mathrm{d}\tau}$$

Communication Cost of $O(t\|H_I\|) = O(t\|H_{\partial A}\|)$



Communication Complexity ≥ Entanglement Spread

$$C_\varepsilon(\mathbf{GS}_{AB}) \geq \mathbf{ES}_\varepsilon(\mathbf{GS}_A) = \mathbf{S}^\varepsilon_{\max}(\mathbf{GS}_A) - \mathbf{S}^\varepsilon_{\min}(\mathbf{GS}_A)$$

Time complexity of Alice and Bob doesn't matter so

Modify LCU [BCC+15] and use EPR-assistance to implement Taylor expansion of e^{iHt}

Also used to share ancillary registers in QPE

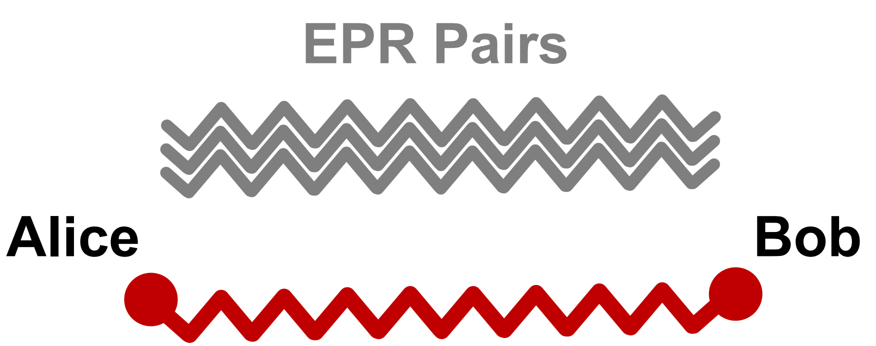
Summary

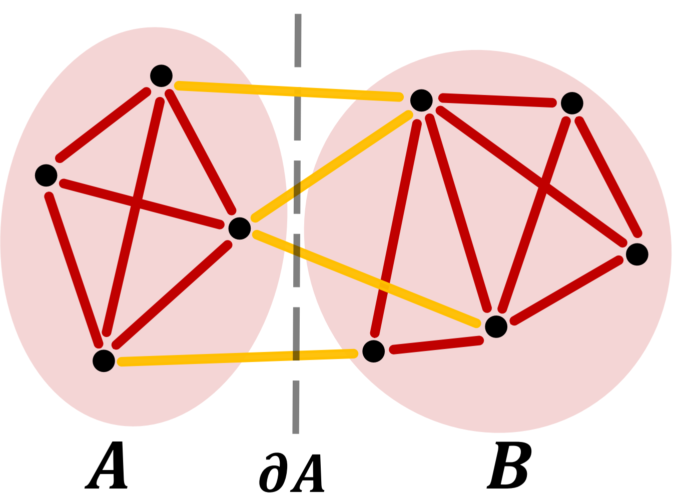
Communication Complexity ≥ Entanglement Spread

$$C_\varepsilon(\Omega_{AB}) \geq ES_\varepsilon(\Omega_A) = S^\varepsilon_{\max}(\Omega_A) - S^\varepsilon_{\min}(\Omega_A)$$

Area law for Entanglement Spread on *any* Graph

$$ES_\varepsilon(GS_A) \leq \widetilde{O}\left(\frac{|\partial A|}{\text{gap}} \cdot \log\frac{1}{\varepsilon}\right)$$





Improvement for Lattices

Sub-Area law for Entanglement Spread on *lattices* (Tight)

$$\mathrm{ES}_{\varepsilon}(\mathrm{GS}_A) \le \tilde{O}\left(\sqrt{\frac{|\partial A|}{\mathrm{gap}}} \cdot \log\frac{1}{\varepsilon}\right)$$

Gives evidence for Li-Haldane Conjecture [LH08] in physics

$\mathrm{GS}_A \approx e^{-H_{\partial A}}$ **Then** $\mathrm{ES}(\mathrm{GS}_A) = O\left(\sqrt{|\partial A|}\right)$

Improvement for Lattices

Sub-Area law for Entanglement Spread on *lattices* (Tight)

$$\mathrm{ES}_\varepsilon(\mathrm{GS}_A) \le \tilde{O}\left(\sqrt{\frac{|\partial A|}{\mathrm{gap}}} \cdot \log\frac{1}{\varepsilon}\right)$$

Implication for Entropy Area Law

Gapped ground states always have small Entanglement Spread

$$S^{\varepsilon}_{\max}(\mathrm{GS_A}) - S^{\varepsilon}_{\min}(\mathrm{GS_A})$$

$S^{\varepsilon}_{\min}(\mathrm{GS_A})$ is small → Entropy Area Law

$S^{\varepsilon}_{\min}(\mathrm{GS_A})$ is large → Violated Entropy Area Law [AHL+14]

Open questions

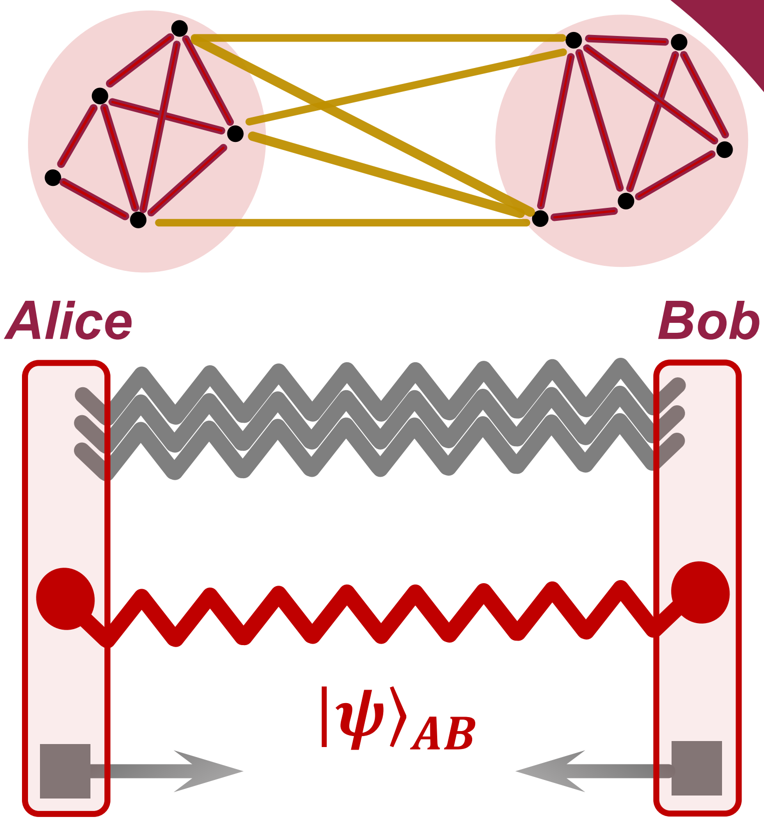
1) Efficient contraction of tensor network representation of gapped ground states from entanglement spread area law?

[AAJ16], [CPSV11]

2) Other applications for our AGSP based on QPE and Hamiltonian simulation?

3) Other universal properties of gapped ground states?

From Communication Complexity to an Entanglement Spread Area Law



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